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On the uniformly perfectness of diffeomorphism groups preserving a submanifold and its applications

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1. Introduction

In this talk we shall describe the recent results on the uniformly perfectness of diffeomorphism groups of smooth manifolds preserving a submanifold.

Let M be a smooth connected manifold without boundary. Let $D^\infty(M)$ denote the group of C^∞-diffeomorphisms of M with compact support which are isotopic to the identity through C^∞-diffeomorphisms with compact support. It is known that M.Herman [5] and W.Thurston [6] proved $D^\infty(M)$ is perfect.

Let (M, N) be a manifold pair and $D^\infty(M, N)$ be the group of C^∞-diffeomorphisms of M preserving N which are isotopic to the identity through compactly supported C^∞-diffeomorphisms preserving N. In [1], we proved that the group $D^\infty(M, N)$ is perfect if the dimension of N is positive. In this talk we consider the conditions for $D^\infty(M, N)$ to be uniformly perfect. A group G is said to be uniformly perfect if each element of G is represented as a product of a bounded number of commutators of elements in G.

In [7], [8] T.Tsuboi obtained an excellent results on the uniform perfectness of the group $D^r(M)$. He proved that it is uniformly perfect $1 \leq r \leq \infty$ $(r \neq \dim M + 1)$ when M is an odd dimensional manifold or an even dimensional manifold with the appropriate conditions.

In [1], [2] we studied the conditions for $D^\infty(M, N)$ to be uniformly perfect when M is a compact manifold. If the group $D^\infty(M, N)$ is uniformly perfect, then both $D^\infty(N)$ and $D^\infty(M - N)$ are uniformly perfect. We need the another conditions for the converse. Let $p : D^\infty(M, N) \to D^\infty(N)$ be the map given by the restriction. If the connected components of $\ker p$ are finite, then $D^\infty(M, N)$ is a uniformly perfect group for $n \geq 1$. There exist many examples satisfying this condition.

If N is the union of circles in M and the connected components of $\ker p$ are infinite, then we can prove that $D^\infty(M, N)$ is not a uniformly perfect group. We can apply the result for various cases. If M is an oriented surface and N a disjoint union of circles in M, we can determine the uniformly perfectness of the group $D^\infty(M, N)$ ([2]). Finally we consider the case when $M = S^3$ and N is a knot in S^3, Then we prove that $D^\infty(S^3, N)$ is uniformly perfect if and only if N is a torus knot.

2. Statement of the main results

Let (M, N) be a manifold pair. Then $D^\infty(M, N)$ is perfect only if $\dim N \geq 1$ ([1], Theorem 1.1). Thus we assume that $\dim N \geq 1$ and investigate the conditions that $D^\infty(M, N)$ is uniformly perfect.

Theorem 2.1 ([1], [2]). *Let M be an m-dimensional compact manifold without boundary and N an n-dimensional C^∞-submanifold such that both groups $D^\infty(M - N)$ and $D^\infty(N)$ are uniformly perfect. If the connected components of* $\ker p$ *are finite, then $D^\infty(M, N)$ is a uniformly perfect group for $n \geq 1$.*

The converse of Theorem 2.1 is valid when N is a disjoint union of circles in M.

Theorem 2.2 ([2]). *Let M be an m-dimensional compact manifold without boundary and N be a disjoint union of circles in M. If the connected components of* $\ker p$ *are infinite, then $D^\infty(M, N)$ is not a uniformly perfect group.*

Now we apply Theorem 2.1 and Theorem 2.2 for studying the uniformly perfectness of the group $D^\infty(M, N)$ when M is an orientable surface and N is a disjoint union of circles.

Theorem 2.3 ([3]). *$D^\infty(M, N)$ is uniformly perfect if and only if*
(1) *$M = S^2$ and $k = 1$ and,*
(2) *$M = T^2$, $k = 1$ and N represents a non-trivial element of $\pi_1(T^2)$.*

Finally we consider the case where K is a knot in S^3. Using the result by G. Burde and H. Zieschang [4], we have the following.

Theorem 2.4. *$D^\infty(S^3, K)$ is uniformly perfect if and only if K is a torus knot.*

References

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