On Topological Blow-up

Taro Yoshino

What is it?

Topological Blow-up is a method

to understand non-Hausdorff spaces.

Motivation (Clifford-Klein form)

$\Gamma \subset G \supset H$	Triple of Lie groups
$\Gamma \curvearrowright G/H$	Natural action
$\Gamma \backslash G/H$	Clifford-Klein form

(Defined by T. Kobayashi)

Motivation (Clifford-Klein form)

$\Gamma \subset G \supset H$	Triple of Lie groups	$\Gamma \subset PSL(2,\mathbb{R}) \supset PSO(2)$
$\Gamma \curvearrowright G/H$	Natural action	$\Gamma \curvearrowright G/H \simeq \mathcal{H}$
$\Gamma\backslash G/H$	Clifford-Klein form	cpt Riemannian surface

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Global Local

Motivation (Deformation Space)

Definition (Deformation sp) (Kobayashi)

$$\mathcal{D} := \{\varphi : \Gamma \to G : \text{inj hom } (+\alpha)\} \, / \, \mathrm{Ad}(G).$$

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In general case, $\mathcal{D}$ is **not** always **Hausdorff**.

What does “understand” mean?

“understand a topology” $\overset{\textit{means}}{\Longleftrightarrow}$

able to imagine how sequences are converging.

Convergence of a Sequence of Subsets

$A_n, A \subset X$

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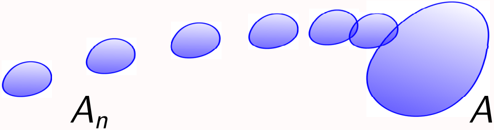
Definition

$A_n \to A$ in $X \iff$



Convergence of a Sequence of Subsets

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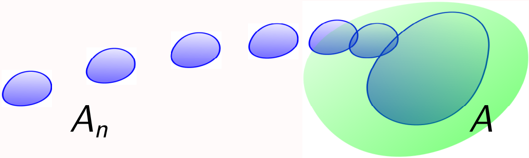
Definition

$$A_n \to A \text{ in } X \iff A \subset {}^{\forall}U \text{ (open)}$$
$$A_n \subset U \ (n \gg 1).$$



Main Theorem

$^\forall$ X: loc. cpt. (non-Hausdorff) sp.

$^\exists$ Y: loc. cpt. Hausdorff sp.

$^\exists$ $\tau : X \to 2^Y$: a map.

Theorem

For $^\forall\{x_n\}$, $x \in X$

$$x_n \to x \text{ in } X \iff \tau(x_n) \to \tau(x) \text{ in } Y.$$

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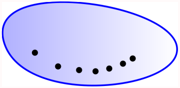
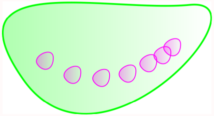
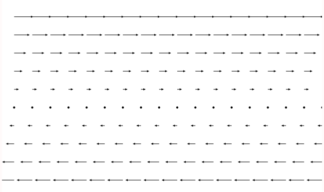




Example 1

Vector Field $y\frac{\partial}{\partial x}$



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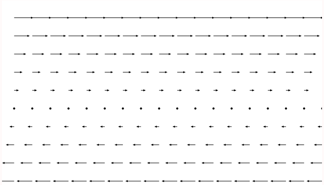

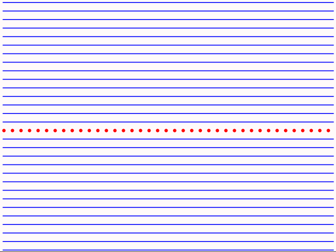

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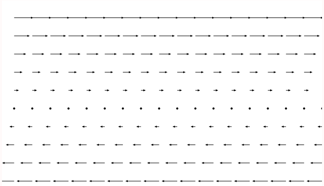

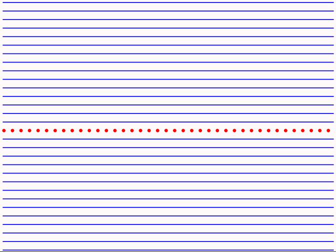

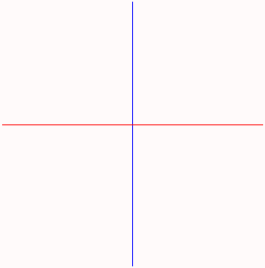

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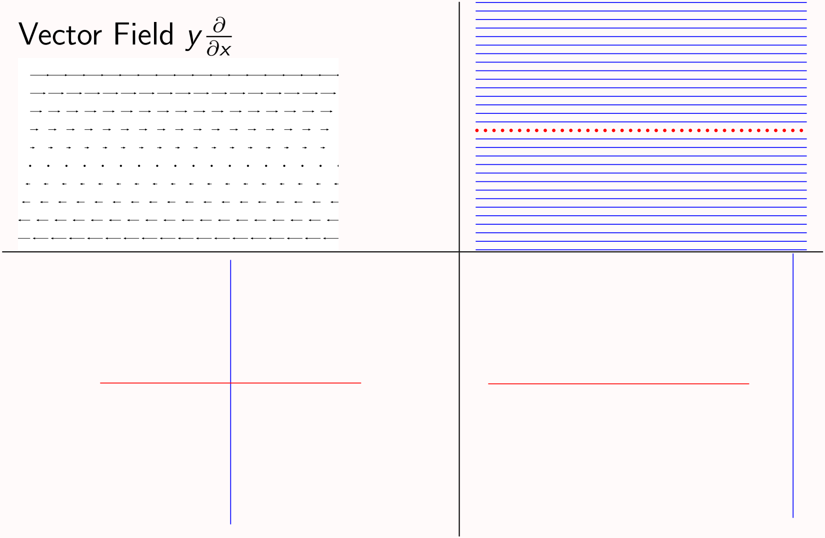

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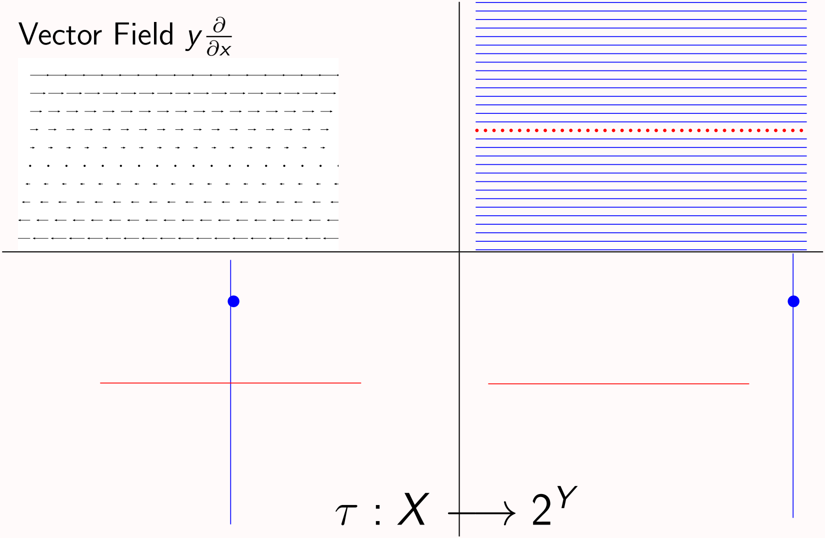
Vector Field $y\frac{\partial}{\partial x}$
$\tau : X \longrightarrow 2^Y$

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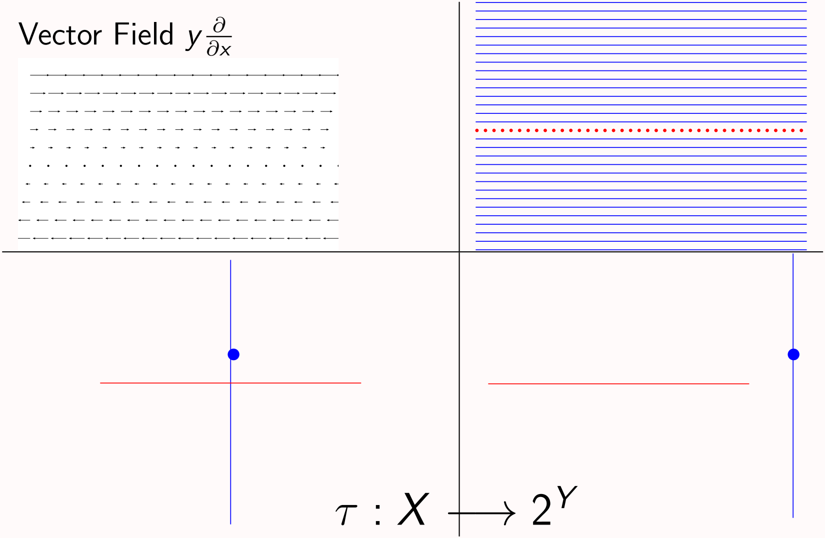

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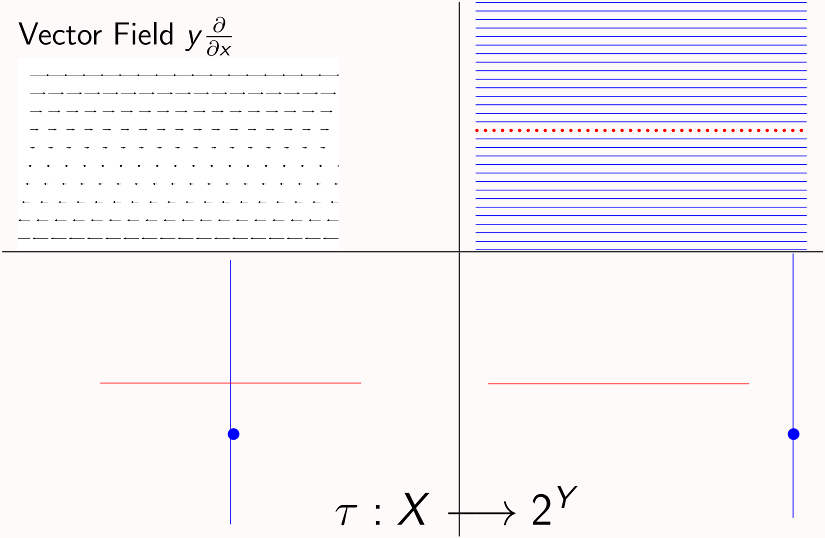

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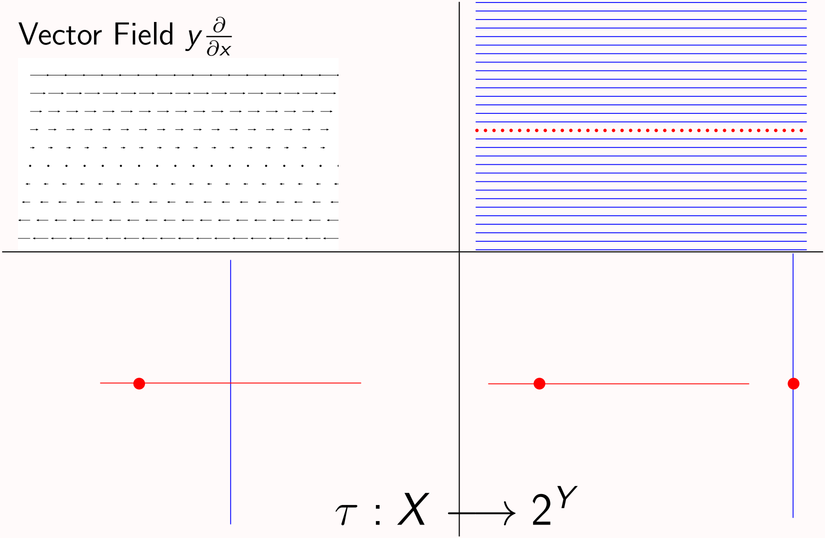

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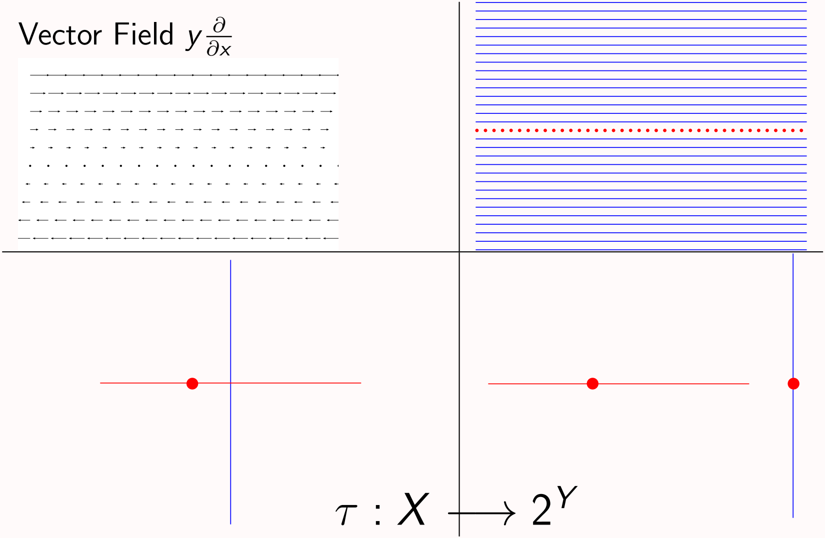
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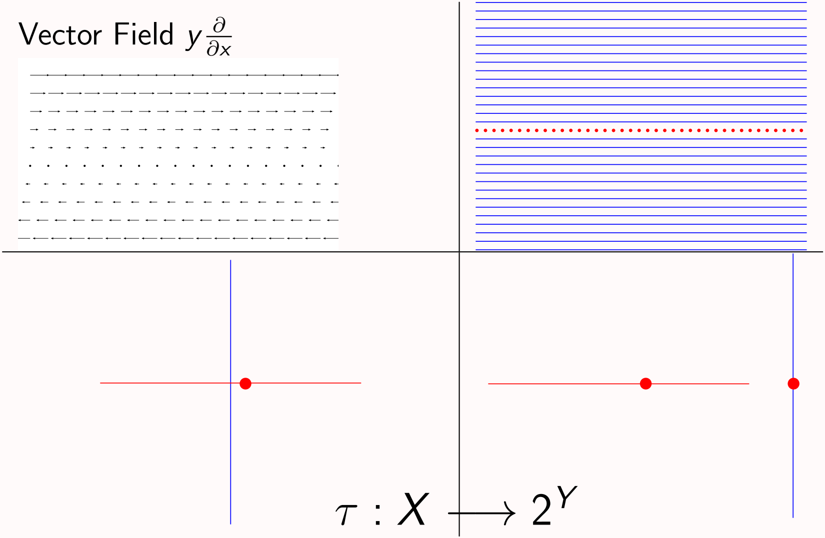
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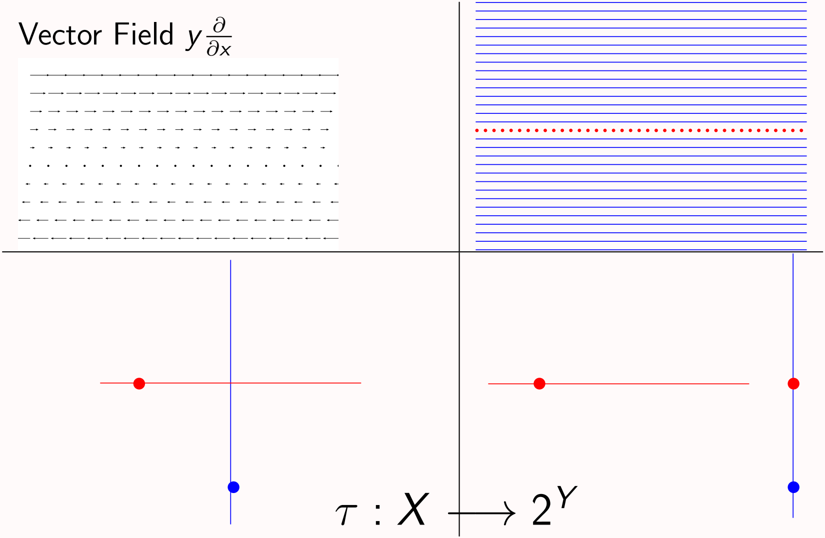
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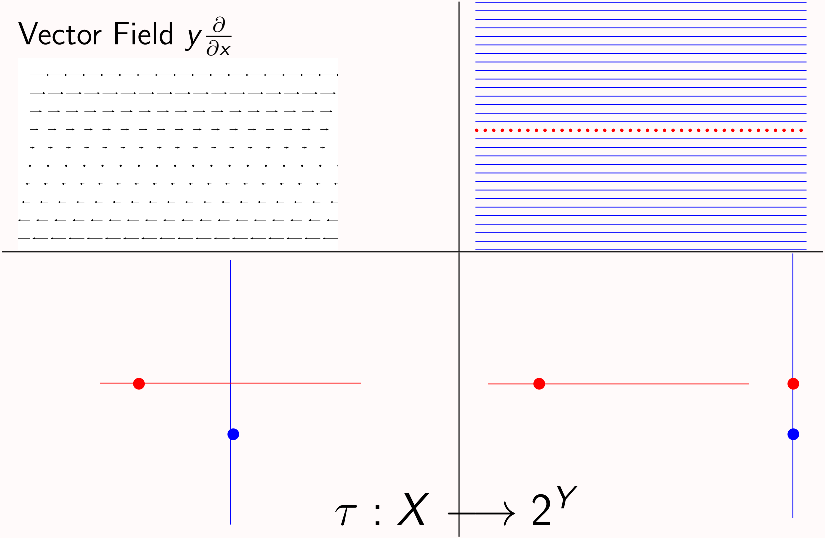

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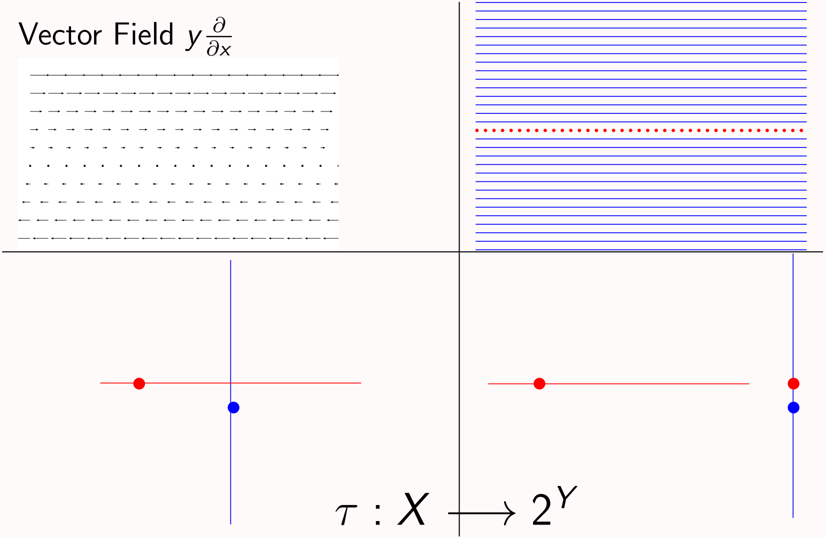

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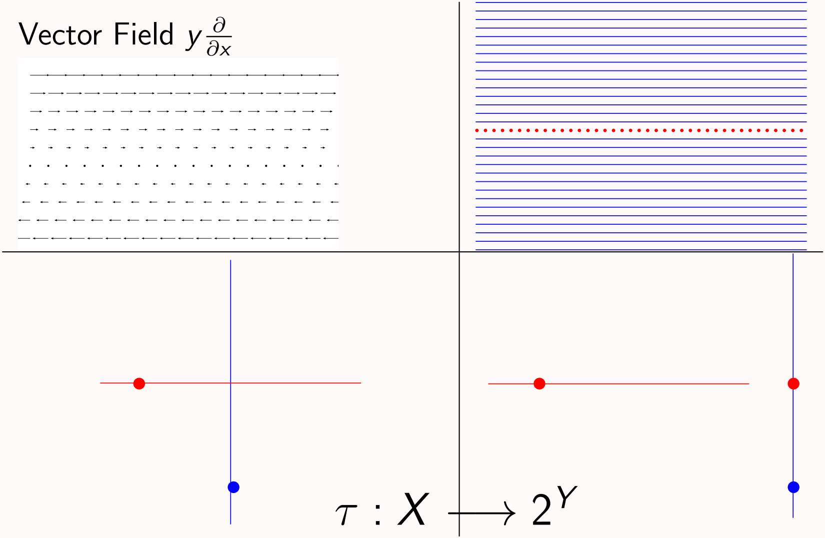

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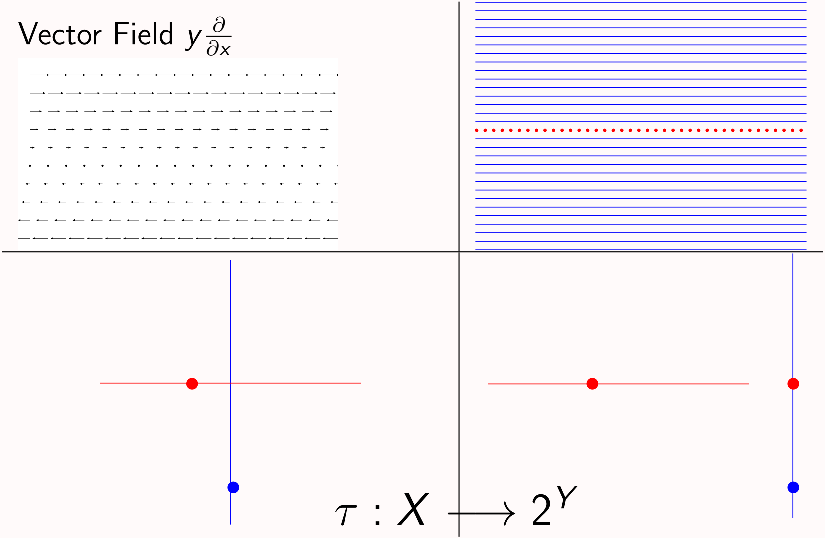

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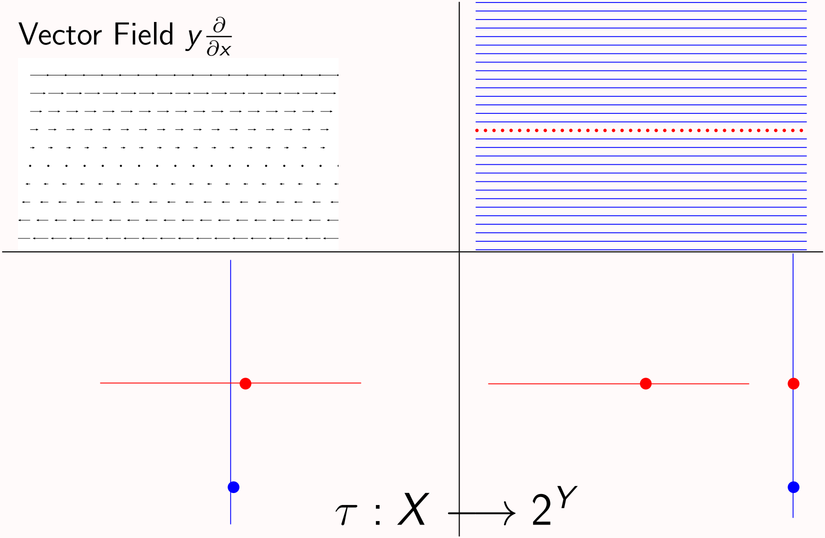
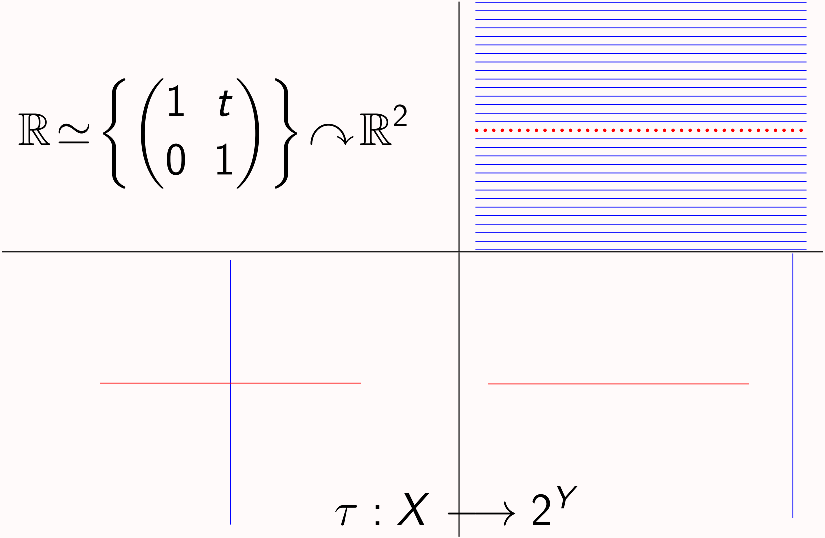


$$\tau : X \longrightarrow 2^Y$$

Example 1

$$\mathbb{R} \simeq \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\} \curvearrowright \mathbb{R}^2$$



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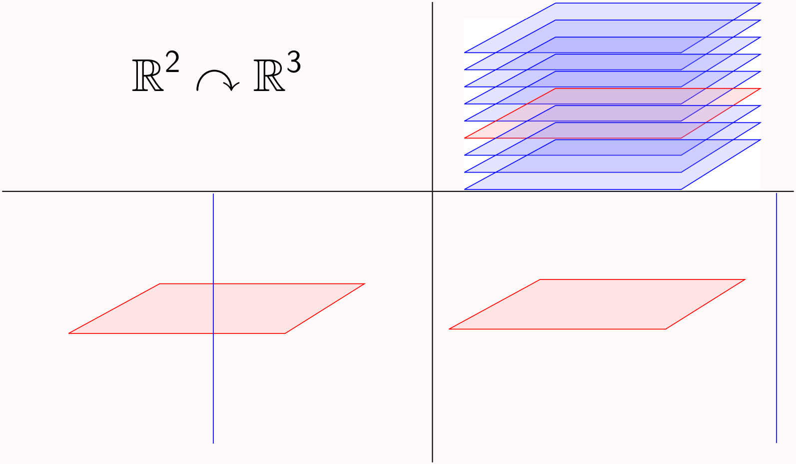
Example 2



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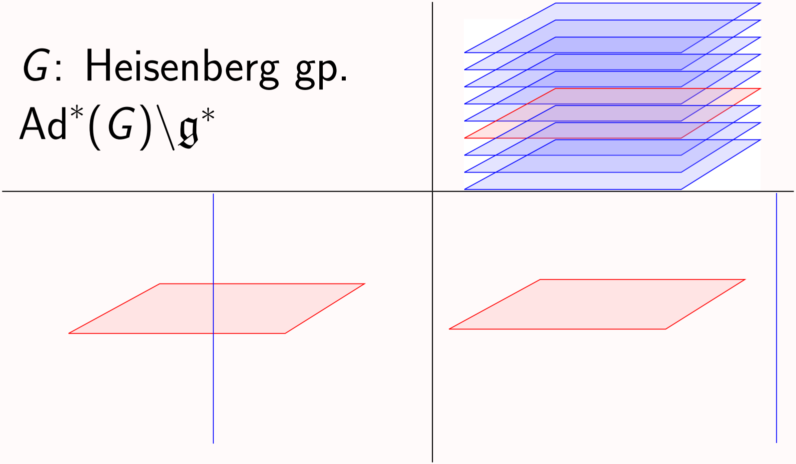
Example 2

G: Heisenberg gp.

$\mathrm{Ad}^*(G)\backslash\mathfrak{g}^*$



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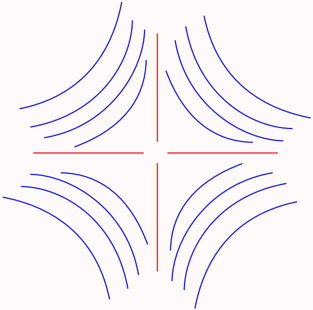
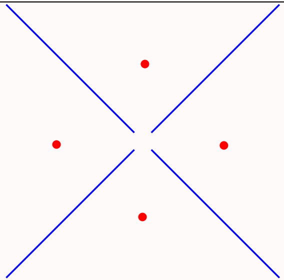
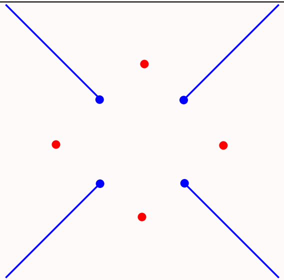
Example 3

$$\mathbb{R} \simeq \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}$$

$$\curvearrowright \mathbb{R}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$



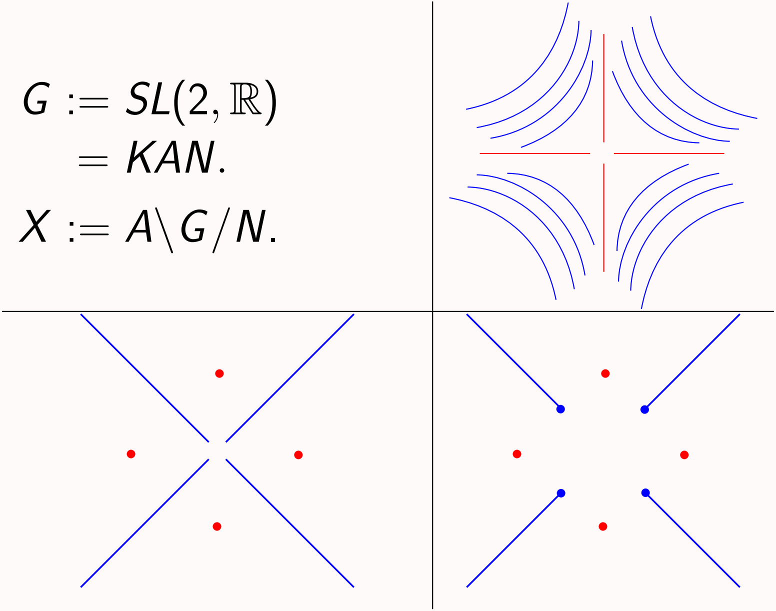




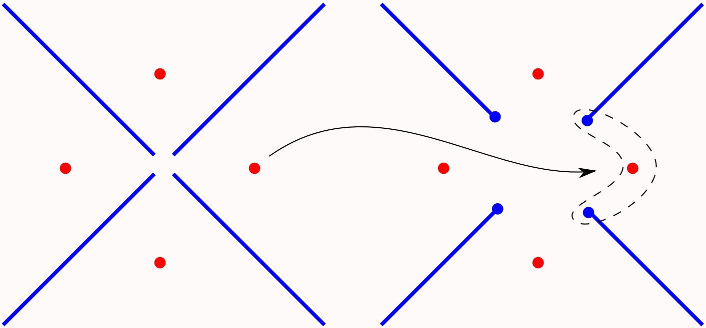
Example 3

$$G := SL(2,\mathbb{R}) = KAN.$$

$$X := A\backslash G/N.$$



The image of τ



$$\tau : X \longrightarrow 2^Y$$

Emulation of Convergence

$$X \xhookrightarrow{\text{Blow-up}} Y$$

X $\qquad\qquad\qquad\qquad$ Y

Emulation of Convergence

$$X \xrightarrow{\text{Blow-up}} Y$$

$$X \qquad\qquad\qquad\qquad Y$$

$$x_n \to x \qquad \Longleftrightarrow \qquad \tau(x_n) \to \tau(x)$$

Emulation of Convergence

$$X \xrightarrow{\text{Blow-up}} Y$$

X (weak topology) $\qquad$ Y (strong topology)

$$x_n \to x \iff \tau(x_n) \to \tau(x)$$

Emulation of Convergence

$$X \xrightarrow{\text{Blow-up}} Y$$

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(weak convergence)

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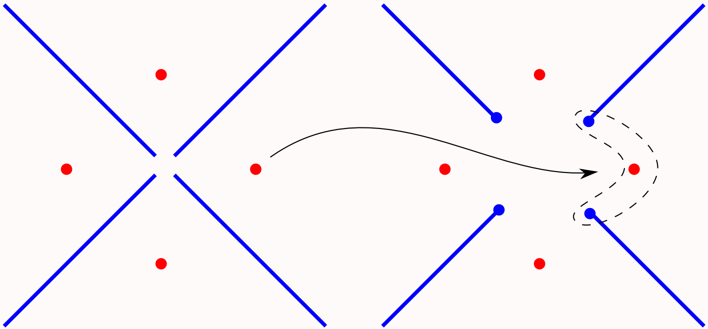

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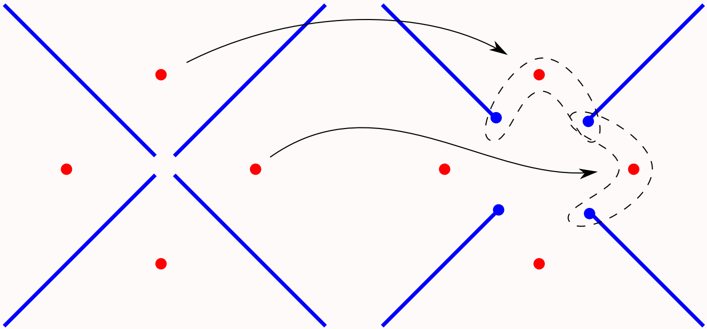

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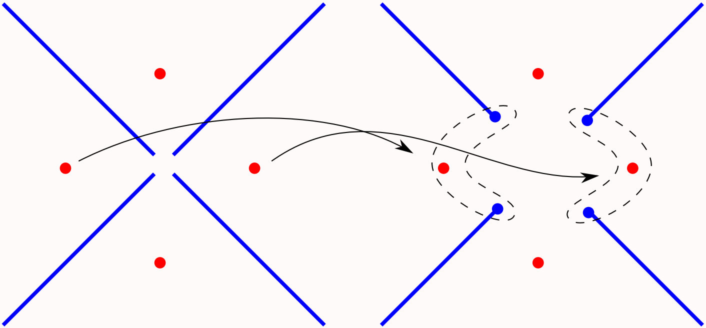

$$\tau : X \longrightarrow 2^Y$$

Properties of τ

$$X \xrightarrow{\text{Blow-up}} Y, \qquad \tau : X \to 2^Y.$$

Property

$$x, x' \text{ are separable in } X \iff \tau(x) \cap \tau(x') = \emptyset.$$

Properties of τ

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Property

x, x' are separable in $X \iff \tau(x) \cap \tau(x') = \emptyset.$

Property

$\tau(x)$ is compact.

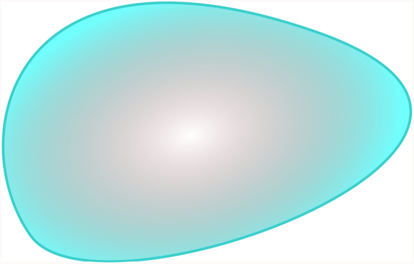
Counterexample to Lipsman's Conjecture

$$\mathbb{R}^2 \simeq \left\{ \begin{pmatrix} 1 & & & & b & \\ & 1 & a & \frac{a^2}{2} & \frac{a^3}{6} & -b \\ & & 1 & a & \frac{a^2}{2} & \\ & & & 1 & a & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\} \curvearrowright \mathbb{R}^5 \simeq \left\{ \begin{pmatrix} * \\ * \\ * \\ * \\ * \\ 1 \end{pmatrix} \right\}$$

(The action is free, but not proper)

$$X := \mathbb{R}^2 \backslash \mathbb{R}^5.$$

Topology on X



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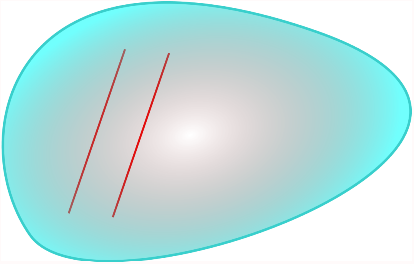

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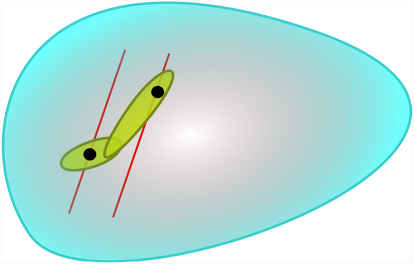
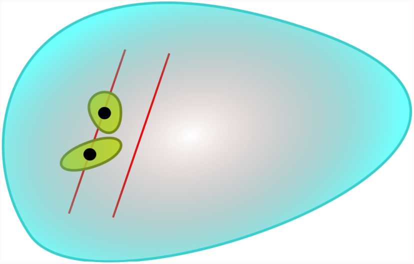


Not Separable

$$X = \mathbb{R}^2 \backslash \mathbb{R}^5$$

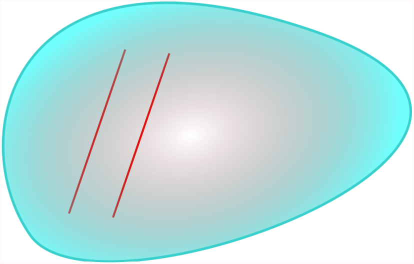
Topology on X

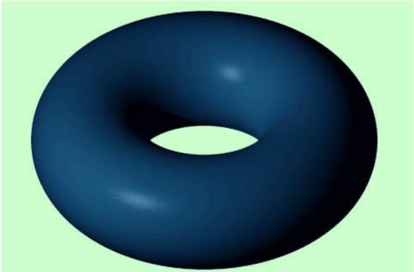


Separable

$$X = \mathbb{R}^2 \backslash \mathbb{R}^5$$

Topology on X

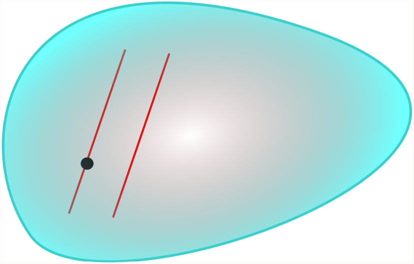


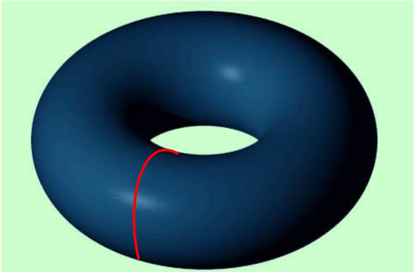


$$\cap$$

$$X = \mathbb{R}^2 \backslash \mathbb{R}^5 \xrightarrow{\text{Blow-up}} Y$$

Topology on X



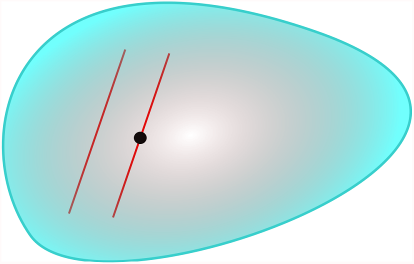


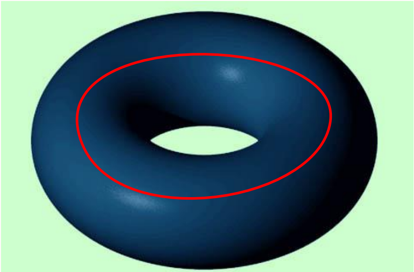
$$\bigcap$$

$$X = \mathbb{R}^2 \backslash \mathbb{R}^5 \xrightarrow{\text{Blow-up}} Y$$

$$\tau : X \longrightarrow 2^Y.$$

Topology on X





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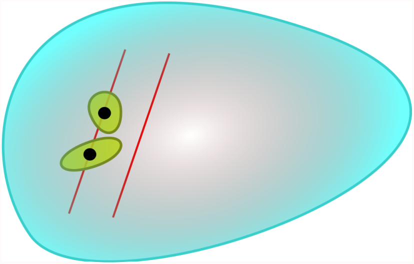
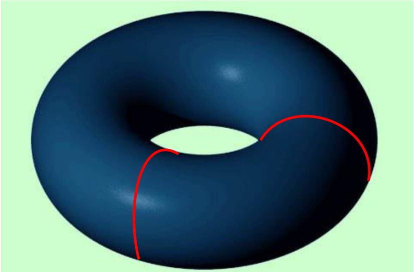
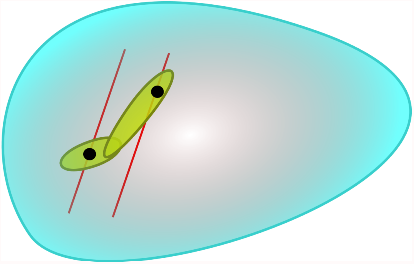
 

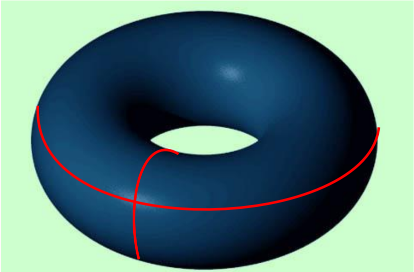
Separable

$$\begin{array}{ccc} & & \cap \\ X = \mathbb{R}^2 \backslash \mathbb{R}^5 & \xrightarrow{\text{Blow-up}} & Y \end{array}$$

$$\tau : X \longrightarrow 2^Y.$$

Topology on X





Not Separable

$$\cap$$

$$X = \mathbb{R}^2 \backslash \mathbb{R}^5 \quad \xrightarrow{\text{Blow-up}} \quad Y$$

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Functor

Proposition

$f : X \to X'$ proper continuous

$\implies {}^{\exists}\hat{f} : Y \to Y'$ proper continuous.

Functor

Proposition

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$\Longrightarrow\ {}^{\exists}\hat{f} : Y \to Y'$ proper continuous.

We may regard $X \xrightarrow{\text{Blow-up}} Y$ as a functor.

Functor

Proposition

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We may regard $X \xrightarrow{\text{Blow-up}} Y$ as a functor.

(loc. cpt. sp. proper cont.)

$\longrightarrow$ (loc. cpt. Hausdorff sp. proper cont.)

Definition blow-up space

How to define blow-up space?

H-lim & nH-lim operators

Observation

X is compact Hausdorff

$$\iff$$

any maximal filter has the unique limit pt.

H-lim & nH-lim operators

Observation

X is compact Hausdorff

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any maximal filter has the unique limit pt.

For a compact Hausdorff space X,

$$\textit{H-lim} : \mathcal{M}_X \to X, \quad \mathcal{F} \mapsto \text{(the uniq limit pt)}.$$

H-lim & nH-lim operators

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For a general topological space X,

$$\textit{nH-lim} : \mathcal{M}_X \to 2^X, \quad \mathcal{F} \mapsto \text{(the set of all limit pts)}.$$

Definition the topological blow-up Y

X: cpt Haus	X: loc cpt
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Then, Y^* is a compact Hausdorff space.

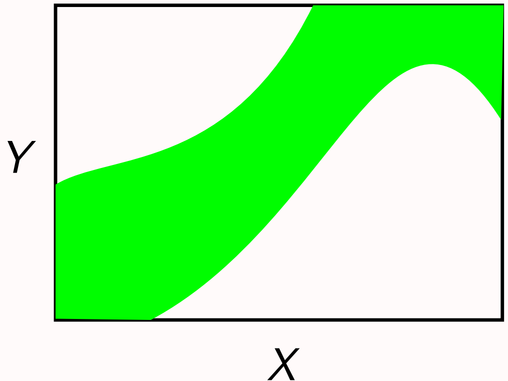
$$Y := Y^* \setminus \{\emptyset\}$$

Observation

X, Y: sets

There is a natural 1:1 correspondence

$$\eta : X \to 2^Y \qquad \longleftrightarrow \qquad \xi : Y \to 2^X.$$



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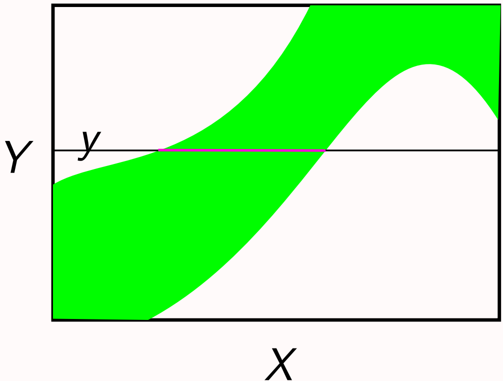

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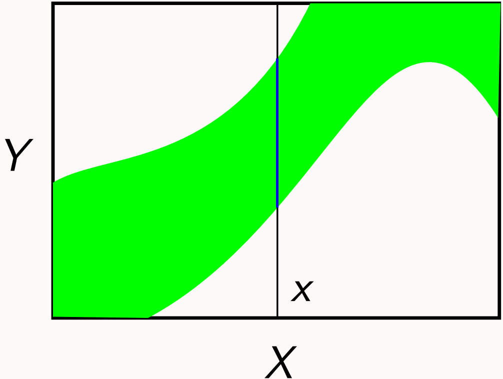

Definition of τ

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$\to$

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Example 5

Vector Field on $\mathbb{R}^4$:

$$x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}$$

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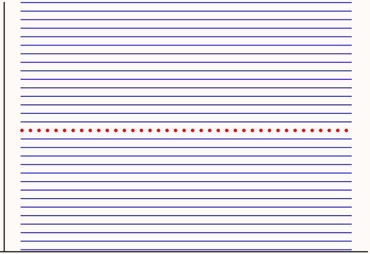

Example 5

Vector Field on $\mathbb{R}^4$:

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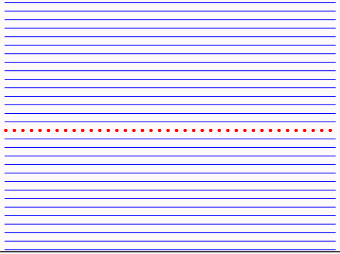

$$X = S \sqcup R$$
$$S \simeq \mathbb{R}^2$$
$$R \simeq \mathbb{R}^3 - \mathbb{R}$$

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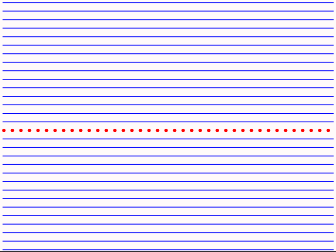
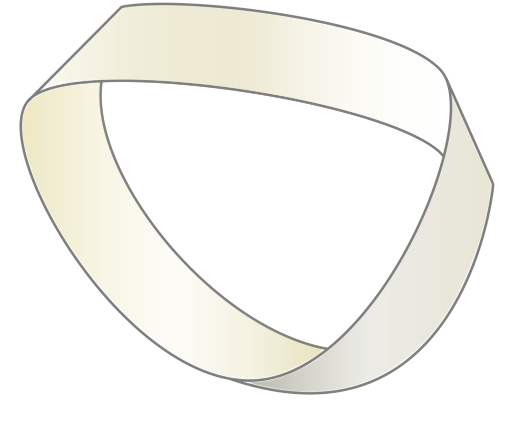


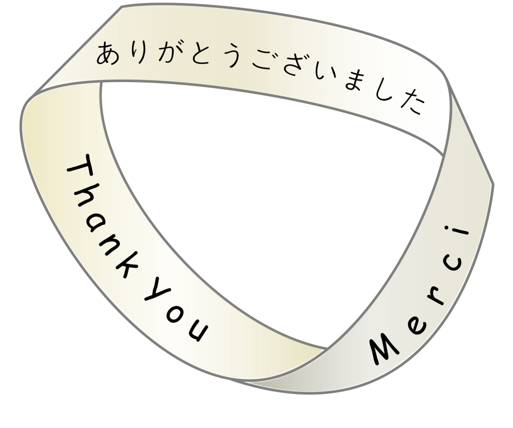
$$X = S \sqcup R$$

$$S \simeq \mathbb{R}^2$$

$$R \simeq \mathbb{R}^3 - \mathbb{R}$$

$$Y = S \sqcup (R \sqcup \boxed{?})$$




ありがとうございました
Thank You
Merci