Projective transformation groups of pseudo-Riemannian manifolds

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Geometry and Dynamics
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Geometry and Dynamics

(a biased point of view?)

Meaning of "Geometry and Dynamics"?

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Goal: Put them in competition and see what happens $\rightarrow$ Classify

Gromov's vague conjecture

*"**Rigid** Geometric Structures on compact manifolds with Large automorphism group are **Classifiable**!"*

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There is a (mathematical) definition of "geometric structure" and "rigid" but none for "large" and "classifiable"

Definition of Rigidity, by examples

Rigid: (solid)
Riemannian metric
Pseudo-Riemannian metric
Affine connection

Definition of Rigidity, by examples

Rigid: (solid)
Riemannian metric
Pseudo-Riemannian metric
Affine connection

Non-rigid: (fluid)
Symplectic structure
Contact structure
Complex structure
A plane field

Steps (in working with the conjecture)

0. Pick a geometric structure (or a class of them).
1. State a precise conjecture: important and non-trivial step
2. Prove it.

Projective structures

• Connection ∇ : it allows derivation of vector fields on M

• Geometrically: a second order differential equation on $M \to$ a vector field $\mathcal{G}_\nabla$ on M whose orbits are the geodesics of ∇

There is equivalence $\nabla \iff \mathcal{G}_\nabla$
∇ is determined by $\mathcal{G}_\nabla$

• Projective structure (or projective connection?): a class of connections having the same non-parametrized geodesics

Morphisms

A diffeomorphism $(M, \nabla) \to (M', \nabla')$ is projective if it sends unparameterized geodesics of ∇ to unparameterized geodesics of ∇'

Metric projective structures

g Riemannian metric
∇_g its Levi-Cevita connection
σ_g the projective structure associated to ∇_g

Levi-Civita connection exists in the pseudo-Riemannian case (although there is no direct variational definition of geodesics)..

Two metrics g and $\bar{g}$ on M are projectively equivalent if Id_m is projective between ∇_g and $\nabla_{\bar{g}}$

Automorphisms groups

• $\mathrm{Aut}(M, g)$: isometries of g: $\mathrm{Iso(M, g)}$

• $Aut(M, \nabla_g)$: diffeomorphisms preserving (parameterized) geodesics of g, classically called $\mathrm{Aff}\,(M, g)$

• $\mathrm{Aut}(M, \sigma_g)$: diffeomorphisms preserving unparameterized geodesics of g, classically $\mathrm{Proj(M, g)}$

Clearly: $\mathrm{Iso(M, g)} \subset \mathrm{Aff\,(M, g)} \subset \mathrm{Proj(M, g)}$

Experimental result

Our first main result is to give a precise statement of Gromov's conjecture for metric projective structures:

Conjecture (Projective Lichnerowicz conjecture)

Let (M, g) be a compact pseudo-Riemannian manifold.
Then $\mathrm{Proj(M, g)/Aff\,(M, g)}$ *is finite unless* (M, g) *is isometrically covered by the standard Riemannian sphere (up to constant)?*

(The new fact here is the consider the full groups instead of their identity components...)

Mathematical Results

Theorem (The conjecture is true in the Riemannian and in the analytic-Lorentz cases)

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Theorem (The conjecture is true in the Riemannian and in the analytic-Lorentz cases)

Let (M, g) be a compact pseudo-Riemannian manifold.

• *If g is Riemannian, then* $\mathrm{Proj(M, g)/Aff\,(M, g)}$ *is finite, unless M is a (finite) quotient of the standard sphere.*

•• *If g is Lorentzian and analytic, then* $\mathrm{Proj(M, g)/Aff\,(M, g)}$ *is finite.*

All objects in a simple case

Geodesics

$\mathbb{R}^n$ Euclidean space

(Parameterized) geodesic: a curve with constant speed and whose support is a straight line segment
$\gamma :]a, b[\to \mathbb{R}^n$, $\frac{d^2\gamma}{dt^2} = 0$

Non-parameterized geodesic: γ has its image in a straight segment $\iff$ there exists function $p(t)$ such $\frac{d^2\gamma}{dt^2} = p(t)\frac{d\gamma}{dt}$

Transformations

U, V open sets in $\mathbb{R}^n$, $f : U \to V$ diffeomorphism,

f **isometric** if it preserves distances,
Then, $\exists\ A \in O(n, (\mathbb{R}))$, $b \in \mathbb{R}^n$, such that $f =$ restriction to U of the map $x \to Ax + b$

f **affine** if it sends (a parameterized) geodesic to a (parametrized) geodesic.
Then, $\exists\ A \in \mathrm{GL}_n(\mathbb{R}))$, $b \in \mathbb{R}^n$, such that $f =$ restriction to U of the map $x \to Ax + b$

f **projective** if it sends (a umparameterized) geodesic to a (umparametrized) geodesic.
(f preserves geometric line segments)

Fundamental Theorem of projective real geometry

A projective transformation (between open sets of the Euclidean space) is a homography:

There exist $\alpha_0, \dots, \alpha_n$ linear forms $\beta_0, \dots, \beta_n$ scalars

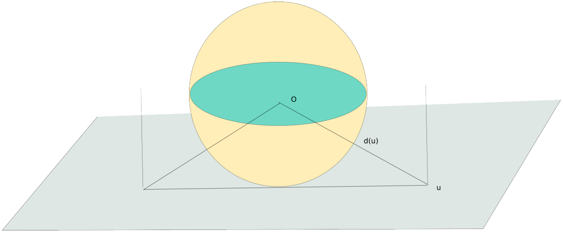
$$f(x) = (\frac{\alpha_1(x) + \beta_1}{\alpha_0(x) + \beta_0}, \dots, \frac{\alpha_n(x) + \beta_n}{\alpha_0(x) + \beta_0})$$

Seen on the sphere

The standard sphere $\mathbb{S}^n$ is **projectively flat:** any point has a neighbourhood projectively diffeomorphic to an open set in $\mathbb{R}^n$

Precisely: any (open) hemisphere is (globally) projectively diffeomorphic to $\mathbb{R}^n$: a perspective map...

perspective



PGL_{n+1}

- Any local projective transformation of $\mathbb{S}^n$ extends globally to $\mathbb{S}^n$ ($\mathbb{S}^n$ is an equivariant projective compactification of $\mathbb{R}^n$)

$\mathrm{Proj}(\mathbb{S}^{\mathrm{n}}) = \mathrm{PGL}_{\mathrm{n}+1}(\mathbb{R})$
$(= \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^* = \mathrm{SL}_{n+1}(\mathbb{R})$ up to index 2)

$A = \mathrm{GL}_{n+1}(\mathbb{R})$ acts as $A.x = \frac{Ax}{\|Ax\|}$
It preserves linear 2-planes of $\mathbb{R}^{n+1}$ and hence great circles of $\mathbb{S}^n$

Alternatively, in the projective space

PGL_{n+1} acts naturally on $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$

If $\mathbb{P}^n(\mathbb{R}) = \mathbb{S}^n / \pm Id$ is endowed with the quotient metric, then affine charts (in $\mathbb{R}^n$) are projective (and not affine)

Beltrami Theorem

The hyperbolic space too is projectively flat:

because of the Klein model (a complete metric on the unit ball of curvature -1 whose geodesics are line segments)

Beltrami: *A locally projectively flat Riemannian manifold has constant sectional curvature*

Some remarks

Regularity: we assumed transformations f diffeomorphic, but the theorem is true for f merely bijective (without measurability hypothesis...)

Complex case: f sends an affine complex line to an affine complex line (precisely open sets in them)

Same conclusion: complex homography

Regularity

Regularity is not automatic in the complex case?!

There is a bijection $f : \mathbb{C}^2 \to \mathbb{C}^2$ sending any complex algebraic curve defined by a polynomial of degree d, to a similar one
(in particular preserving affine complex lines),
but f non continuous...

Some historical facts

What is new?

• In the Riemannian case, the theorem is proved by V. Matveev, but for the identity components:

$\mathrm{Proj}^0(\mathrm{M}, \mathrm{g}) = \mathrm{Aff}^{\,0}(\mathrm{M}, \mathrm{g})$, unless M is covered by the sphere,
In other words

- **Hypothesis:** (M, g) admits a projective non-affine one parameter group of transformations

-**Conclusion** M is a quotient of the sphere

Here:

- **Hypothesis:** (M, g) has a projective transformation none of which powers is affine

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Conclusion M is a quotient of the sphere

•• The Lorentz case (in the analytic case) is new: only partial (technical) results are known.

The projective group: a classical Problem

A Russian speciality

Current names: Bolsinov (UK), Topalov (?), Matveev (Germany)

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Italian:
Beltrami, Dini, Fubini, Levi-Civita....

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Older: Solodovnokov, Sinjukov, Aminova,

Italian:
Beltrami, Dini, Fubini, Levi-Civita....
Others,
Weyl, Eisenhart, Painlevé, Darboux, Lagrange, Cartan, Lie,...
Hall (relativity)
Bryant,

Kähler version

A Japanese speciality: Hasegawa, Fujimura, Ishihara, Yano, Hiramatu, Yoshimatsu...

Here, one defines h-planar curves and h-projective transformations accordingly,

A curve $\gamma : [a, b] \to M$ is h-planar if its complexified tangent direction field is parallel:
$T(t) = \mathbb{C}\gamma'(t)$ is parallel along γ

Theorem

Let (M, g) be a compact Kähler manifold.
Then $\mathrm{Proj}^{\mathrm{hol}}(\mathrm{M}, \mathrm{g})$ *is a finite extension of* $\mathrm{Aff}^{\,hol}(M, g)$*, unless* (M, g) *is (isometrically and holomorphically) covered by* $\mathbb{P}^n(\mathbb{C})$ *endowed with its Fubini-Study metric (up to a constant).*

The identity component case "$\mathrm{Proj}^{\mathrm{hol}^0} = \mathrm{Aff}^{\,hol^0}$ unless (M, g) is covered by the projective space", was proved by Matveev and Rosemann..

Further remarks, Projective vs Projective

- In complex geometry: M is projective if it embeds holomorphically in $\mathbb{P}^N(\mathbb{C})$,
say $M \subset \mathbb{P}^N(\mathbb{C})$

- $G_M = \{A \in \mathrm{PGL}_{N+1}(\mathbb{C}) / \ A(M) = M\}$ could be called the projective group of M !?
It may be large, e.g non-compact and acts transitively

- On the other hand, Induce the Fubini-Study metric on M: $(M, g_{FS_{|M}})$

$\mathrm{Proj}^{\mathrm{hol}}(\mathrm{M}, \mathrm{g}_{\mathrm{FS}_{|\mathrm{M}}}) = ?$

Theorem

$\mathrm{Proj}^{hol}(\mathrm{M}, \mathrm{g}_{\mathrm{FS}|\mathrm{M}})$ *is a finite extension of* $\mathrm{Iso}^{hol}(\mathrm{M}, \mathrm{g}_{\mathrm{FS}|\mathrm{M}})$ *unless M is a Veronese submanifold.*

Corollary: for non Veronese submanifolds G_M does not act projectively!

Proof: by our theorem if $\mathrm{Proj}^{hol}(\mathrm{M}, \mathrm{g}_{\mathrm{FS}|\mathrm{M}})$ is not a finite extension of $\mathrm{Iso}^{hol}(\mathrm{M}, \mathrm{g}_{\mathrm{FS}|\mathrm{M}})$, then
$(M, g_{FS|M})$ is isometric to $(\mathbb{P}^n(\mathbb{C}), cg_{FS}))$, for some constant c,
$n = \dim M$ (up to a cover, but forget it)

Nash $\cap$ Kodaira

Question: Find holomorphic and isometric immersions
$(\mathbb{P}^n(\mathbb{C}), cg_{FS})) \to (\mathbb{P}^N(\mathbb{C}), g_{FS})$

Answer: up to composition with an ambient isometry (in SU_{N+1}), this is a Veronese map:

$$v_k : [X_0, \ldots, X_n] \to [\ldots X^I \ldots]$$

where X^I ranges over all monomials of degree k in $X_0, \ldots, X_n$.

- In particular $c = k$ (quantic fact!)

Calabi Egregium Theorem

Theorem

(Extrinsic = Intrinsic)
Let $F(N, b)$ denote the simply connected Hermitian space of dimension N and constant holomorphic sectional curvature b.
Let M be a Kähler manifold (not necessarily complete) and $f : M \to F(N, b)$ a holomorphic isometric immersion.
Then, f is rigid in the sense that any other immersion f' is deduced from f by composing with an element of $\mathrm{Iso}(\mathrm{F}(\mathrm{N}, \mathrm{b}))$

Example: the induced metric on an elliptic curve can never be flat (but almost)!
Same fact but by other methods for Calabi-Yau manifolds... (but Fano could be at some scalings?)

Idea

(Calabi's thesis)
Diastasis (diastatic function) $D(p, q)$....

Geometry creates Dynamics

(back to motivations...)

- M a smooth manifold:
- • Action of $\mathrm{Diff}(M)$ on $Met(M)$ its space of metrics...

(M, g) Riemannian (or pseudo-) $\rightarrow$
•• Geodesic flow

• • • Symmetry groups, e.g. $\mathrm{Iso(M, g)}$, $\mathrm{Aff}\,(M, g)$, $\mathrm{Proj(M, g)}$,
But they are generically trivial.
Goal: Look for and characterize the special non-generic cases!

(like $\mathrm{SL}\,(n, \mathbb{Z})$ in groups!)

Hierarchy of Groups

(M, g) pseudo-Riemannian
– $\mathrm{Iso(M, g)}$ the group of all isometries, i.e. diffeomorphisms such that $f^*g = g$

– Aff (M, g) ... preserving parametrized geodesics of (M, g).
– $\mathrm{Proj(M, g)}$ ... preserving unparameterized geodesics of (M, g).

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– $\mathrm{Proj(M, g)}$ ... preserving unparameterized geodesics of (M, g).

- $\mathrm{Sim(M, g)}$ the group of similarities (or homotheties) i.e. maps f such that $f^*g = ag$ for some constant.

- $\mathrm{Conf(M, g)}$ the group of conformal transformations such that $f^*g = ag$, where a is a function on M.

Inclusions
$\mathrm{Iso} \subset \mathrm{Sim} \subset \mathrm{Conf}$,

$\mathrm{Iso} \subset \mathrm{Sim} \subset \mathrm{Aff} \subset \mathrm{Proj}$.

Special spaces are those for which inclusion is non-trivial

Focus today on the question: when is $\mathrm{Aff} \subsetneq \mathrm{Proj}$?

Remark: they are all Lie groups with identity components $\mathrm{Iso}^0, \ldots, \mathrm{Aff}^0, \mathrm{Proj}^0$.

Example 1: Conformal beauty of the sphere (a central substratum)

• Observation: The sphere is beautiful, meaning: the inclusion chain is non-trivial!
$\mathsf{Iso}(\mathbb{S}^n) = \mathsf{O}(n+1)$, $\mathsf{Conf}(\mathbb{S}^n) = \mathsf{O}(1, n+1)$

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$\mathrm{Iso}(\mathbb{S}^n) = \mathrm{O}(n+1)$, $\mathrm{Conf}(\mathbb{S}^n) = \mathrm{O}(1, n+1)$

• Rigidity: the sphere is uniquely beautifull, Lichnerowicz conjecture (solved by Ferrand and Obatta): $\mathbb{S}^n$ is the unique compact Riemannian manifold for which the conformal group is not equal to the isometry group of any metric in its conformal class?

Example 2: Projective beauty of the sphere

$\mathrm{Proj}(\mathbb{S}^{\mathrm{n}}) = \mathrm{PGL}_{\mathrm{n}+1}(\mathbb{R})$
$(= \mathrm{GL}_{n+1}(\mathbb{R})/\mathbb{R}^* = \mathrm{SL}_{n+1}(\mathbb{R})$ up to index 2)
$A = \mathrm{GL}_{n+1}(\mathbb{R})$, $A.x = \frac{Ax}{\|Ax\|}$
Alternatively,
PGL_{n+1} acts on $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$
$Aff(\mathbb{S}^n) = \mathrm{Iso}(\mathbb{S}^{\mathrm{n}})$

Some finite quotients of the sphere may have Proj non-compact,

- Our question here: projective rigidity of the sphere?

Example 3: Affine beauty of the torus

$\mathbb{T}^n = \mathbb{R}^n/\Lambda$, Λ lattice in $\mathbb{R}^n$, e.g. $\Lambda = \mathbb{Z}^n$
$\mathrm{Iso} = \mathbb{T}^{\mathrm{n}}$, up to a finite index
$\mathrm{Aff} = \mathrm{GL}_n(\mathbb{Z})$ ($=\mathrm{SL}_n(\mathbb{Z})$ up to index 2),

But $\mathrm{Proj}(\mathbb{T}^{\mathrm{n}}) = \mathrm{Aff}(\mathbb{T}^{\mathrm{n}})$

- Affine rigidity of the torus is not easy to state...

Projective flatness

$f : (M, g) \to (M', g')$ projective diffeomorphism

A metric (M, g) is (locally) projectively flat if it is projectively diffeomorphic to the Euclidean space.

Betrami: in this case (M, g) has constant sectional curvature

One also defines projectively flat (non-metric) connections

This leads to a (G, X)-structure with $G = \mathrm{PGL}_{n+1}(\mathbb{R})$, $X = \mathbb{P}^n(\mathbb{R})$

Terminology: "projective structure" is sometimes used to mean a "flat projective structure"...

Space of metrics, equivalence relations, without group actions

Restrict discussion to the Riemannian case $\rightarrow$ generalize to pseudo-...

$\mathcal{M}et(M)$ be the space of all Riemannian metrics on M.

(projective) Equivalence relation: $g \sim g'$ iff they have the same unparameterized geodesics

$\mathcal{M}et^{\mathrm{Proj}}(g)$= equivalence class of g
$\mathcal{M}et^{\mathrm{Proj}}(g) \supset \mathbb{R}g$

Let $g \to \nabla_g$ (its Levi-Civita connection) $\to \sigma_g$ (associated projective structure), then the classes are levels of $g \to \sigma_g$

When $\mathrm{Diff}(M)$ acts on $\mathcal{M}et(M)$, the stabilizer of g is $\mathrm{Iso(M, g)}$, and the stabilizer of the (projective) class of g is $\mathrm{Proj(M, g)}$.

The action of $\mathrm{Proj(M, g)}$ on $\mathcal{M}et^{\mathrm{Proj}}(g)$, is a priori, neither trivial, nor transitive... (it may happen that $\mathrm{Proj(M, g)}$ is trivial but not is $\mathcal{M}et^{\mathrm{Proj}}(g)$).

Remark: affine, conformal... equivalence classes can be defined similarly.

Rank

Fundamental Fact: $\mathcal{M}et^{\mathrm{Proj}}(g)$ has a finite dimension, bounded by that of the standard metric on the sphere of same dimension.

$\dim \mathcal{M}et^{\mathrm{Proj}}(g) =$ Degree of (projective) mobility of (M, g)
≥ 1 (since $\mathcal{M}et^{\mathrm{Proj}}(g) \supset \mathbb{R}g$)

Rank $=$ degree of mobility -1

Hint in the affine case:

Write $\bar{g}_x(u, v) = g_x(T_x(u), v)$,
$x \to T_x \in \mathrm{End}\,(T_x M)$ an endomorphism of TM (symmetric with respect to g)

Write $\bar{g} = gT$ and $T = g^{-1}\bar{g}$

If $\bar{g}$ affinelly equivalent to g,
then $\bar{g}$ is parallel with respect to ∇_g, hence T is parallel (w.r.t. ∇_g)
Hence $\dim \mathcal{M}et^{\mathrm{Aff}}(g) \leq n(n+1)/2$, $n = \dim M$

(Details for the projective case latter on)

Philosophy

Let $f \in \mathrm{Diff}(M)$ act naturally on $\mathcal{M}et(M)$

- The f- action has a fixed point $\iff$ f is an isometry for some Riemannian metric on M.

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QUESTION What is the dynamical counterpart of the fact that the f-action preserves some (finite dimensional) manifold V in $\mathcal{M}et(M)$. (special case dim $V = 2$)

Reminiscent of mapping class group action on the Teichmuller space...

Metric on the space of metrics

$g \in \mathcal{M}et(M)$,
$T_g\mathcal{M}et(M) =$ space of vector fields $\chi(M)$
Endow $\chi(M)$ with the L^2_g-metric

Special case $\dim \mathcal{M}et^{\mathrm{Proj}}(M, g) = 2$

We need to consider only this case because in the higher rank case we have:

Theorem (Kiosak - Matveev, Mounoud)

Let (M, g) a compact pseudo-Riemannian that is not covered by the standard (Riemannian) sphere.
If the degree of projective mobility of (M, g) is ≥ 3, then any projectively equivalent metric to g is affinely equivalent to it.

Challenge!

Corollary: all what remains to consider is the challenging case when $\dim \mathcal{M}et^{\mathrm{Proj}}(M, g) = 2$

$\mathrm{Proj}(\mathrm{M}, \mathrm{g})$ acts on the surface $\mathcal{M}et^{\mathrm{Proj}}(M, g)$

Challenge!

Corollary: all what remains to consider is the challenging case when $\dim \mathcal{M}et^{\mathrm{Proj}}(M, g) = 2$

Proj(M, g) acts on the surface $\mathcal{M}et^{\mathrm{Proj}}(M, g)$

Paradoxical thoughts:

$\rightarrow$ The rank 1 case is easier to handle since Proj acts on a smaller space $\mathcal{M}et^{\mathrm{Proj}}$?

$\leftarrow$ The higher rank case is easier since when $\mathcal{M}et^{\mathrm{Proj}}$ is big then Proj will be big.

- In rank one case, one proves Proj is small, say generated by one diffeomorphism f

- Such f is dissipative, say a north-south dynamics with an attractor and repulser being submanifolds of some codimension.

- Associated are many geodesic and umbilical (but singular) foliations...

- But, how to detect that you are on a (rigorously) round sphere!?

Some similar apparently easy geometric problems:

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• Warning: co-existence of "dissipative" and chaotic dynamics: there exist Riemannian metrics on compact 3-manifolds whose geodesic flow is completely integrable but chaotic !

(Ref: Bolsinov, Taimanov: Integrable geodesic flows with positive topological entropy)
(Both work on the projective equivalence and projective transformation problems)

Some projective differential geometry

(emphasising on its non-linear character as is the classical Schwarzian derivative..., and avoiding Cartan connections theory...)

Examples

Dini

On surfaces, near a generic point, g are $\bar{g}$ are projectively equivalent $\iff$ in some co-ordinate system:

$$g = (X(x) - Y(y))(dx^2 + dy^2)$$

and

$$\bar{g} = (\frac{1}{Y(y)} - \frac{1}{X(x)})(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)})$$

$X(x) > Y(y)$

Remarks

X and Y are (essentially) eigen-values of the tensor T defined by: $\bar{g}(.,.) = g(T.,.)$.

Observe the Commutation: $X(x,y) = X(x)$, $Y(x,y) = Y(y)$

In a domain with a Dini normal form: strong simplicity: $X(x_1,y_1) > Y(x_2,y_2)$: $\inf X > \sup Y$

Compactness forces confluence of eigenvalues and leads to rigidity...

A projective transformation of the torus

On $\mathbb{T}^2$

$$g = (f(x) - \frac{1}{f(y)})(\sqrt{f(x)}dx^2 + \frac{1}{\sqrt{f(y)}}dy^2)$$

$$\phi(x, y) = (y, x)$$

is projective, but not affine (except f very special)

Ellipsoid: a recent example, by means of integrable systems theory

(Topalov, Matveev, Tabachnikov)
Ellipsoid:

$$\Sigma_{i=1}^{i=n} \frac{(x_i)^2}{a_i} = 1$$

$$g = \text{ the metric induced from } \mathbb{R}^n$$

$$\bar{g} = \frac{1}{\Sigma(\frac{x_i}{a_i})^2} (\Sigma \frac{dx_i^2}{a_i})$$

Affine equivalence

$g \to \nabla_g$ is non linear...
g and $\bar{g}$ affinely equivalent
$\iff$
$\nabla_g = \nabla_{\bar{g}}$ (Levi-Civita connections)
Equation: $\nabla_g - \nabla_{\bar{g}} = 0$ is non-linear on g

$\bar{g}(u, v) = g(Tu, v)$, write $T = g^{-1}\bar{g}$
$T : TM \to TM$ endomorphism, a $(1, 1)$-tensor

Linearization: $\nabla_g - \nabla_{\bar{g}} = 0 \iff T$ parallel $\iff \nabla_g T = 0$

Projective equivalence

$\nabla \to (\Gamma_{ij}^k)$ and $\bar{\nabla} \to (\bar{\Gamma}_{ij}^k)$ are projectively equivalent if the have the same umparametrized geodesics

The Geodesic equations $\ddot{x}^k = \Gamma_{ij}^k(x)\dot{x}^i\dot{x}^j$ and $\ddot{x}^k = \bar{\Gamma}_{ij}^k(x)\dot{x}^i\dot{x}^j$ have same geometric solutions

$A = \nabla - \bar{\nabla}$ tensor, $A : TM \times TM \to TM$

Projective equivalence: A traceless, i.e. $A(u, u)$ is parallel to $u = 0$
$\iff$ A has the form: $A(u, v) = l(u)v + l(v)u$ for some form l

Non-linear equation on g

$$\nabla_u \bar{g}(\xi, \eta) = \bar{g}(\xi, \eta) d\theta(u) + \frac{1}{2}\bar{g}(\xi, u) d\theta(\eta) + \frac{1}{2}\bar{g}(\eta, u) d\theta(\xi)$$

$$\theta = \ln(\frac{\det \bar{g}}{\det g})^{\frac{1}{1+n}}$$

"Linearization!"

$T = g^{-1}\bar{g}$

Define L such that $T = \frac{L^{-1}}{\det L}$
i.e. $\bar{g}(u, v) = \frac{1}{\det L} g(L^{-1}u, v)$ (write $\bar{g} = \frac{1}{\det L} g L^{-1}$), so,

$$L = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g$$

$T \to L$ is a (partially defined) transform on the sections of $End(TM)$

Proposition

g and $\bar{g}$ are projectively equivalent $\iff$ L satisfies the linear equation:

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v, u) d\mathrm{trace(L)(w)} + \frac{1}{2}\mathrm{g(w, u)dtrace(L)(v)}$$

Say that such L is a $\mathcal{P}$-tensor, and denote their space $\mathcal{P}(M, g)$.

So, $\bar{g} \in \mathcal{M}et^{\mathrm{Proj}}(M, g) \to L \in \mathcal{P}(M, g)$ is a bijection onto its image ($=$ the invertible L).

Remarks

1. For g and $\bar{g}$ projectively equivalent
$\bar{g} \in \mathcal{M}et^{\mathrm{Proj}}(M, g) = \in \mathcal{M}et^{\mathrm{Proj}}(M, \bar{g})$, but $\mathcal{P}(M, g) \neq \mathcal{P}(M, \bar{g})$

2. The $\mathrm{Diff}(M)$-action on the sections of $\mathrm{End}\,(TM)$ obtained by composing with the $T \to L$ transform is still linear!

3. We have in particular a linear representation
$\rho : \mathrm{Proj}(\mathrm{M}, \mathrm{g}) \to \mathrm{GL}\,(\mathcal{P}(\mathrm{M}, \mathrm{g}))$

Nijenhaus tensor

Elements of $\mathcal{P}$ are not parallel, nevertheless have some special properties: For L section of $\mathrm{End}\,(TM)$,

$$N_L(u,v) = [Lu, Lv] - L[Lu, v] - L[u, Lv] - L^2[u, v]$$

A $\mathcal{P}$-tensor has $N_L = 0$

Integrability

An almost complex structure has a vanishing Nijenhaus $N_L = 0 \iff$ it is integrable, i.e. it defines a complex structure.

In general, if L diagonalizable,

- Eigen- distributions are integrable

(define $\lambda : M \to C$ as eigenfunction it is continuous and $\lambda(x)$ is a spectral value of $L(x)$, $\forall x$)

- An eigenfunction is constant along the leaves of the other distributions

- In particular eigenfunctions with higher multiplicity (> 1) are constant

- L is symmetric (auto-adjoint w.r.t. g), thus eigen-distributions are orthogonal.

- L is parallel $\iff$ all eigenfunctions are constant.

Foliations

If constant eigenvalue exist, then we get a festival of geodesic and umbilical... foliations, but singular.

In the pseudo-Riemannian non-Riemannian case, foliations may degenerate: the restriction of the metric on leaves is a degenerate.

Example: The weak stable foliation of the geodesic flow on negatively curved metric on a surface, is a lightlike geodesic foliation for some natural Lorentz metric on the unit tangent bundle.

Elements of proof, Actions

Stress tensor

(M, g) pseudo-Riemannian
$f \in \mathrm{Diff}(M)$

Usual stress (or strength):
$T_f = g^{-1} f^* g$, (i.e $f^* g(.,.) = g(T_f .,.)$)

Use the transform $T \to L$, define an adapted stress K_f as:

$$f^* g = \frac{1}{\det K_f} g K_f^{-1}$$

Action

Proj(M, g) acts on $\mathcal{M}et^{\mathrm{Proj}}(M, g)$

$$(f, g) \in \mathrm{Proj(M, g)} \times \mathcal{M}\mathrm{et}^{\mathrm{Proj}}(\mathrm{M, g}) \to \mathrm{f^*g}$$

Transported action on $\mathcal{P}(M, g)$ via the map
$L \to g_L = \frac{1}{\det L} g L^{-1}$

$$(f, L) \in \mathrm{Proj(M, g)} \times \mathcal{P}(\mathrm{M, g}) \to \mathrm{f^*L.K_f} \in \mathcal{P}(\mathrm{M, g})$$

The action is linear!

Representation

$$\rho : \mathrm{Proj}(\mathrm{M},\mathrm{g}) \to \mathrm{GL}\,(\mathcal{P}(\mathrm{M},\mathrm{g})) = \mathrm{GL}\,_2(\mathbb{R})$$

$\rho(f) = 1 \iff K_f = I \iff f$ isometry

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Say

$$\rho : \mathrm{Proj}(\mathrm{M}, \mathrm{g}) \to \mathrm{SL}_2(\mathbb{R})$$

Homography

Fix f, $K = K_f$

$\{I, K\}$ a basis of $\mathcal{P}(M, g)$
$\rho(f)K = f^*L.K = aI + bK$

$$f^*K = \frac{aI + bK}{K}$$

Let

$$A = A_f = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$$

It acts homographically on $\mathbb{C}$:

$$A \centerdot z = \frac{az + b}{z}$$

$Sections(\mathrm{End}\,(TM))$ is an algebra, and $A\centerdot$ acts on it

Previous equation $f^*K = A \centerdot K$

Consequently

$$f^{n*}K = A^n \centerdot K$$

(one knows the action of f on its stress tensor...)

Spectrum of K

$x \to Sp(x)$ spectrum of $K(x)$
$Sp(x)$ subset $\mathbb{C} \times \ldots \mathbb{C}$...

$$Sp(f^n x) = A^n . Sp(x)$$

If $\lambda : M \to \mathbb{R}$ is eigenfunction: $\lambda(x) \in Sp(x)$,
Up to taking a power,

$$\lambda(f^n) = A^n . \lambda(x)$$

λ semi-conjugates the two dynamical systems $(M, f) \to (\mathbb{C}, A)$

Classification of elements of SL_2

elliptic
parabolic
hyperbolic

Let λ real eigenfunction
$\lambda(M) \subset \mathbb{R}$ is a compact $A_\bullet$-invariant interval

Cases $\rho(f)$ elliptic or parabolic

- Elliptic and preserving an interval implies Thus $A^2 = 1$

- Parabolic with an invariant interval implies $\lambda(M)$ is a fixed point.

- Hyperbolic case: if λ non-constant, then $\lambda(M) = [\lambda_-, \lambda_+]$
λ_- and λ_+ fixed points of A.

South-North dynamics between them,

Hyperbolic case

(M, g) is Riemannian,

- K is diagonalizable

- As suggested by Dini normal form (in dimension 2), Matveev and Topolov prove that that a somewhere inequality between eigenfunctions extends to the whole M:

$$\lambda_1(x_0) < \lambda_2(x_0) \Longrightarrow \sup \lambda_1 \leq \inf \lambda_2$$

- It follows that there exists exactly one non-constant eigenfunction λ with range $[\lambda_-, \lambda_+]$

- λ_- and λ_+ could be (constant) eigenfunctions,

- λ is simple since by the general theory (of Nijenhaus tensors) a non-simple eigenfunction is constant.

Weyl tensor

- All this allows understanding topological (i.e. without measure) Lyapunov exponents...

- $\rightarrow$ vanishing of the projective Weyl tensor

- (M, g) is projectively flat.... $\square$

Lorentz case

- K is not, a priori, diagonalizable: an auto-adjoint endomorphism w.r.t. a non-definite non-degenerate form is not necessarily diagonalizable....

- Global ordering of eigenfunctions is no longer valid.

- The stress tensor has no dynamical meaning, e.g. $K = 1$ means f is isometric, but may have non-trivial Lyapunov exponents...!

... $\square$

Conclusion: Cultural Content

1. There is a Gromov's vague conjecture on classifibiability of geometric substrata with a large symmetry group

2. Fundamental Theorem of projective geometry

3. Beltrami Theorem: projectively flat Riemannian metrics have constant sectional curvature

4. There are non measurable projective bijections of $\mathbb{C}^n$

5. Experimental Fact (= conjecture): besides the standard Riemannian sphere, and up to finite objects, any projective transformation of a compact pseudo-Riemannian manifold is affine.

6. There is Theorema Egregium of Calabi for holomorphic-isometric immersions: extrinsic quantities turn out to be intrinsic.

7. There are geodesic flows of compact Riemannian manifolds which are completely integrable and chaotic!

5. Experimental Fact (= conjecture): besides the standard Riemannian sphere, and up to finite objects, any projective transformation of a compact pseudo-Riemannian manifold is affine.

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7. There are geodesic flows of compact Riemannian manifolds which are completely integrable and chaotic!

8. Tokyo is a very beautiful geometric structure!