Signatures on the braid groups

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$$\begin{tikzcd}
\widetilde{\mathrm{Diff}}_+(S^1) \arrow[r, "\tau"] \arrow[d] & \mathbb{R} \arrow[d] \\
\mathrm{Diff}_+(S^1) \arrow[r, "\mathrm{rot}"] & S^1.
\end{tikzcd}$$

Theorie der Zöpfe.

Von EMIL ARTIN in Hamburg.

THE THEORY OF BRAIDS

By EMIL ARTIN
Princeton University

THE theory of braids shows the interplay of two disciplines of pure mathematics—topology, used in the definition of braids, and the theory of groups, used in their treatment.

The fundamentals of the theory can be understood without too much technical knowledge. It originated from a much older problem in pure mathematics—the classification of knots. Much progress has been achieved in this field; but all the progress seems only to emphasize the extreme difficulty of the problem. Today we are still very far from a complete solution. In view of this fact it is advisable to study objects that are in some fashion similar to knots, yet simple enough so as to make a complete classification possible. Braids are such objects.

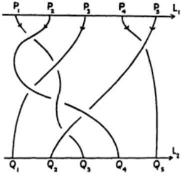
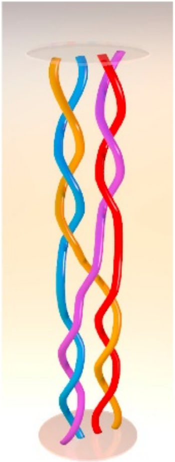


FIGURE 1

In order to develop the theory of braids we first explain what we call a *weaving pattern* of order n (n being an ordinary integral number which is taken to be 5 in Figure 1).

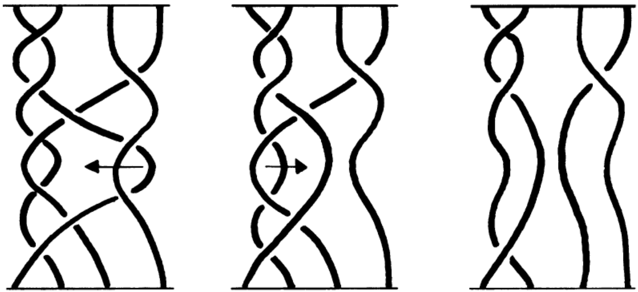
Let L_1 and L_2 be two parallel straight lines in space with given orientation in the same sense (indicated by arrows). If P is a point on L_1, Q a point on L_2, we shall sometimes join P and Q by a curve c. In our drawings we can only indicate the projection of c onto the plane containing L_1 and L_2, since c itself may be a winding curve in space.

A *n-strand geometrical braid* is the embedding of n disjoint monotone arcs in $\mathbb{C} \times [0, 1]$ going from $(z_i, 0)$ to $(z_i, 1)$.



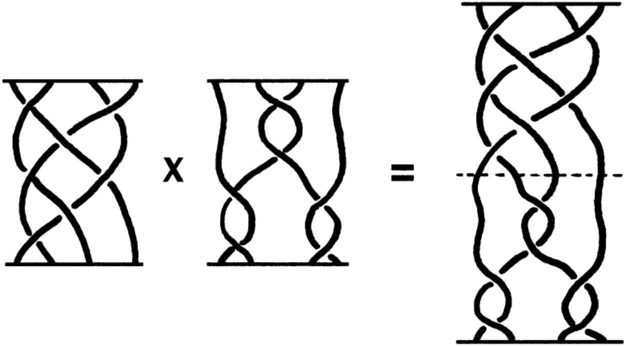
Pic: M. A. Berger

A *braid* is an isotopy class of geometrical braids.



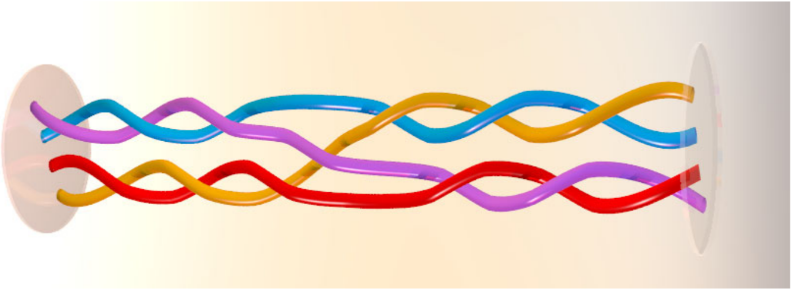
Pic: A. Sossinsky

We can stack one braid upon another to get a third one.



Up to isotopy, this gives the *n-strand braid group* B(n).

Pic: A. Sossinsky



Pic: M. A. Berger

$$\begin{tikzcd} & \widetilde{\mathrm{Diff}}_+(S^1) \arrow[d] \\ B(n) \arrow[r] & \mathrm{Diff}_+(S^1) \end{tikzcd}$$

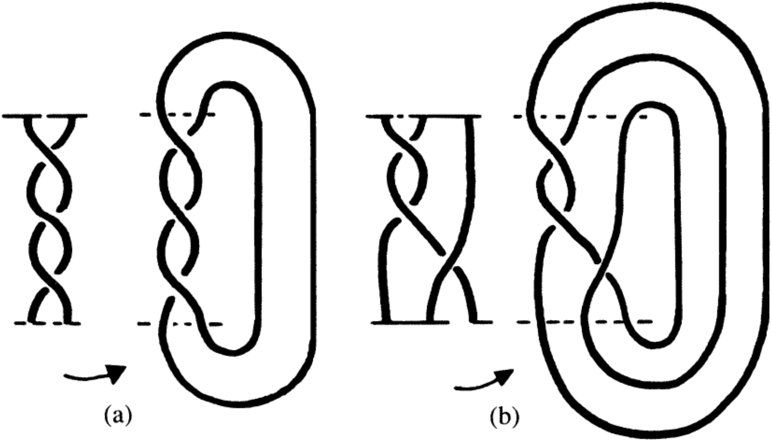
(Dehornoy, Thurston...)

$$
\begin{tikzcd}
 & \widetilde{\mathrm{Diff}}_+(S^1) \arrow[d] \\
B(n) \arrow[r] \arrow[ur, dashed] & \mathrm{Diff}_+(S^1)
\end{tikzcd}
$$

(Dehornoy, Thurston...)

So there is a translation number!

One can *close* a braid to get a link.



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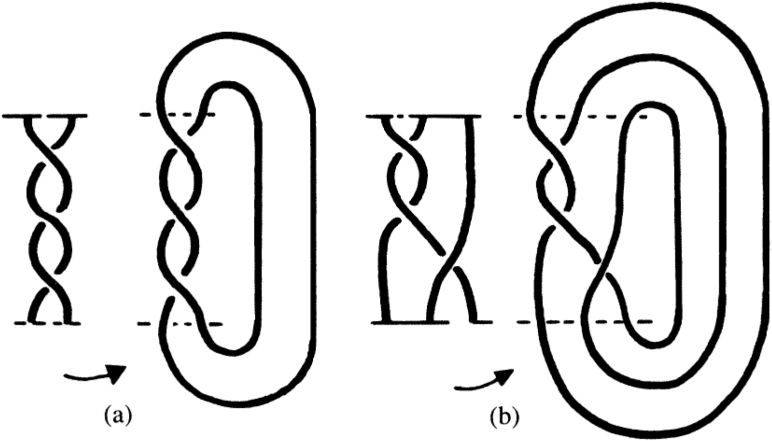

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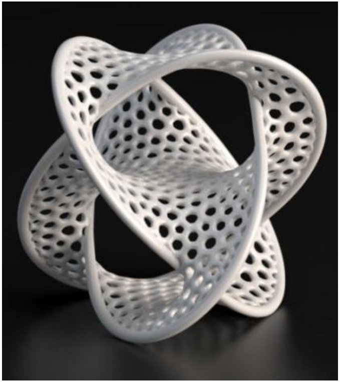
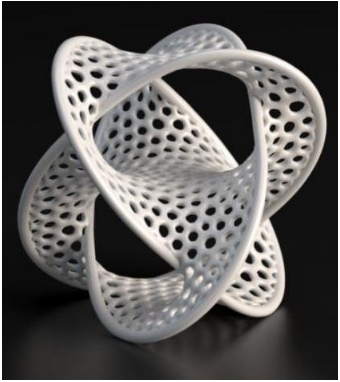
Every link is the closure of a braid (Alexander, 1923).



Pic: www.shapeways.com

Every link $L \subset S^3$ has a *Seifert surface:* an oriented surface $F \subset S^3$ such that $\partial F = L$.



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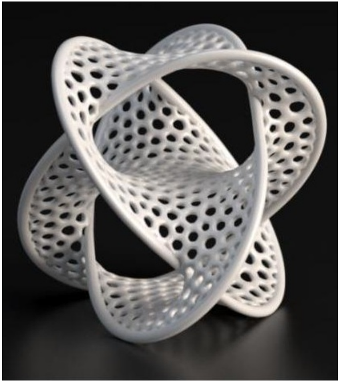
Every link $L \subset S^3$ has a *Seifert surface:* an oriented surface $F \subset S^3$ such that $\partial F = L$.

The orientation gives a morphism

$$i_+ : H_1(F) \to H_1(S^3 \setminus F).$$



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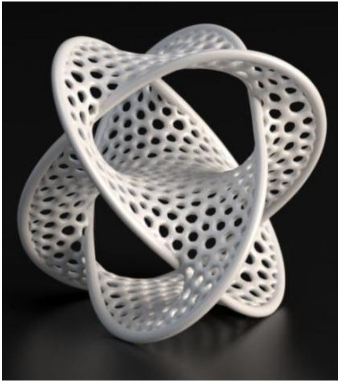
We get a bilinear form Θ on $H_1(F)$.

$$\Theta(\xi, \eta) = \mathrm{lk}(\xi, i_+(\eta)).$$



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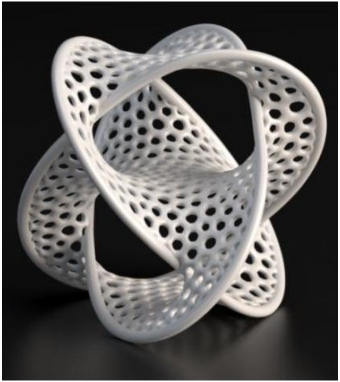
For $\omega \in S^1$,

$$(1-\omega)\Theta + (1-\overline{\omega})\Theta^T$$

is Hermitian.



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$(1-\omega)\Theta + (1-\overline{\omega})\Theta^T$ is Hermitian.

Its signature

$$\mathrm{sign}_\omega(L) \in \mathbb{Z}$$

is an invariant of the link.

Theorem (Gambaudo, Ghys, 2005). Let $x, y \in \mathrm{B}(n)$. Then

$$\operatorname{sign}_\omega(\widehat{xy}) - \operatorname{sign}_\omega(\widehat{x}) - \operatorname{sign}_\omega(\widehat{y}) = -\operatorname{Meyer}(\operatorname{Bur}_\omega(x), \operatorname{Bur}_\omega(y)).$$

Theorem (Gambaudo, Ghys, lighter version)

$\mathrm{sign}_\omega(\widehat{xy}) - \mathrm{sign}_\omega(\widehat{x}) - \mathrm{sign}_\omega(\widehat{y}) =$ sth. bounded and 4-dim.

58.

Análisis situs combinatorio

Revista Matematica Hispano-Americana 5, 43 p. (1923)

§ 1. Aritmética de las ecuaciones lineales.

A la exposición general del Análisis Situs conviene hacer preceder las principales proposiciones de la Aritmética de las ecuaciones lineales; es decir, el estudio de las soluciones en números enteros de un sistema de ecuaciones lineales con coeficientes enteros y n incógnitas. Un sistema de valores de $\mathbf{x} = (x^1, x^2, \ldots\ldots, x^n)$ será designado con el nombre de *vector;* la adición y la multiplicación por un número λ determinado seguirá las leyes expresadas por las siguientes igualdades:

$$(x^1, x^2, \ldots\ldots, x^n) + (y^1, y^2, \ldots\ldots, y^n) = (x^1 + y^1, x^2 + y^2, \ldots\ldots, x^n + y^n)$$
$$\lambda (x^1, x^2, \ldots\ldots, x^n) = (\lambda x^1, \lambda x^2, \ldots\ldots, \lambda x^n).$$

Theorie der quadratischen Formen in beliebigen Körpern.

Von *Ernst Witt* in Göttingen.

Witt groups are groups of bilinear objects up to sublagrangian reduction $E \rightsquigarrow I^{\perp}/I$:

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- Symmetric bilinear forms over fields

$$\mathbf{W}(\mathbb{R}) \simeq \mathbb{Z}.$$

- Hermitian forms over fields with involution

$$\mathbf{W}(\mathbb{C}) \simeq \mathbb{Z}.$$

Blanchfield form

$\mathscr{A}_L$: torsion part of $H_1(\widetilde{E}(L); \mathbb{Q})$.

$$\begin{aligned} \mathrm{Bl} : \mathscr{A}_L \times \mathscr{A}_L &\to \qquad \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}] \\ ([c_1], [c_2]) &\mapsto \frac{\sum \langle t^n S, c_2 \rangle t^n}{p} \bmod 1 \end{aligned} \qquad (\partial S = pc_1)$$

(cf. *linking form* of a 3-manifold)

Blanchfield signature

$$\mathrm{Bl} : \mathscr{A}_L \times \mathscr{A}_L \to \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}].$$

The Blanchfield form is an Hermitian form on a torsion $\mathbb{Q}[t^{\pm 1}]$-module. It defines a *Blanchfield signature*

$$\beta_L = [\mathrm{Bl}] \in \mathbf{WT}(\mathbb{Q}[t^{\pm 1}])$$

in the Witt group of such objects.

$$\mathbf{WT}(\mathbb{Q}[t^{\pm 1}]) \simeq \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty \,.$$

The projection of β_L in

$$\mathbf{WT}(\mathbb{Q}[t^{\pm 1}])/\text{torsion} \simeq \mathbb{Z}^\infty$$

is essentially equivalent to the ω-signatures.

Theorem. Let $x, y \in \mathrm{B}(n)$. Then

$$\beta_{\widehat{xy}} - \beta_{\widehat{x}} - \beta_{\widehat{y}} = -\partial\,\mathrm{Meyer}(\mathrm{Bur}(x), \mathrm{Bur}(y)).$$

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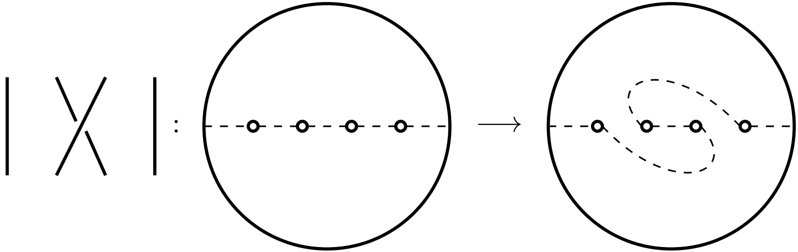
Better:

$$\beta_{\widehat{xy}} - \beta_{\widehat{x}} - \beta_{\widehat{y}} = \text{sth. “bounded” and 4-dim.}$$

This equality takes place in $\mathbf{WT}(\mathbb{Q}[t^{\pm 1}])$.

$$B(n) = \pi_0 \operatorname{Diff}_+(D_n, \partial_{\mathrm{ext}} D_n).$$



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ご清聴ありがとうございました。

Merci pour votre attention.