Homotopy and Isotopy Finiteness of Tight Contact Structures

Contact Triangulations

Joint work with
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Finiteness Results

Theorem (homotopy finiteness, Colin-Honda-G)

On a closed 3-manifold, only finitely many homotopy classes of plane fields contain tight contact structures.

Theorem (isotopy finiteness, Colin-Honda-G)

On a closed atoroidal 3-manifold, there are only finitely many isotopy classes of tight contact structures.

Theorem (Colin, Honda-Kazez-Matić)

On a closed, toroidal, and irreducible 3-manifold, there are infinitely many conjugation classes of tight contact structures.

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Torsion of Contact Structures

Definition

The torsion of a contact structure ξ is the largest integer $n \geq 0$ for which (V, ξ) contains $\mathbf{T}^2 \times [0,1]$ with the (induced) contact structure defined by

$$\cos(2n\pi t)\, dx_1 - \sin(2n\pi t)\, dx_2 = 0, \quad (x,t) \in \mathbf{T}^2 \times [0,1].$$

Examples

- Overtwisted contact structures have infinite torsion while tight contact structures on atoroidal 3-manifolds have zero torsion.
- The contact structure ξ_n defined on $\mathbf{T}^3$ by

$$\cos(2n\pi t)\, dx_1 - \sin(2n\pi t)\, dx_2 = 0, \quad n \geq 1,$$

  has torsion $n - 1$.
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Finiteness and Torsion

Theorem (Colin-Honda-G)

On a closed 3-manifold, there are only finitely many conjugation classes of tight contact structures with fixed bounded torsion.

Remark

However, there can be infinitely many isotopy classes.
For instance, consider the diffeomorphism

$$\phi\colon \mathbf{T}^3 \to \mathbf{T}^3, \quad (x_1, x_2, t) \mapsto (x_1, x_2, t + x_1).$$

For any fixed $n \geq 1$, the contact structures $(\phi^k)_*\xi_n$, $k \in \mathbf{Z}$, are conjugate to one another but pairwise non-isotopic.

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Characteristic Foliations

On any surface $S \subset (V, \xi)$, the singular line field $\xi \cap TS$ integrates to a singular foliation ξS called the characteristic foliation of S.
This foliation is singular at points p where ξ is tangent to S, *i.e.*, where $T_pS = \xi_p$.

Example

Let $W = \mathbf{D}^2 \times \mathbf{S}^1$ be a standard neighborhood of a closed transversal curve $C = \{0\} \times \mathbf{S}^1$.
The contact structure ξ defined by

$$dz + \varepsilon r^2 d\theta$$

is transverse to each torus $T_r = r\mathbf{S}^1 \times \mathbf{S}^1$ and the characteristic foliation ξT_r is a linear foliation with slope $dz/d\theta = -\varepsilon r^2$.

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Lutz Modification Revisited

(V, ξ): any contact 3-manifold;
$T \subset V$: an embedded torus transversal to ξ;
$W = T \times [0, 1] \subset V$: a tubular neighborhood of T with Legendrian fibers.

In W, ξ is defined by

$$\cos\theta(x,t)\,dx_1 - \sin\theta(x,t)\,dx_2 = 0,$$

where $(x, t) \in \mathbf{R}^2/\mathbf{Z}^2 \times [0, 1]$ and $\partial_t\theta > 0$.
For all $n \geq 0$, the 1-form

$$\cos(\theta(x,t) + 2n\pi t)\,dx_1 - \sin(\theta(x,t) + 2n\pi t)\,dx_2$$

defines another contact structure in W which coincides with ξ on ∂W.

This Lutz modification with coefficient n along T yields a contact structure ξ_n on V which is homotopic to ξ among plane fields.

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Lutz Modification and Compressibility

$\theta\colon W = T \times [0,1] \to \mathbf{S}^1$: the angle between ξ and a fixed "horizontal" direction.

Case of Linear Characteristic Foliations

If θ is homotopic to a constant then Lutz modification tends to increase torsion.
In particular, if T is compressible, ξ_n is overtwisted for every $n \geq 1$.

Characteristic Foliations with Reeb Components

If θ is not homotopic to a constant then Lutz modification does not change the conjugation class: ξ_n is the image of ξ by a Dehn twist.
In particular, if T bounds a solid torus, ξ_n is isotopic to ξ.

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A Finite Generating Set

Theorem (Colin-Honda-G)

On a closed 3-manifold V, there are finitely many tight contact structures $\xi_1, \ldots, \xi_n$ and, for each $i \in \{1, \ldots, n\}$, finitely many tori $T^i_1, \ldots, T^i_{k_i}$ transverse to ξ_i such that every tight contact structure ξ on V, up to isotopy, is obtained from one of the ξ_i's by Lutz modification with coefficients $n^i_j(\xi) \in \mathbf{N}$ along the T^i_j's.

Orientation of Characteristic Foliations

$S \subset (V, \xi)$: a compact oriented surface;
ω: a positive area form on S;
λ: the 1-form induced on S by a defining contact form for ξ.

Orientation Convention

The leaves of ξS are oriented by the vector field η such that

$$\eta \lrcorner \omega = \lambda.$$

Equivalently, the leaves are oriented so that the positive side of ξ is on their left-hand side.

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Convex Surfaces

$S \subset (V, \xi)$: a compact oriented surface with (possibly empty) Legendrian boundary.
S is ξ-convex if it admits a thickening

$$\mathbf{R} \times S \supset S \times \{0\} = S$$

in which ξ is invariant by ∂_t, $t \in \mathbf{R}$.

Proposition

S is ξ-convex iff there is a multi-curve $\Gamma \subset S$ such that:

- *Γ is transverse to ξS;*
- *in each component of $S \setminus \Gamma$, the foliation $\pm\xi S$ is spanned by a vector field which expands area and points outward along Γ.*

The dividing set Γ of a ξ-convex surface completely determines ξ in the homogeneous thickening $\mathbf{R} \times S$.

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Genericity of Convex Surfaces

L: a Legendrian cuve on a compact surface $S \subset (V, \xi)$.
The Thurston-Bennequin number $\mathrm{tb}(L, S)$ of L relative to S is the winding number of ξ around S along L.

If S is ξ-convex with dividing set Γ then

$$\mathrm{tb}(L, S) = -\frac{1}{2}\mathrm{Card}(L \cap \Gamma) \leq 0.$$

Proposition

Let S be a compact surface with (possibly empty) Legendrian boundary. If S is closed or if $\mathrm{tb}(\partial S) \leq 0$ then S can be perturbed to a ξ-convex surface by an arbitrarily small isotopy.

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Tightness and Convex Surfaces

Lemma (Legendrian Realization Lemma)

Let S be a ξ-convex surface, Γ its dividing set and $L \subset S$ a graph such that each component of $S \setminus L$ meets Γ. Then S can be isotoped among ξ-convex surfaces to a surface S' on which L is a Legendrian graph.

Corollary

If ξ is tight and if S is a xi-convex surface different from $\mathbf{S}^2$ then Γ has no contractible closed component.

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Legendrian Approximation

Lemma

Any smooth curve in a contact manifold (V, ξ) can be made Legendrian by an arbitrarily $\mathcal{C}^0$-small isotopy. Moreover, this isotopy can be chosen fixed on any sub-arc which is already Legendrian.

Proof. Consider $\mathbf{R}^3$ with the contact form $dy - z\,dx$. A parameterized arc $(x(t), y(t), z(t))$ is Legendrian iff

$$z(t) = \dot{y}(t)/\dot{x}(t)$$

and, in particular, $(x(t), y(t))$ has no vertical tangent line. A typical example is $(3t^2, 2t^3, t)$ whose projection to the xy-plane has a cusp in 0.
Take any curve $(x_0(t), y_0(t), z_0(t))$. We can approximate $(x_0(t), y_0(t))$ by a curve $(x(t), y(t))$ (with cusps but no vertical tangent line) whose slope $\dot{y}/\dot{x}$ approximates z_0.

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Triangulations

What is called here a triangulation of V is a simplicial decomposition of V (as a topological space) satisfying the following regularity conditions:

- every 1- or 2-simplix is smooth;
- every 3-simplex has, along its edges except at vertices, a dihedral angle in $(0, \pi)$.

Any smooth triangulation of V satisfies these conditions.

An isotopy of triangulations is a continuous 1-parameter family of triangulations.

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Contact Triangulations

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- 2-simplices are ξ-convex and ξ-disciplined;
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The Thurston-Bennequin number of a contact triangulation Δ is

$$\mathrm{TB}(\Delta) = -\sum_{F} \mathrm{tb}(\partial F)$$

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Minimal Contact Triangulations

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A set of tight contact structures on V is complete if it represents all isotopy classes of such structures.

Proposition

There exist a triangulation Δ of V and a complete set $\mathcal{X}$ of tight contact structures on V such that Δ is a minimal contact triangulation of (V, ξ) for every $\xi \in \mathcal{X}$.

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Minimal Contact Triangulations

A contact triangulation Δ of (V, ξ) is minimal if it has the smallest Thurston-Bennequin number among all contact triangulations which are isotopic to Δ rel. a neighborhood of vertices.

A set of tight contact structures on V is complete if it represents all isotopy classes of such structures.

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Existence of Contact Triangulations

ξ: a tight contact structure on V;
Δ_0: an arbitrary smooth triangulation of V.
Isotope Δ_0 to a contact triangulation Δ of (V, ξ) as follows:

- approximate the 1-simplices by ξ-Legendrian arcs;
- make the 2-simplices ξ-convex by a small perturbation relative to their boundaries;
- note that 3-simplices are contained in Darboux charts because ξ is tight (Eliashberg's Theorem).

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Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ_F: the dividing set of F.

Lemma

If a component C of Γ_F has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma_F \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma_F \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number. Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Dividing Sets of 2-simplices

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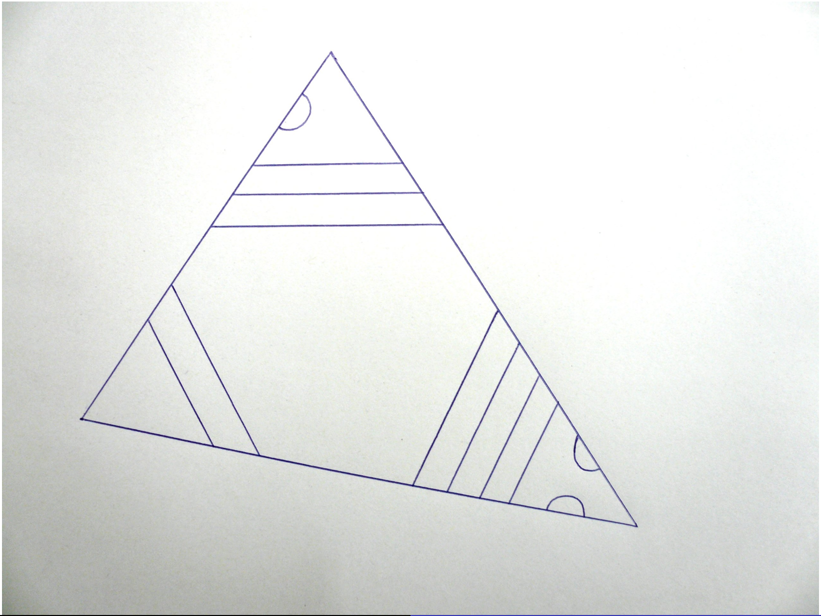
Typical Dividing Set



Associated Normalized Foliation