Geometric representation theory of braid groups related to quantum groups and hypergeometric integrals

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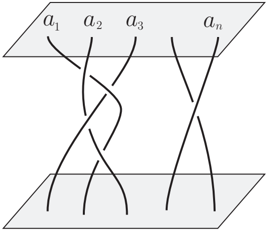
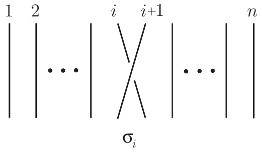
Braid groups

Braid groups were introduced by E. Artin in 1920's.



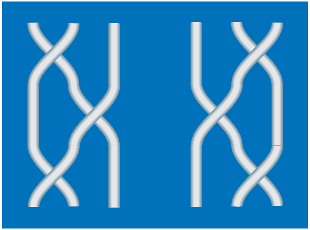
The isotopy classes of geometric braids as above form a group by composition. This is the braid group with n strands denoted by B_n.

Braid relations

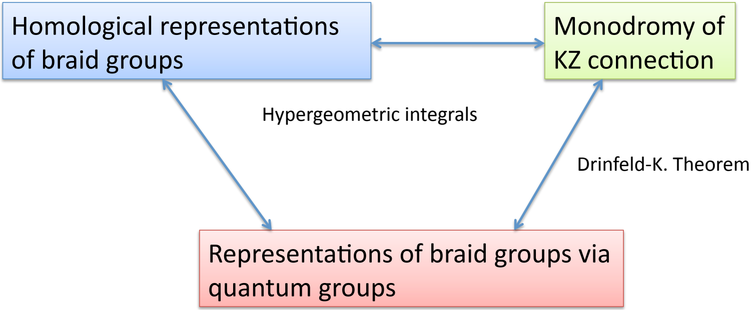


B_n is generated by σ_i, $1 \le i \le n-1$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1$$



Quantum symmetry in representations of braid groups



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- Knizhnik-Zamolodchikov (KZ) connection

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- Space of conformal blocks and Gauss-Manin connection
- Homological representations and dual Garside structures
- Categorification of KZ connection

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$
$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

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Suppose $X = \mathbf{C}$.

$$\pi_1(\mathcal{F}_n(\mathbf{C})) = P_n, \quad \pi_1(\mathcal{C}_n(\mathbf{C})) = B_n$$

We set $X_n = \mathcal{F}_n(\mathbf{C})$

Logarithmic forms

We set

$$\omega_{ij} = d\log(z_i - z_j), \quad 1 \leq i \neq j \leq n.$$

Consider a total differential equation of the form $d\phi = \omega\phi$ for a logarithmic form

$$\omega = \sum_{i<j} A_{ij}\omega_{ij}$$

with $A_{ij} \in M_m(\mathbf{C})$.

Infinitesimal pure braid relations

As the integrability condition we infinitesimal pure braid relations

$$[A_{ik}, A_{ij} + A_{jk}] = 0, \quad (i, j, k \ \text{ distinct}),$$
$$[A_{ij}, A_{k\ell}] = 0, \quad (i, j, k, \ell \ \text{ distinct})$$

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- As the holonomy of the flat connection ω we obtain linear representation of the pure braid group P_n.

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- The horizontal section of ω is expressed as an infinite sum of iterated integrals of logarithmic forms (hyperlogarithms).
- Any unipotent representations of P_n are give by hyperlogarithms (Aomoto).
- Infinitesimal pure braid relations describe the nilpotent completion of the pure braid group P_n over $\mathbf{Q}$ (Malcev algebra).

Theorem

For any linear representation $\rho : P_n \to GL_m(\mathbf{C})$ sufficiently close to identity, there exists a flat connection

$$\omega = \sum_{i<j} A_{ij} d\log(z_i - z_j)$$

with $A_{ij} \in M_m(\mathbf{C})$ such that the monodromy of the connection ω is given by ρ.

Poincaré's paper

SUR

LES GROUPES DES ÉQUATIONS LINÉAIRES[1]

Acta mathematica, t. 4, p. 201-311; 1884.

Dans trois Mémoires [*Théorie des groupes fuchsiens* ([2]) (*Acta mathematica*, t. 1, p. 1-62); *Mémoire sur les fonctions fuchsiennes* ([3]) (*Acta*, t. 1, p. 193-294); *Mémoire sur les groupes kleinéens* ([4]) (*Acta*, t. 3, p. 49-92)] j'ai étudié les groupes discontinus formés par des substitutions linéaires et les fonctions uniformes qui ne sont pas altérées par les substitutions de ces groupes. Avant de montrer comment ces fonctions et d'autres analogues donnent les intégrales des équations linéaires à coefficients algébriques, il est nécessaire de résoudre deux problèmes importants :

1° *Étant donnée une équation linéaire à coefficients algébriques, déterminer son groupe.*

2° *Étant donnée une équation linéaire du second ordre dépendant de certains paramètres arbitraires, disposer de ces paramètres de manière que le groupe de l'équation soit fuchsien.*

Soit maintenant w_{imn}^k le coefficient de $\alpha^m\beta^n$ dans le développement de w_i^k. Ce coefficient s'exprimera pour la même raison par des quadratures, et il en sera de même des coefficients qui entrent dans le développement des invariants, car ce sont des polynomes entiers par rapport aux w_{imn}^k. On peut d'ailleurs pousser plus loin l'étude du développement de la fonction v_i. A cet effet posons

$$\Lambda(x, \alpha_1) = \int_0^x \frac{dx}{x-\alpha_1},$$

$$\Lambda(x, \alpha_1, \alpha_2) = \int_0^x \frac{dx\,\Lambda(x, \alpha_1)}{x-\alpha_2},$$

. . . . ,

$$\Lambda(x, \alpha_1, \alpha_2, \ldots, \alpha_{q-1}, \alpha_q) = \int_0^x \frac{dx\,\Lambda(x, \alpha_1, \alpha_2, \ldots, \alpha_{q-1})}{x-\alpha_q}.$$

Remarquons maintenant qu'on peut mettre l'équation (1) sous la forme de p équations simultanées du premier ordre. Si les intégrales sont régulières dans le voisinage de chacun des points singuliers, nous pourrons introduire p variables simultanées $u_1 = v$, u_2, ..., u_p et remplacer l'équation (1) par les p équations simultanées

$$\frac{du_i}{dx} = \Sigma\, \varphi_{ik} u_k.$$

Les φ_{ik} seront des fonctions rationnelles de la forme suivante :

(2) $$\varphi_{ik} = \sum \frac{A_{i.k.\lambda}}{x-a_\lambda},$$

Aomoto's paper

Fonctions hyperlogarithmiques et groupes de monodromie unipotents

Par Kazuhiko Aomoto

Dans cet article *le problème de Riemann-Hilbert dans le sens restreint* est formulé dans l'espace projectif $P^l(\boldsymbol{C})$ en utilisant les équations de Schlesinger et Lappo-Danilevski et résolu moyennant "*fonctions hyperlogarithmiques*" dans le cas de monodromie unipotente (Théorème 1). Ce résultat est analogue à celui de K. T. Chen [5] mais ici plus précis. Soit S un ensemble analytique de codimension une dans $P^l(\boldsymbol{C})$ et soit $\pi_1(P^l(\boldsymbol{C})-S)$ le groupe fondamental de $P^l(\boldsymbol{C})-S$ par rapport à un point base x_0. On montrera ensuite que *l'anneau de groupe*

$$\varprojlim_{s=\infty} \boldsymbol{C}[\pi_1(P^l(\boldsymbol{C})-S)]/\mathfrak{J}_s$$

est isomorphe au completé $\hat{\mathfrak{E}}(\mathfrak{G})$ *de l'algèbre enveloppante* $\mathfrak{E}(\mathfrak{G})$ *de l'algèbre d'holonomie* $\mathfrak{G}$ où $\mathfrak{J}_s$ désigne le s-ième puissance de l'idéal d'augmentation $\mathfrak{J}$ de l'anneau de groupe $C[\pi_1(P^l(C)-S)]$ (Théorème 2). En particulier on aura *la dualité de* $\hat{\mathfrak{E}}(\mathfrak{G})$ *et la cohomologie à* 0 *dimension des intégrales itérées de K. T. Chen* (voir aussi [4]).

K. T. Chen's iterated integals of differential forms

$\omega_1, \cdots, \omega_k$: differential forms on M
ΩM : loop space M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \; ; \; 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$

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The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

Iterated integrals as differential forms on loop space

The expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along fiber with respect to the projection $p : \Delta_k \times \Omega M \to \Omega M$.

differential form on the loop space ΩM
with degree $p_1 + \cdots + p_k - k$.

Differentiation on loop spaces

As a differential form on the loop space $d\int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \; \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

$\mathcal{A} = \{H_1, \cdots, H_m\}$: arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$ defined by linear forms f_j, $1 \leq j \leq m$. Consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

A be the Orlik-Solomon algebra generated by the logarithmic forms $\omega_j = d \log f_j$, $1 \leq j \leq m$. .

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The reduced bar complex of the Orlik-Solomon algebra is the tensor algebra defined by

$$\overline{B}^*(A) = \bigoplus_{k \geq 0} \left(\bigotimes^k \overline{A} \right)$$

($\overline{A}$: degree shifted by 1.) There is a natural filtration defined by

$$\mathcal{F}^{-k}(\overline{B}^*(A)) = \bigoplus_{\ell \leq k} (\bigotimes^{\ell} \overline{A})$$

Bar complex and fundamental group

Theorem

For the reduced bar complex for the Orlik-Solomon algebra there is an isomorphism

$$\mathcal{F}^{-k} H^0(\overline{B}^*(A)) \cong \mathrm{Hom}(\mathbf{Z}\pi_1(M, \mathbf{x}_0)/J^{k+1}, \mathbf{C})$$

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In the case $M = \mathcal{F}_n(\mathbf{C})$ the above space is isomorphic to the space order k invariants for the pure braid group P_n (a prototype for Kontsevich integrals).

Holonomy Lie algebra

Define the holonomy Lie algebra as a quotient of free Lie algebra by

$$\mathfrak{h}(M) = \mathcal{L}(X_1, \cdots, X_m)/\mathfrak{a}$$

where $\mathfrak{a}$ is an ideal generate by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_k}], \quad 1 \leq p < k$$

for maximal family of hyperplanes $\{H_{j_1}, \cdots, H_{j_k}\}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_k}) = 2$$

Universal holonomy map

We put

$$\omega = \sum_{j=1}^{m} \omega_j X_j .$$

Then there is a universal holonomy map

$$\Theta_0 : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_1, \cdots , X_m\rangle\rangle/\mathfrak{a}$$

defined by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}$$

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This induces an isomorphism

$$\mathbf{C}\widehat{\pi}_1(M, \mathbf{x}_0) \cong \mathbf{C}\langle\langle X_1, \cdots, X_m \rangle\rangle / \mathfrak{a}$$

By taking the primitive part we have an isomorphism between nilpotent completion of the fundamental group and the holonomy Lie algebra over $\mathbf{Q}$.

The following statement was first obtained by Aomoto.

Theorem

Let

$$\rho : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

be a unipotent representation of the fundamental group of the complement of hyperplane arrangement. Then there exists an integrable connection

$$\omega = \sum_{j=1}^{\ell} A_j \omega_j, \quad A_j \in \mathrm{End}(V_j)$$

such that each A_j is nilpotent and the monodromy representation of ω coincides with ρ.

Proof

For a unipotent representation $\rho : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$ there exists k such that ρ induces a homomorphism

$$\widetilde{\rho} : \mathbf{C}\pi_1(M, \mathbf{x}_0)/J^{k+1} \longrightarrow \mathrm{End}(V).$$

The universal holonomy homomorphism of the connection $\sum_{j=1}^{\ell} \omega_j X_j$ induces an isomorphism

$$\theta : \mathbf{C}\pi_1(M, \mathbf{x}_0)/J^{k+1} \cong \mathbf{C}\langle\langle X_1, \cdots, X_\ell \rangle\rangle/(\mathfrak{a} + \widehat{J}^{k+1})$$

where $\widehat{J}$ denotes the completed augmentation ideal. Define a homomorphism

$$\alpha : \mathbf{C}\langle\langle X_1, \cdots, X_\ell \rangle\rangle/(\mathfrak{a} + \widehat{J}^{k+1}) \longrightarrow \mathrm{End}(V)$$

by $\alpha = \widetilde{\rho} \circ \theta^{-1}$ and put $A_j = \alpha(X_j)$, $1 \leq j \leq \ell$.

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

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We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

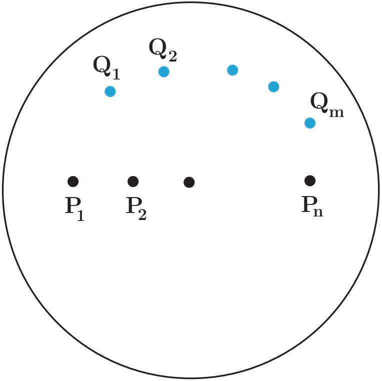
$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$: fiber of π, $\quad Y_{n,m} = X_{n,m}/\mathfrak{S}_m$

Relative configuration spaces

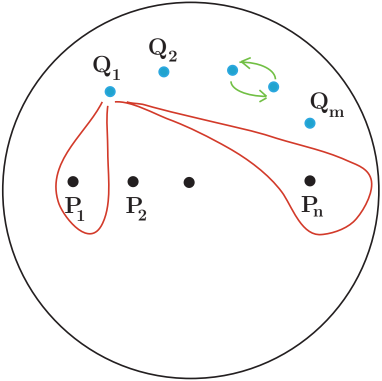
Fix $P = \{(1,0), \cdots, (n,0)\} \subset D$, where D is a 2 dimensional disc.
$\Sigma = D \setminus P$

$$\mathcal{F}_{n,m}(D) = \mathcal{F}_m(\Sigma), \quad \mathcal{C}_{n,m}(D) = \mathcal{F}_m(\Sigma)/\mathfrak{S}_m$$



Homology of relative configuration spaces

$$H_1(\mathcal{C}_{n,m}(D);\mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z}$$



Consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.

Abelian coverings

Consider the homomorphism

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defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.
Composing with the abelianization map
$\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}.$$

$\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$: the covering corresponding to $\operatorname{Ker} \beta$.

$H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module by deck transformations.

Express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$.

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$H_{n,m}$ is a free R-module of rank

$$d_{n,m} = \begin{pmatrix} m+n-2 \\ m \end{pmatrix}.$$

$\rho_{n,m} : B_n \longrightarrow \operatorname{Aut}_R H_{n,m}$: homological representations $(m > 1)$ extensively studied by Bigelow and Krammer ; they are faithful representations.

$\mathfrak{g} = sl_2(\mathbf{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda \in \mathbf{C}$

M_λ : Verma module of $sl_2(\mathbf{C})$ with highest weight vector v such that

$$Hv = \lambda v, \ Ev = 0$$

M_λ is spanned by

$$v, Fv, F^2 v, \cdots$$

Space of null vectors

$\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n$
Consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$.

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m : non-negative integer

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \; ; \; Hx = (|\Lambda| - 2m)x\}$$

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The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.
The monodromy of KZ connection

$$\theta_{\kappa,\lambda} : P_n \longrightarrow \mathrm{Aut}\, N[|\Lambda| - 2m]$$

Comparison theorem

We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$. We have

$$\theta_{\kappa,\lambda} : B_n \longrightarrow \mathrm{Aut}\, N[n\lambda - 2m].$$

Theorem

There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Local system over the configuration space

$\pi : X_{n+m} \to X_n$: projection defined by
$(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.
$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \times \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{2}{\kappa}}$$

(multi-valued function on X_{n+m}).
Consider the local system $\mathcal{L}$ associated with Φ.

Notation:
$W[|\Lambda| - 2m]$ has a basis

$$F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$$

with $|J| = j_1 + \cdots + j_n = m$ and $v_{\lambda_j} \in \mathcal{M}_{\lambda_j}$ the highest weight vector.

Solutions to KZ equation

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Theorem (Schechtman-Varchenko, Date-Jimbo-Matsuo-Miwa, ...)

The hypergeometric integral

$$\sum_{|J|=m} \left(\int_\Delta \Phi R_J(z,t) dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in $N[|\Lambda| - 2m]$ and is a solution of the KZ equation, where Δ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$.

Homology basis

For generic λ, κ,

$$H_j(Y_{n,m}, \mathcal{L}^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m}, \mathcal{L}^*) \cong H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

(homology with locally finite chains)

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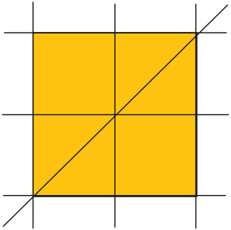
The above homology is spanned by bounded chambers.



bounded chambers : basis of twisted homology
(the case $n = 3, m = 2$).

Homology basis (continued)

For non-negative integers $m_1, \cdots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1,\cdots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned} &1 < t_1 < \cdots < t_{m_1} < 2 \\ &2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3 \\ &\cdots \\ &n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n. \end{aligned}$$

Put $M = (m_1, \cdots, m_{n-1})$ and write Δ_M for $\Delta_{m_1,\cdots,m_{n-1}}$.
The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H_m^{lf}(X_{n,m}, \mathcal{L})$ and its image $\overline{\Delta}_M = \pi_{n,m}(\Delta_M)$ defines a homology class $[\overline{\Delta}_M] \in H_m^{lf}(Y_{n,m}, \mathcal{L})$.
Under a genericity condition $[\overline{\Delta}_M]$ form a basis of $H_m^{lf}(Y_{n,m}, \mathcal{L})$.

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is give by the matrix of the form

$$\left(\int_{\overline{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. with $\omega_{M'}$ a multivalued m-form on $X_{n,m}$.

The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \operatorname{Aut} H_m(Y_{n,m}, \mathcal{L}^*)$ is equivalent to the action of B_n on the solutions of the KZ equation with values in $N[n\lambda - 2m]$.

Theorem

There is an isomorphism

$$N_h[\lambda n - 2m] \cong H_m(Y_{n,m}, \mathcal{L}^*)$$

which is equivariant with respect to the action of the braid group B_n, where $N_h[\lambda n - 2m]$ is the space of null vectors for the corresponding $U_h(\mathfrak{g})$-module with $h = 1/\kappa$.

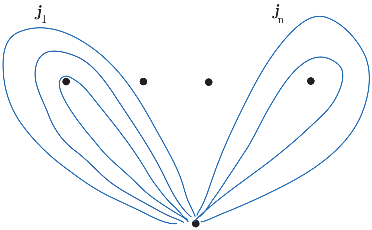
Quantum symmetry for twisted chains

There is the following correspondence:

twisted multi-chains $\Longleftrightarrow$ weight vectors $F^{j_1}v_1 \otimes \cdots \otimes F^{j_n}v_n$

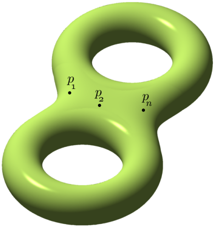
twisted boundary operator $\Longleftrightarrow$ the action of $E \in U_h(\mathfrak{g})$

$$H_m(Y_{n,m}, \mathcal{L}^*) \Longleftrightarrow N_h[\lambda n - 2m]$$



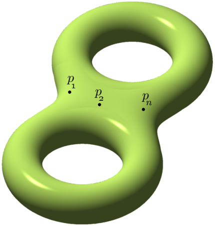
twisted multi-chains

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
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$\mathcal{H}_\Sigma(p, \lambda)$: **space of conformal blocks**
vector space spanned by holomorphic parts of the WZW partition function.

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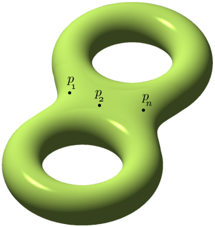

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vector space spanned by holomorphic parts of the WZW partition function.
Geometry : vector bundle over the moduli space of Riemann surfaces with n marked points with projectively flat connection.

The space of conformal blocks

Conformal field theory

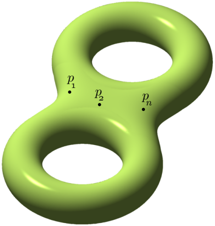
$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points

$\mapsto$

$\mathcal{H}_\Sigma$: complex vector space - the space of conformal blocks

The mapping class group $\Gamma_{g,n}$ acts on $\mathcal{H}_\Sigma$:

Quantum representations



$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra with commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\, g\, \langle X, Y \rangle c$$

K a positive integer (level)
$\widehat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$
c acts as $K \cdot \mathrm{id}$.

Representations of an affine Lie algebra

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λ : an integer with $0 \leq \lambda \leq K$
$\mathcal{H}_\lambda$: irreducible quotient of $\mathcal{M}_\lambda$ called the integrable highest weight modules.

Geometric background

G : the Lie group $SL(2, \mathbf{C})$
$LG = \mathrm{Map}(S^1, G)$: loop group
$\mathcal{L} \longrightarrow LG$: complex line bundle with $c_1(\mathcal{L}) = K$

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The affine Lie algebra $\widehat{\mathfrak{g}}$ acts on the space of sections $\Gamma(\mathcal{L})$.
The integrable highest weight modules $\mathcal{H}_\lambda$, $0 \leq \lambda \leq K$, appears as sub representations.
As the infinitesimal version of the action of the central extension of $\mathrm{Diff}(S^1)$ the Virasoro Lie algebra acts on $\mathcal{H}_\lambda$.

The space of conformal blocks - definition -

Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
$p_1, \cdots, p_n \in \Sigma$
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

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The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$.

Conformal block bundle

Σ_g : Riemann surface of genus g
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The union

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for any complex structures on Σ_g forms a vector bundle on $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

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This vector bundle is called the conformal block bundle and is equipped with a natural projectively flat connection. The holonomy representation of the mapping class group is called the quantum representation.

The flat connection is the KZ connection.
$\mathcal{L}$: rank 1 local system over $Y_{n,m}$ associated with Φ
$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})$
$p = (p_1, \cdots, p_n, \infty)$
There is a surjective period map

$$H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow \mathcal{H}^*_{\mathbf{C}P^1}(p, \lambda)$$

The above period map is not injective in general, which is related to fusion rules, resonance at infinity etc.

Gauss-Manin connection

$\mathcal{H}_{n,m}$: local system over X_n with fiber $H_m(Y_{n,m}, \mathcal{L}^*)$

Theorem

There is surjective bundle map to the conformal block bundle

$$\mathcal{H}_{n,m} \longrightarrow \bigcup \mathcal{H}^*_{\mathbf{C}P^1}(p, \lambda)$$

via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.

cf. Looijenga's work

Σ_g : Riemann surface of genus g
$\Gamma_g = \mathrm{Diff}^+(\Sigma_g)/\text{isotopy}$: mapping class group
$\mathcal{H}_{\Sigma_g}$: the space of conformal blocks at level K

Quantum representations of mapping class groups

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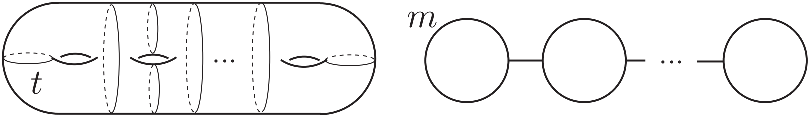
$\Gamma_g = \mathrm{Diff}^+(\Sigma_g)/\text{isotopy}$: mapping class group
$\mathcal{H}_{\Sigma_g}$: the space of conformal blocks at level K

There is a projectively linear action of Γ_g on the space of conformal blocks $\mathcal{H}_{\Sigma_g}$, which is called the quantum representation of the mapping class group.

Properties of quantum representations

A basis of the space of conformal block is described by colored trivalent graphs which are dual to pants decomposition of a surface. There is a combinatorial description of the action of the mapping class group.



The Dehn twist τ_t along t acts as a diagonal matrix.
The representation behaves nicely with respect to a stabilization for Heegaard splitting.

Images of quantum representations

The quantum representations are projectively unitary.

$$\rho_K : \Gamma_g \longrightarrow PU(\mathcal{H}_{\Sigma_g})$$

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The image of the quantum representation is "big" in the following sense.

Theorem (Funar-K.)

Suppose $g \geq 4$ and K sufficiently large. Then the image of any Johnson subgroup by ρ_K contains a non-abelian free group.

Index finite subgroups of mapping class groups

Put $K_g = I_g(2)$, the second Johnson subgroup.
$\rho([K_g, K_g])$ is of finite index in the image of the quantum representation $\rho(\Gamma_g)$.
Let U_g be the kernel of

$$\Gamma_g \longrightarrow \rho(\Gamma_g)/\rho([K_g, K_g])$$

Then, U_g is a finite index subgroup of Γ_g. This construction gives a systematic way to construct many finite index subgroups of Γ_g.

Question : Does U_g have infinite abelianization?

Dual Garside structure

The bounded chamber basis Δ_M plays an important role in detecting the dual Garside structure from the homological representation with respect to this basis.

Theorem (T. Ito and B. Wiest)

The dual Garside length of a braid word β with respect to the Birman-Ko-Lee band generators is expressed as

$$\mathit{max}\ \mathit{deg}_q\ \rho_{n,m}(\beta) - \mathit{min}\ \mathit{deg}_q\ \rho_{n,m}(\beta).$$

Categorification of KZ connections

There is a work in progress to construct 2-holonomy of KZ connection for braid cobordism based on the 2-connection investigated by L. Cirio and J. Martins of the form

$$A = \sum_{i<j} \omega_{ij} \Omega_{ij}$$

$$B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} \ P_{jik} + \omega_{ij} \wedge \omega_{jk} \ P_{ijk}),$$

where A has values in the algebra of 2-chord diagrams, a categorification of the algebra of horizontal chord diagrams and

$$\partial B = dA + \frac{1}{2} A \wedge A.$$