Homotopy and Isotopy Finiteness of Tight Contact Structures

Lutz Modification

Joint work with
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Plane Fields

ξ: a plane field on a 3-manifold;
τ: a nonsingular vector field in ξ.

There are local coordinates (x, t, z) in which $\tau = \partial_t$ and $\xi = \ker \alpha$ where

$$\alpha = dz - y(x, t, z)\, dx\, .$$

Then

$$\alpha \wedge d\alpha = \partial_t y\, dx \wedge dt \wedge dz,$$

so $\alpha \wedge d\alpha$ does not vanish iff (x, y, z) are local coordinates.

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Contact Structures

- A contact form on an *oriented* 3-manifold is a 1-form α such that $\alpha \wedge d\alpha$ is everywhere positive.
- A contact structure is a (cooriented) plane field ξ defined by a contact form.

Equivalently (Darboux' Theorem):

- A contact strucutre is a plane field ξ locally defined by $dz - y\,dx = 0$, where (x, y, z) are coordinates compatible with the ambient orientation.
- A contact manifold is a manifold equipped with a contact structure.

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Second Model

The 1-form on $\mathbf{R}^3$ defined in cylindrical coordinates by

$$\alpha = f(r)\,dz + g(r)\,d\theta$$

is a contact form iff the parameterized curve

$$r \mapsto \big(f(r), g(r)\big) \in \mathbf{R}^2 \setminus \{0\}$$

revolves about 0 counterclockwise. Indeed,

$$\alpha \wedge d\alpha = \big(f(r)g'(r) - f'(r)g(r)\big)\,dr \wedge d\theta \wedge dz.$$

Example

The contact form $dz + r^2 d\theta$ is equivalent to $dz - y\,dx$:

$$dz + r^2 d\theta = dz + \tfrac{1}{2}(x\,dy - y\,dx) = d(z + \tfrac{1}{2}xy) - y\,dx\,.$$

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Third Model

The 1-form on $\mathbf{R}^3$ defined in Cartesian coordinates by

$$\alpha = \cos\varphi(t)\, dx_1 - \sin\varphi(t)\, dx_2$$

is a contact form iff $\varphi' > 0$:

$$\alpha \wedge d\alpha = \varphi'(t)\, dx_1 \wedge dx_2 \wedge dt\,.$$

Moreover, α is equivalent to $dz - y\,dx$ via the change of variables:

$$x = t, \quad y = \big(\sin\varphi(t)\, x_1 + \cos\varphi(t)\, x_2\big)\, \varphi'(t),$$
$$z = -\big(\cos\varphi(t)\, x_1 - \sin\varphi(t)\, x_2\big).$$

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The contact form $\cos(2n\pi t)\, dx_1 - \sin(2n\pi t)\, dx_2$, for every $n \geq 1$, descends to $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$.

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Conjugation and Isotopy

- Two contact structures are conjugate if there is a diffeomorphism ϕ whose differential takes one to the other;
- They are isotopic if there exists such a ϕ which is isotopic to the identity.
  Then they are homotopic among contact structures: take the path $\xi_t = (\phi_t)_* \xi_0$.

Theorem (Gray)

Two contact structures on a closed manifold are isotopic iff they are homotopic among contact structures.

Corollary

On a closed manifold, there are at most countably many isotopy classes of contact structures.

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Tight / Overtwisted Contact Structures

Definition (Eliashberg)

- A contact structure ξ on V is overtwisted if there is an embedded disk $D \subset V$ which is tangent to ξ at all boundary points:

$$T_pD = \xi_p \quad \text{for every } p \in \partial D.$$

  D itself is called an overtwisted disk.
- A contact structure ξ is tight if it is not overtwisted.

Example

The contact structure on $\mathbf{R}^3$ defined by

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Ground Results

Theorems

Let V be a closed oriented 3-manifold.

(Lutz-Martinet) *Every plane field on V is homotopic to an overtwisted contact structure.*

(Bennequin) *The standard contact structures on $\mathbf{R}^3$ and $\mathbf{S}^3$ are tight.*

(Gromov) *Every fillable contact structure on V is tight.*

(Eliashberg) *If two overtwisted contact structures on V are homotopic among plane fields then they are isotopic.*

(Eliashberg) $\mathbf{R}^3$ *and* $\mathbf{S}^3$ *have a unique tight contact structure up to isotopy.*

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Transversal Triangulations

Theorem (Lutz-Martinet)

Any plane field on a closed oriented manifold V is homotopic to a contact structure.

Lemma (Thurston's Jiggling Lemma)

Given a plane field ξ_0 on V, there exists an arbitrarily fine smooth triangulation such that:

- *ξ_0 is transverse to the 1- and 2-simplices;*
- *the direction of ξ_0 is nearly constant on each 3-simplex.*

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Deformation near the Skeleton

Near any 1- or 2-simplex Σ, there are coordinates (x, t, z) in which

- ξ_0 is defined by $dz = y_0(x, t, z)\, dx$;
- $\Sigma \subset \{t = 0\}$.

Now replace y_0 by a function y_1 satisfying

$$y_1(x, 0, z) = y_0(x, 0, z) \quad \text{and} \quad \partial_t y_1 > 0.$$

This yields a plane field ξ_1 which is contact on a neighborhood U of the 2-skeleton and is an arbitrarily C^0-small deformation of ξ_0.

In each 3-simplex Δ, choose a ball $B = \mathbf{D}^3$ such that

- $\partial B = \mathbf{S}^2$ is contained in U and transverse to ξ_1 except at its poles;
- ξ_1 is transverse to the vector field ∂_z in $B = \mathbf{D}^3$.

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Digging Circular Holes

Connect the north pole of B to its south pole by a path $A \subset U$ transverse to ξ_1.
Take a small tube $N = \mathbf{D}^2 \times A$ around A and consider the solid torus

$$W = B \cup N = \mathbf{D}^2 \times \mathbf{S}^1.$$

Then:

- ξ_1 is a contact structure near ∂W;
- ξ_1 is transverse to the vector field ∂_z, $z \in \mathbf{S}^1$.

So ξ_1 can be defined in W by

$$dz + u(r, \theta, z)\, d\theta + v(r, \theta, z)\, dr = 0$$

where v vanishes near ∂W — provided we suitably identified W to $\mathbf{D}^2 \times \mathbf{S}^1$.

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Filling the Holes

Consider on $\mathbf{R}^2 \times \mathbf{S}^1$ the overtwisted contact structure ξ defined by

$$\cos r\, dz + r \sin r\, d\theta = 0.$$

The function

$$r \in (3\pi/2, 5\pi/2) \longmapsto r \tan r \in (-\infty, +\infty)$$

is a diffeomorhism.
Therefore, for any point $(\theta, z) \in \mathbf{S}^1 \times \mathbf{S}^1 = \partial W$, there is a unique $r(\theta, z) \in (3\pi/2, 5\pi/2)$ such that

$$r(\theta, z) \tan r(\theta, z) = u(1, \theta, z)\,.$$

The restriction of ξ to the solid torus

$$\{(r, \theta, z) \mid r \leq r(\theta, z)\}$$

gives a contact structure on W which coincides with ξ_1 on ∂W.

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Lutz Modification

Any closed transversal curve $C \subset (V, \xi)$ has a standard neighborhood

$$W = \mathbf{D}^2 \times \mathbf{S}^1 \supset C = \{0\} \times \mathbf{S}^1$$

in which ξ is defined by

$$\alpha_0 = 0 \quad \text{where} \quad \alpha_0 = dz + \varepsilon r^2 d\theta\,.$$

Let $r \mapsto (f(r), g(r)) \in \mathbf{R}^2 \setminus \{0\}$ be an arc revolving about 0 and joining $(1,0)$ to $(1, \varepsilon^2)$ after one complet turn.
Then the contat structure defined on W by the form

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Irreducible Manifolds

Definition

A 3-manifold V is irreducible if every embedded 2-sphere in V bounds a 3-ball.

Examples

- $\mathbf{R}^3$ and $\mathbf{S}^3$ are irreducible, as well as all manifolds covered by $\mathbf{R}^3$ or $\mathbf{S}^3$.
- $\mathbf{S}^2 \times \mathbf{S}^1$ and the connected sum of two closed 3-manifolds different from $\mathbf{S}^3$ are not irreducible.

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Compressible Surfaces

Definition

An embedded surface $S \subset V$ is compressible if there exists an embedded disk $D \subset V$ such that $D \cap S = \partial D$.

Theorem (Dehn-Papakyriakopoulos)

An oriented surface $S \subset V$ is incompressible iff the homomorphism $\pi_1 S \to \pi_1 V$ induced by the inclusion map is injective.

Example

If S is a compressible torus in an irreducible 3-manifold V then either S bounds a solid torus or S is contained in a 3-ball inside V.

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Atoroidal Manifolds

Definition

A 3-manifold V is toroidal or atoroidal depending on whether it contains an incompressible torus or not.

Examples

- $\mathbf{T}^3$, torus bundles over the circle, circle bundles over surfaces of positive genus and most Seifert fibered manifolds are toroidal.
- $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{S}^1$ and most closed 3-manifolds (in particular closed hyperbolic 3-manifolds) are atoroidal.

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Finiteness Results

Theorem (homotopy finiteness, Colin-Honda-G)

On a closed 3-manifold, only finitely many homotopy classes of plane fields contain tight contact structures.

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On a closed atoroidal 3-manifold, there are only finitely many isotopy classes of tight contact structures.

Theorem (Colin, Honda-Kazez-Matić)

On a closed, toroidal, and irreducible 3-manifold, there are infinitely many conjugation classes of tight contact structures.

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