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On the ampleness of positive CR line bundles over Levi-flat manifolds

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1. Background

A closed real hypersurface M in a complex manifold X is said to be *Levi-flat* if M has a foliation $\mathcal{F}$ (called the *Levi foliation*) whose leaves are non-singular complex hypersurfaces of X. By the Frobenius theorem, this definition is equivalent to saying that M is locally pseudoconvex from both sides. Therefore, by its definition, the study of Levi-flat real hypersurfaces is of two natures: intrinsic one of the theory of foliations, and extrinsic one of function theory of several complex variables.

A problem in the study of Levi-flat real hypersurfaces is to understand the interplay between complexity of the Levi foliation $\mathcal{F}$ and pseudoconvexity of the complement $X \setminus M$, which was first pointed out explicitly by Barrett [2]. He studied several explicit families of Levi-flat real hypersurfaces in compact complex surfaces, and showed that, in these examples, the existence of a leaf with non-trivial holonomy corresponds to the 1-convexity of the complement. Our motivation of this study is to refine this suggested connection and to describe it in quantitative way.

On the other hand, complexity of the Levi foliation $\mathcal{F}$ should also be reflected on transverse regularity of leafwise meromorphic functions on M. Here, what we mean by a *leafwise meromorphic function* is a leafwise holomorphic section of a $\mathcal{C}^\infty$ *CR line bundle* L over M (a $\mathcal{C}^\infty$ $\mathbb{C}$-line bundle over M with $\mathcal{C}^\infty$ leafwise holomorphic transition functions) and when we say *leafwise holomorphic*, it is in the distribution sense. If the Levi foliation $\mathcal{F}$ is enough complicated, leafwise meromorphic functions may lose transverse regularity since they are analytically continued along leaves and can behave wildly in the transverse direction. Actually, Inaba [4] showed that if we impose continuity on leafwise holomorphic functions on compact Levi-flat manifolds, they must be constant along leaves.

In this note, we take the latter viewpoint and focus on the following

Question 1.1. How does pseudoconvexity of the complement $X \setminus M$ affect transverse regularity of leafwise meromorphic functions on M?

This short note is an announcement of [1], to which we refer the reader for the details.

2. Results

First we recall a Kodaira type embedding theorem of Ohsawa and Sibony, which holds for not only compact Levi-flat hypersurfaces but also abstract compact Levi-flat manifolds.

Theorem 2.1 ([7] Theorem 3, refined in [5]). *Let M be a compact $\mathcal{C}^\infty$ Levi-flat manifold equipped with a $\mathcal{C}^\infty$ CR line bundle L. Suppose L is positive along leaves, i.e., there exists a $\mathcal{C}^\infty$ hermitian metric on L such that the restriction of the curvature form to each leaf is everywhere positive definite. Then, for any $\kappa \in \mathbb{N}$, L is $\mathcal{C}^\kappa$-ample, i.e., there exists $n_0 \in \mathbb{N}$ such that one can find leafwise holomorphic sections $s_0, \cdots, s_N$ of $L^{\otimes n}$, of class $\mathcal{C}^\kappa$, for any $n \geq n_0$, such that the ratio $(s_0 : \cdots : s_N)$ embeds M into $\mathbb{CP}^N$.*

For arbitrarily large $\kappa \in \mathbb{N}$, by taking n_0 sufficiently large depending on κ, we can obtain so many $\mathcal{C}^\kappa$ leafwise holomorphic sections of $L^{\otimes n_0}$; in fact, they form an infinite dimensional vector space. Note that the existence of a positive-along-leaves $\mathcal{C}^\infty$ CR line bundle over M is equivalent to the tautness of the Levi foliation of M; our setting is not too restrictive.

A natural question on the Ohsawa–Sibony embedding theorem is whether or not we can improve the regularity to $\kappa = \infty$. This question asks the dependence of n_0 on κ as $\kappa \to \infty$, and at this point we will face a subtle interplay between transverse regularity of leafwise meromorphic functions, and complexity of the Levi foliation or pseudoconvexity of the complement.

Now we introduce a notion of pseudoconvexity that we are going to focus on.

Definition 2.2 (Takeuchi 1-complete space). Let D be a relatively compact domain in a complex manifold X with $\mathcal{C}^2$ boundary. We say that D is *Takeuchi 1-complete* if there exists a $\mathcal{C}^2$ defining function r of ∂D defined on a neighborhood of D with $D = \{z \mid r(z) < 0\}$ such that, with respect to a hermitian metric on X, all of the eigenvalues of the Levi form of $-\log(-r)$ are bounded from below by a strictly positive constant entire on D.

Takeuchi 1-completeness not only implies that the domain is Stein, but also implies that it behaves as if it is in complex Euclidean space.

Theorem 2.3 (cf. [6] Theorem 1.1). *Let D be a Takeuchi 1-complete domain with defining function r. Then, $-\partial\overline{\partial}\log(-r)$ gives a complete Kähler metric on D, and it follows that $-(-r)^\varepsilon$ with sufficiently small $\varepsilon > 0$ becomes a strictly plurisubharmonic bounded exhaustion function on D, i.e., D is hyperconvex.*

We do not have any compact Levi-flat real hypersurface in $\mathbb{C}^n$ if $n \geq 2$. Nevertheless, there exist compact Levi-flat real hypersurfaces in compact complex surfaces whose complements are Takeuchi 1-complete.

Theorem 2.4 ([1])**.** *Let Σ be a compact Riemann surface of genus ≥ 2 and $\rho : \pi_1(\Sigma) \to \mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$ a group homomorphism. Denote by $\mathcal{D}_\rho = \Sigma \times_\rho \mathbb{D}$ the holomorphic unit disc bundle obtained by the suspension of ρ over Σ. Suppose there exists a unique non-holomorphic harmonic section $h : \Sigma \to \mathcal{D}_\rho$ whose $\mathrm{rank}_{\mathbb{R}}\, dh = 2$ on a non-empty open set. Then, $\mathcal{D}_\rho$ is Takeuchi 1-complete in its associated $\mathbb{CP}^1$-bundle $X_\rho = \Sigma \times_\rho \mathbb{CP}^1$.*

The boundary of $\mathcal{D}_\rho$ is a flat S^1-bundle $M_\rho = \Sigma \times_\rho S^1$, thus, a Levi-flat real hypersurface. The assumption is fulfilled for any non-trivial quasiconformal deformation ρ of Γ where Γ is a Fuchsian representation of $\Sigma = \mathbb{D}/\Gamma$.

The proof of Theorem 2.4 is by explicitly constructing a suitable defining function, in which the harmonic section h is the essential ingredient. This technique originates in the work of Diederich and Ohsawa [3].

We can observe the following Bochner–Hartogs type phenomenon for Levi-flat real hypersurfaces with Takeuchi 1-complete complements.

Theorem 2.5. *Let X be a compact complex surface, L a holomorphic line bundle over X, and M a $\mathcal{C}^\infty$ compact Levi-flat real hypersurface of X which splits X into two Takeuchi 1-complete domains $D \sqcup D'$. Then, there exists $\kappa \in \mathbb{N}$ such that any $\mathcal{C}^\kappa$ leafwise holomorphic section of $L|M$ extends to a holomorphic section of L.*

This theorem tells us that the space of $\mathcal{C}^\kappa$ leafwise holomorphic sections of $L|M$ is finite dimensional for sufficiently large κ; in particular, the space of $\mathcal{C}^\infty$ leafwise holomorphic sections of $L|M$ is always finite dimensional. This description is a qualitative answer to Question 1.1 for Levi-flat real hypersurfaces with Takeuchi 1-complete complements.

The proof of Theorem 2.5 can be done with established techniques in function theory of several complex variables. A simple proof is given in [1].

As a corollary, we give an example that shows that the Ohsawa–Sibony embedding theorem cannot hold for $\kappa = \infty$ in general.

Corollary 2.6 ([1])**.** *Let Σ be a compact Riemann surface of genus ≥ 2, and $\rho : \pi_1(\Sigma) \to \mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$ a group homomorphism. Denote the suspended $\mathbb{CP}^1$ bundle by $\pi \colon X_\rho \to \Sigma$. Take a positive line bundle L over Σ. Suppose $\mathcal{D}_\rho$ has a unique non $\pm$holomorphic harmonic section h whose $\mathrm{rank}_{\mathbb{R}}\, dh = 2$, then $\pi^* L|M_\rho$ is positive along leaves, but never $\mathcal{C}^\infty$ ample.*

3. Further Questions

We conclude this short note with further questions. The following notion is a quantitative version of Takeuchi 1-completeness according to Theorem 2.3.

Definition 3.1. Let D be a Takeuchi 1-complete domain with defining function r. We denote by $\varepsilon_{DF}(r)$ the supremum of $\varepsilon \in (0, 1)$ such that $-(-r)^{\varepsilon}$ is a strictly plurisubharmonic bounded exhaustion function on D, and call it the *Diederich–Fornaess exponent* of the defining function r.

Our questions are quantitative or intrinsic versions of Question 1.1.

Question 3.2. Can we estimate the κ in Theorem 2.5 in terms of the Diederich–Fornaess exponent of some defining function?

Question 3.3. What is the counterpart of the Diederich–Fornaess exponent in the theory of foliations? By using it, can we prove Corollary 2.6 without looking the natural Stein filling $\mathcal{D}_\rho$?

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