Homotopy and Isotopy Finiteness of Tight Contact Structures

Branched Surfaces

Joint work with
Vincent Colin and Ko Honda

ENS-Lyon — CNRS

June 1st, 2007

A Finite Generating Set

Theorem (Colin-Honda-G)

On a closed 3-manifold V, there are finitely many tight contact structures $\xi_1, \ldots, \xi_n$ and, for each $i \in \{1, \ldots, n\}$, finitely many tori $T^i_1, \ldots, T^i_{k_i}$ transverse to ξ_i such that every tight contact structure ξ on V, up to isotopy, is obtained from one of the ξ_i's by Lutz modification with coefficients $n^i_j(\xi) \in \mathbf{N}$ along the T^i_j's.

Contact Triangulations

Definition

A contact triangulation of (V, ξ) is a triangulation Δ of V such that:

- 1-simplices are Legendrian;
- 2-simplices are ξ-convex and ξ-disciplined;
- 3-simplices are contained in Darboux charts.

The Thurston-Bennequin number of a contact triangulation Δ is

$$\mathrm{TB}(\Delta) = -\sum_{F} \mathrm{tb}(\partial F)$$

where F ranges over 2-simplices.

Contact Triangulations

Definition

A contact triangulation of (V, ξ) is a triangulation Δ of V such that:

- 1-simplices are Legendrian;
- 2-simplices are ξ-convex and ξ-disciplined;
- 3-simplices are contained in Darboux charts.

The Thurston-Bennequin number of a contact triangulation Δ is

$$\mathrm{TB}(\Delta) = -\sum_{F} \mathrm{tb}(\partial F)$$

where F ranges over 2-simplices.

Contact Triangulations

Definition

A contact triangulation of (V, ξ) is a triangulation Δ of V such that:

- 1-simplices are Legendrian;
- 2-simplices are ξ-convex and ξ-disciplined;
- 3-simplices are contained in Darboux charts.

The Thurston-Bennequin number of a contact triangulation Δ is

$$\mathrm{TB}(\Delta) = -\sum_F \mathrm{tb}(\partial F)$$

where F ranges over 2-simplices.

Contact Triangulations

Definition

A contact triangulation of (V, ξ) is a triangulation Δ of V such that:

- 1-simplices are Legendrian;
- 2-simplices are ξ-convex and ξ-disciplined;
- 3-simplices are contained in Darboux charts.

The Thurston-Bennequin number of a contact triangulation Δ is

$$\mathrm{TB}(\Delta) = -\sum_{F} \mathrm{tb}(\partial F)$$

where F ranges over 2-simplices.

Minimal Contact Triangulations

A contact triangulation Δ of (V, ξ) is minimal if it has the smallest Thurston-Bennequin number among all contact triangulations which are isotopic to Δ rel. a neighborhood of vertices.

A set of tight contact structures on V is complete if it represents all isotopy classes of such structures.

Proposition

There exist a triangulation Δ of V and a complete set $\mathcal{X}$ of tight contact structures on V such that Δ is a minimal contact triangulation of (V, ξ) for every $\xi \in \mathcal{X}$.

Minimal Contact Triangulations

A contact triangulation Δ of (V, ξ) is minimal if it has the smallest Thurston-Bennequin number among all contact triangulations which are isotopic to Δ rel. a neighborhood of vertices.

A set of tight contact structures on V is complete if it represents all isotopy classes of such structures.

Proposition

There exist a triangulation Δ of V and a complete set $\mathcal{X}$ of tight contact structures on V such that Δ is a minimal contact triangulation of (V, ξ) for every $\xi \in \mathcal{X}$.

Minimal Contact Triangulations

A contact triangulation Δ of (V, ξ) is minimal if it has the smallest Thurston-Bennequin number among all contact triangulations which are isotopic to Δ rel. a neighborhood of vertices.

A set of tight contact structures on V is complete if it represents all isotopy classes of such structures.

Proposition

There exist a triangulation Δ of V and a complete set $\mathcal{X}$ of tight contact structures on V such that Δ is a minimal contact triangulation of (V, ξ) for every $\xi \in \mathcal{X}$.

Dividing Sets of 2-Simplices

Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ: the dividing set of F.

Lemma

If a component C of Γ has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number. Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Dividing Sets of 2-Simplices

Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ: the dividing set of F.

Lemma

If a component C of Γ has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number. Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Dividing Sets of 2-Simplices

Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ: the dividing set of F.

Lemma

If a component C of Γ has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number.
Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Dividing Sets of 2-Simplices

Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ: the dividing set of F.

Lemma

If a component C of Γ has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number. Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Dividing Sets of 2-Simplices

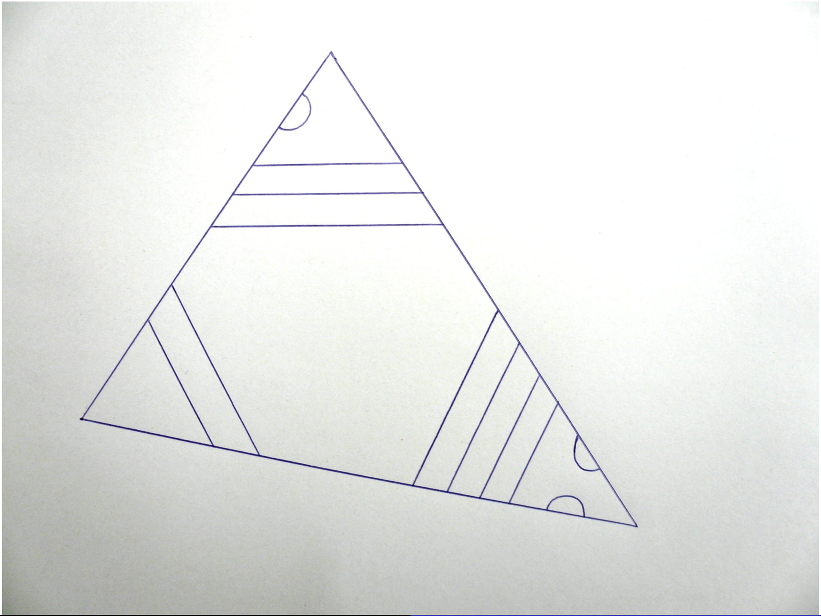
Δ: a minimal contact triangulation of (V, ξ);
F: a 2-simplex of Δ.
Γ: the dividing set of F.

Lemma

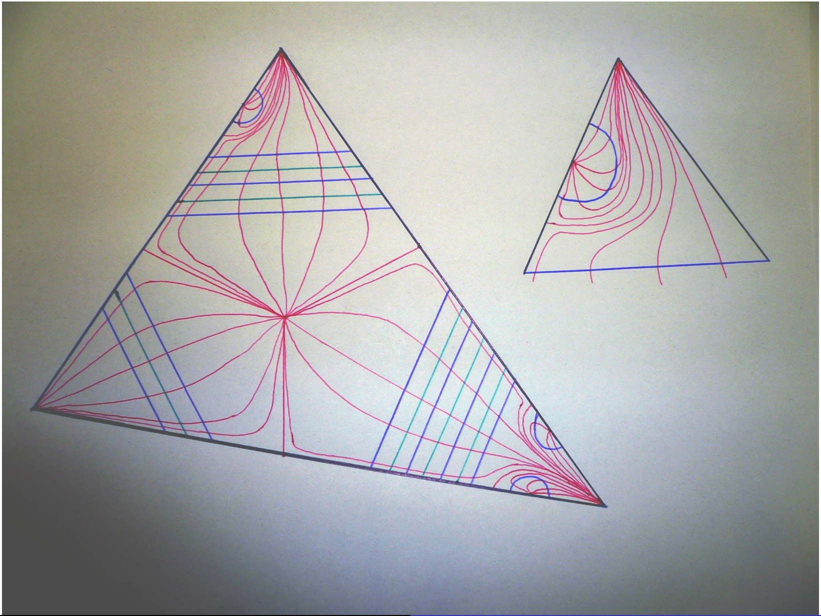
If a component C of Γ has its two endpoints on the same edge E then one of these endpoints is outermost on $\Gamma \cap E$.

Proof. If no endpoint of C is outermost in $\Gamma \cap E$, deform Δ (by an isotopy supported in the open star of E) to a contact triangulation with smaller Thurston-Bennequin number.
Isotope F to $F' = F \setminus U$ where U is a small open neighborhood in F of the half-disk cut by C and make $\partial F'$ Legendrian using the LR-Lemma.

Typical Dividing Set



Associated Normalized Foliation



Flat Legendrian Arcs

G: a 3-simplex of a minimal contact triangulation.
A piecewise smooth Legendrian arc $L \subset \partial G$ is flat if it avoids the vertices and if ξ is a supporting hyperplane of ∂G at all points of L.
Such an arc is a union of edge segments and singularity arcs of 2-simplices.

Lemma (Holonomy Lemma)

Let $L \subset \partial G$ be a flat Legendrian arc. Assume there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and let v_a, v_b denote the inward looking tangent vectors to L at a and b, respectively. Then $(v_a, b - a, v_b)$ is a positive basis and the Legendrian curve $\widehat{L} = L \cup [a, b]$ has Thurston-Bennequin number -1.

Flat Legendrian Arcs

G: a 3-simplex of a minimal contact triangulation.
A piecewise smooth Legendrian arc $L \subset \partial G$ is flat if it avoids the vertices and if ξ is a supporting hyperplane of ∂G at all points of L.
Such an arc is a union of edge segments and singularity arcs of 2-simplices.

Lemma (Holonomy Lemma)

Let $L \subset \partial G$ be a flat Legendrian arc. Assume there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and let v_a, v_b denote the inward looking tangent vectors to L at a and b, respectively. Then $(v_a, b - a, v_b)$ is a positive basis and the Legendrian curve $\widehat{L} = L \cup [a, b]$ has Thurston-Bennequin number -1.

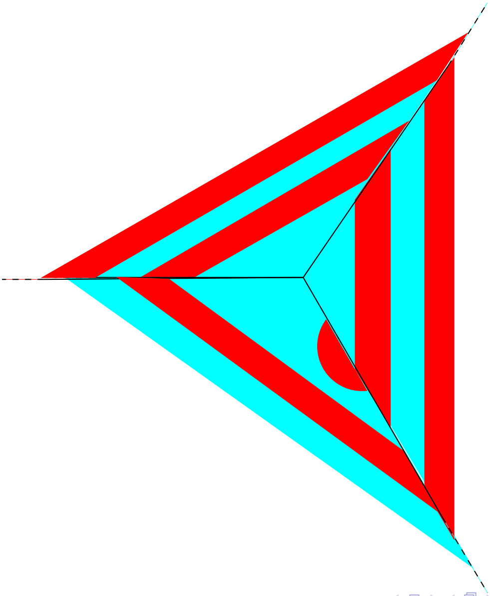
Flat Legendrian Arcs

G: a 3-simplex of a minimal contact triangulation.
A piecewise smooth Legendrian arc $L \subset \partial G$ is flat if it avoids the vertices and if ξ is a supporting hyperplane of ∂G at all points of L.
Such an arc is a union of edge segments and singularity arcs of 2-simplices.

Lemma (Holonomy Lemma)

Let $L \subset \partial G$ be a flat Legendrian arc. Assume there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and let v_a, v_b denote the inward looking tangent vectors to L at a and b, respectively. Then $(v_a, b - a, v_b)$ is a positive basis and the Legendrian curve $\widehat{L} = L \cup [a, b]$ has Thurston-Bennequin number -1.

The Holonomy Phenomenon



Fibered Domains

M: a compact domain with boundary and edges but no corners;
τ: a foliation of M by compact intervals.
(M, τ) is a fibered domain if ∂M is the union of two smooth compact surfaces $\partial_h M, \partial_v M$ such that:

- $\partial_h M$ is transverse to τ while $\partial_v M$ is tangent to τ;
- $\partial_h M$ and $\partial_v M$ have the same boundary.

Proposition

There exist finitely many fibered domains (M_i, τ_i) in V, each given with a contact structures ζ_i on $V \setminus \mathrm{Int} M_i$, such that every tight contact structure on V, up to isotopy and for some i, is equal to ζ_i out of M_i and tangent to τ_i in M_i.

Fibered Domains

M: a compact domain with boundary and edges but no corners;
τ: a foliation of M by compact intervals.
(M, τ) is a fibered domain if ∂M is the union of two smooth compact surfaces $\partial_h M, \partial_v M$ such that:

- $\partial_h M$ is transverse to τ while $\partial_v M$ is tangent to τ;
- $\partial_h M$ and $\partial_v M$ have the same boundary.

Proposition

There exist finitely many fibered domains (M_i, τ_i) in V, each given with a contact structures ζ_i on $V \setminus \mathrm{Int} M_i$, such that every tight contact structure on V, up to isotopy and for some i, is equal to ζ_i out of M_i and tangent to τ_i in M_i.

Fibered Domains

M: a compact domain with boundary and edges but no corners;
τ: a foliation of M by compact intervals.
(M, τ) is a fibered domain if ∂M is the union of two smooth compact surfaces $\partial_h M, \partial_v M$ such that:

- $\partial_h M$ is transverse to τ while $\partial_v M$ is tangent to τ;
- $\partial_h M$ and $\partial_v M$ have the same boundary.

Proposition

There exist finitely many fibered domains (M_i, τ_i) in V, each given with a contact structures ζ_i on $V \setminus \mathrm{Int} M_i$, such that every tight contact structure on V, up to isotopy and for some i, is equal to ζ_i out of M_i and tangent to τ_i in M_i.

Branched Surfaces

$S_0, S_1, S_2 \subset \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$: the respective graphs of the functions on $\mathbf{R}^2$ defined by

$$f_0(x,y) = 0, \qquad f_1(x,y) = -f_2(y,x), \qquad f_2(x,y) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{1/x} & \text{if } x < 0; \end{cases}$$

$\mathbf{X} = (S_0 \cup S_1 \cup S_2) \cap \{x \geq -1\} \subset \mathbf{R}^3$.

A homeomorphism between two open subsets $U, U' \subset \mathbf{X}$ is smooth if its restriction to each $S_i \cap U$ is.

A branched surface (with boundary) is a topological space X locally modeled on $\mathbf{X}$ with smooth transition maps.

$\mathrm{Reg} X \subset X$ is the open set of points where X is a genuine surface with boundary.

Example

If (M, τ) is a fibered domain then $X = M/\tau$ is a branched surface.

Branched Surfaces

$S_0, S_1, S_2 \subset \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$: the respective graphs of the functions on $\mathbf{R}^2$ defined by

$$f_0(x,y) = 0, \qquad f_2(x,y) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{1/x} & \text{if } x < 0; \end{cases}$$
$$f_1(x,y) = -f_2(y,x),$$

$\mathbf{X} = (S_0 \cup S_1 \cup S_2) \cap \{x \geq -1\} \subset \mathbf{R}^3$.

A homeomorphism between two open subsets $U, U' \subset \mathbf{X}$ is smooth if its restriction to each $S_i \cap U$ is.

A branched surface (with boundary) is a topological space X locally modeled on $\mathbf{X}$ with smooth transition maps.

$\mathrm{Reg}X \subset X$ is the open set of points where X is a genuine surface with boundary.

Example

If (M, τ) is a fibered domain then $X = M/\tau$ is a branched surface.

Branched Surfaces

$S_0, S_1, S_2 \subset \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$: the respective graphs of the functions on $\mathbf{R}^2$ defined by

$$f_0(x,y) = 0, \quad f_1(x,y) = -f_2(y,x), \qquad f_2(x,y) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{1/x} & \text{if } x < 0; \end{cases}$$

$\mathbf{X} = (S_0 \cup S_1 \cup S_2) \cap \{x \geq -1\} \subset \mathbf{R}^3$.
A homeomorphism between two open subsets $U, U' \subset \mathbf{X}$ is smooth if its restriction to each $S_i \cap U$ is.
A branched surface (with boundary) is a topological space X locally modeled on $\mathbf{X}$ with smooth transition maps.
$\mathrm{Reg}X \subset X$ is the open set of points where X is a genuine surface with boundary.

Example

If (M, τ) is a fibered domain then $X = M/\tau$ is a branched surface.

Branched Surfaces

$S_0, S_1, S_2 \subset \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$: the respective graphs of the functions on $\mathbf{R}^2$ defined by

$$f_0(x,y) = 0, \qquad f_2(x,y) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{1/x} & \text{if } x < 0; \end{cases}$$
$$f_1(x,y) = -f_2(y,x),$$

$\mathbf{X} = (S_0 \cup S_1 \cup S_2) \cap \{x \geq -1\} \subset \mathbf{R}^3$.

A homeomorphism between two open subsets $U, U' \subset \mathbf{X}$ is smooth if its restriction to each $S_i \cap U$ is.

A branched surface (with boundary) is a topological space X locally modeled on $\mathbf{X}$ with smooth transition maps.

$\mathrm{Reg}X \subset X$ is the open set of points where X is a genuine surface with boundary.

Example

If (M, τ) is a fibered domain then $X = M/\tau$ is a branched surface.

Branched Surfaces

$S_0, S_1, S_2 \subset \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$: the respective graphs of the functions on $\mathbf{R}^2$ defined by

$$f_0(x,y) = 0, \qquad f_1(x,y) = -f_2(y,x), \qquad f_2(x,y) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{1/x} & \text{if } x < 0; \end{cases}$$

$\mathbf{X} = (S_0 \cup S_1 \cup S_2) \cap \{x \geq -1\} \subset \mathbf{R}^3$.
A homeomorphism between two open subsets $U, U' \subset \mathbf{X}$ is smooth if its restriction to each $S_i \cap U$ is.
A branched surface (with boundary) is a topological space X locally modeled on $\mathbf{X}$ with smooth transition maps.
$\mathrm{Reg}X \subset X$ is the open set of points where X is a genuine surface with boundary.

Example

If (M, τ) is a fibered domain then $X = M/\tau$ is a branched surface.

Construction of Branched Surfaces

For each 3-simplex G, consider all flat Legendrian arcs $L \subset \partial G$ for which there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and set $\widehat{L} = L \cup [a, b]$.

Any such $\widehat{L}$ bounds a disk in G. Identify two such disks if their boundaries $\widehat{L}$ and $\widehat{L}'$ are isotopic on ∂G in the complement of vertices.

Now glue the disks so-obtained in adjacent 3-simplices iff they intersect the common facet along isotopic arcs (again in the complement of vertices).

Construction of Branched Surfaces

For each 3-simplex G, consider all flat Legendrian arcs $L \subset \partial G$ for which there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and set $\widehat{L} = L \cup [a, b]$.

Any such $\widehat{L}$ bounds a disk in G. Identify two such disks if their boundaries $\widehat{L}$ and $\widehat{L}'$ are isotopic on ∂G in the complement of vertices.

Now glue the disks so-obtained in adjacent 3-simplices iff they intersect the common facet along isotopic arcs (again in the complement of vertices).

For each 3-simplex G, consider all flat Legendrian arcs $L \subset \partial G$ for which there is an edge E such that $L \cap E = \partial L = \{a, b\}$ and set $\widehat{L} = L \cup [a, b]$.
Any such $\widehat{L}$ bounds a disk in G. Identify two such disks if their boundaries $\widehat{L}$ and $\widehat{L}'$ are isotopic on ∂G in the complement of vertices.
Now glue the disks so-obtained in adjacent 3-simplices iff they intersect the common facet along isotopic arcs (again in the complement of vertices).

Adjusted Contact Structures

(M, τ): a fibered domain in V;
$X = M/\tau$: the quotient branched surface;
ζ: a contact structure on $V \setminus \mathrm{Int} M$.

A contact structure is adjusted to (M, τ, ζ) if it is equal to ζ out of M and tangent to τ in M.

Every adjusted contact structure ξ is determined, up to homotopy among such structures, by the function

$$a_\xi : \partial_h M \to (0, \infty)$$

which maps each point p to the total rotation angle of ξ along the leaf of τ starting at p.
(This angle is measured using an auxiliary metric and the holonomy of τ.)

Adjusted Contact Structures

(M, τ): a fibered domain in V;
$X = M/\tau$: the quotient branched surface;
ζ: a contact structure on $V \setminus \mathrm{Int} M$.

A contact structure is adjusted to (M, τ, ζ) if it is equal to ζ out of M and tangent to τ in M.

Every adjusted contact structure ξ is determined, up to homotopy among such structures, by the function

$$a_\xi : \partial_h M \to (0, \infty)$$

which maps each point p to the total rotation angle of ξ along the leaf of τ starting at p.
(This angle is measured using an auxiliary metric and the holonomy of τ.)

Adjusted Contact Structures

(M, τ): a fibered domain in V;
$X = M/\tau$: the quotient branched surface;
ζ: a contact structure on $V \setminus \mathrm{Int} M$.

A contact structure is adjusted to (M, τ, ζ) if it is equal to ζ out of M and tangent to τ in M.

Every adjusted contact structure ξ is determined, up to homotopy among such structures, by the function

$$a_\xi : \partial_h M \to (0, \infty)$$

which maps each point p to the total rotation angle of ξ along the leaf of τ starting at p.
(This angle is measured using an auxiliary metric and the holonomy of τ.)

(M, τ, ζ): as before;
ξ_0: a fixed adjusted contact structure.

Proposition

Homotopy classes of adjusted contact structures ξ are in one-to-one correspondence with functions

$$w_\xi : \pi_0(\mathrm{Reg}\, X) \to \mathbf{Z}$$

satisfying the condition

- $w_\xi(R) > -\frac{1}{2\pi} \inf_{\widetilde{R}} a_{\xi_0}$ *and the adjacency relations*
- $w_\xi(R_1) + w_\xi(R_2) = w_\xi(R_3)$ *for every pair of regular sheets R_1, R_2 of X which merge to give R_3.*

Every solution of this system is obtained from a minimal solution by adding a positive solution.

(M, τ, ζ): as before;
ξ_0: a fixed adjusted contact structure.

Proposition

Homotopy classes of adjusted contact structures ξ are in one-to-one correspondence with functions

$$w_\xi : \pi_0(\mathrm{Reg} X) \to \mathbf{Z}$$

satisfying the condition

- $w_\xi(R) > -\frac{1}{2\pi} \inf_{\tilde{R}} a_{\xi_0}$ *and the adjacency relations*
- $w_\xi(R_1) + w_\xi(R_2) = w_\xi(R_3)$ *for every pair of regular sheets R_1, R_2 of X which merge to give R_3.*

Every solution of this system is obtained from a minimal solution by adding a positive solution.

(M, τ, ζ): as before;
ξ_0: a fixed adjusted contact structure.

Proposition

Homotopy classes of adjusted contact structures ξ are in one-to-one correspondence with functions

$$w_\xi : \pi_0(\mathrm{Reg} X) \to \mathbf{Z}$$

satisfying the condition

- $w_\xi(R) > -\frac{1}{2\pi} \inf_{\widetilde{R}} a_{\xi_0}$ *and the adjacency relations*
- $w_\xi(R_1) + w_\xi(R_2) = w_\xi(R_3)$ *for every pair of regular sheets R_1, R_2 of X which merge to give R_3.*

Every solution of this system is obtained from a minimal solution by adding a positive solution.

Linear Algebra

(M, τ, ζ): as before;
ξ_0: a fixed adjusted contact structure.

Proposition

Homotopy classes of adjusted contact structures ξ are in one-to-one correspondence with functions

$$w_\xi : \pi_0(\mathrm{Reg} X) \to \mathbf{Z}$$

satisfying the condition

- $w_\xi(R) > -\frac{1}{2\pi} \inf_{\widetilde{R}} a_{\xi_0}$ *and the adjacency relations*
- $w_\xi(R_1) + w_\xi(R_2) = w_\xi(R_3)$ *for every pair of regular sheets R_1, R_2 of X which merge to give R_3.*

Every solution of this system is obtained from a minimal solution by adding a positive solution.

Conclusion

The minimal solutions of the system give finitely many adjusted contact structures.

Any positive solution of the system corresponds to a surface in M transverse to τ: over each component of $\mathrm{Reg}X$, put the number of sheets given by the solution and glue them along the singular locus.

Since any such surface is transverse to any adjusted contact structure, it is a union of tori and Klein bottles.

A suitable Lutz modification along this surface yields an adjusted contact structure with the right weights.

Conclusion

The minimal solutions of the system give finitely many adjusted contact structures.

Any positive solution of the system corresponds to a surface in M transverse to τ: over each component of $\mathrm{Reg}X$, put the number of sheets given by the solution and glue them along the singular locus.

Since any such surface is transverse to any adjusted contact structure, it is a union of tori and Klein bottles.

A suitable Lutz modification along this surface yields an adjusted contact structure with the right weights.

Conclusion

The minimal solutions of the system give finitely many adjusted contact structures.

Any positive solution of the system corresponds to a surface in M transverse to τ: over each component of $\mathrm{Reg}X$, put the number of sheets given by the solution and glue them along the singular locus.

Since any such surface is transverse to any adjusted contact structure, it is a union of tori and Klein bottles.

A suitable Lutz modification along this surface yields an adjusted contact structure with the right weights.

Conclusion

The minimal solutions of the system give finitely many adjusted contact structures.

Any positive solution of the system corresponds to a surface in M transverse to τ: over each component of $\mathrm{Reg}X$, put the number of sheets given by the solution and glue them along the singular locus.

Since any such surface is transverse to any adjusted contact structure, it is a union of tori and Klein bottles.

A suitable Lutz modification along this surface yields an adjusted contact structure with the right weights.