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A cocycle rigidity lemma for Baumslag-Solitar actions and its applications

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1. A cocycle rigidity lemma

Let $\mathrm{Diff}(\mathbf{R}^n, 0)$ be the group of local diffeomorphisms of $\mathbf{R}^n$ at the origin. In many situations in study of foliations, we encounter with $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycles over a group action. A typical case is the following. Consider an action of simply connected Lie group whose orbits form a smooth codimension-n foliation with trivial normal bundle. Then, the holonomy map of the foliation with respect to a fixed family of transverse coordinates defines a $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycle. In this case, the existence of a transverse geometric structure is equivalent to the condition that the cocycle can be reduced to a subgroup of $\mathrm{Diff}(\mathbf{R}^n, 0)$ which preserves the geometric structure.

In this talk, we show a rigidity lemma for $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycle over actions of the Baumslag-Solitar group $BS(1, k)$. We also apply it to rigidity problem of several group actions.

For integers $k \geq 2$, **the Baumslag-Solitar group** $BS(1, k)$ is the group presented as

$$\langle a, b \mid aba^{-1} = b^k \rangle.$$

There are many copies of $BS(1, k)$ is contained in the group $\mathrm{CAff}(\mathbf{R}^n)$ of conformal affine transformations of $\mathbf{R}^n$. In fact, let f_k and g_v be elements of $\mathrm{CAff}(\mathbf{R}^n)$ given by $f_k(x) = kx$ and $g_v(b) = x+v$. Then, the correspondence $a \mapsto f_k$ and $b \mapsto g_v$ gives an inclusion from $BS(1, k)$ to $\mathrm{CAff}(\mathbf{R}^n)$.

Let Γ and H be topological groups and X a topological space. For a given action $\rho : \Gamma \times X \to X$, a map $\alpha : \Gamma \times X \to H$ is called a **cocycle** over ρ if $\alpha(1_\Gamma, x) = x$ and $\alpha(\gamma\gamma', x) = \alpha(\gamma, \gamma' x) \cdot \alpha(\gamma', x)$ for any $\gamma, \gamma' \in G$ and $x \in X$ (1_Γ is the unit element of Γ). The space of H-valued cocycle over ρ admits a topology as a subspace of $C^0(\Gamma \times X, H)$. Let H' be a subgroup of H. Two H-valued cocycles α and β over ρ are H'-equivalent if there exists $h \in H'$ such that $\beta(\gamma, x) = h \cdot \alpha(\gamma, x) \cdot h^{-1}$ for any $\gamma \in \Gamma$ and $x \in X$.

For an element F of $\mathrm{Diff}(\mathbf{R}^n 0)$, we denote the r-jet of F at the origin by $j_0^r F$. Let $j^r \mathrm{Diff}(\mathbf{R}^n, 0)$ is the group of r-jets of elements of $\mathrm{Diff}(\mathbf{R}^n, 0)$ at the origin. The group $\mathrm{Diff}(\mathbf{R}^n, 0)$ is endowed with the weakest topology such that the projection to $j^r \mathrm{Diff}(\mathbf{R}^n, 0)$ is continuous for any $r \geq 1$ (it is not Hausdorff). We denote the identity map of $\mathbf{R}^n$ by Id. For $r \geq 1$, let $G^{(r)}$ be the subgroup of $\mathrm{Diff}(\mathbf{R}^n, 0)$ consisting of elements with trivial r-jet.

Cocycle Rigidity Lemma *There exists a universal constant $\epsilon_k > 0$ such that the following assertion holds: Let X be a topological space, $\rho : BS(1,k) \times X \to X$ a continuous $BS(1,k)$-action. If continuous cocycles $\alpha, \beta : BS(1,k) \times X \to \mathrm{Diff}(\mathbf{R}^n, 0)$ over ρ satisfies that*

1. *$j_0^2(\alpha(\gamma, x)) = j_0^2(\beta(\gamma, x))$ for any $\gamma \in BS(1,k)$ and $x \in X$, and*
2. *$\|j_0^1(\alpha(a,x)) - (1/k)\,\mathrm{Id}\,\| < \epsilon_k$ and $\|j_0^1(\alpha(b,x)) - \mathrm{Id}\,\| < \epsilon_k$, where* Id *is the identity map on $\mathbf{R}^n$,*

then two cocycles α and β are $G^{(2)}$-equivalent. If $\alpha(a, \cdot) = \beta(a, \cdot)$ in addition then α and β coincide as cocycles.

In other words, a cocycle whose linear part is close to the linear representation given by $a \mapsto (1/k)I$ and $b \mapsto I$ is determined by its 2-jet up to $G^{(2)}$-equivalence.

2. Applications

The first application of the above cocycle rigidity lemma is rigidity of certain conformal local action of a Baumslag-Solitar-like group. For $k \geq 2$ and $n \geq 1$, let $\Gamma_{n,k}$ be the discrete group presented as

$$\langle a, b_1, \ldots, b_n \mid a b_i a^{-1} = b_i^k, b_i b_j = b_j b_i \ (i, j = 1, \ldots, n) \rangle.$$

Each subgroup generated by a and b_i is isomorphic to $BS(1,k)$. Let f_k and g_v be conformal affine maps on $\mathbf{R}^n$ given in the previous section. They naturally extends to the sphere $S^n = \mathbf{R}^n \cup \{\infty\}$. For a basis $B = (v_1, \ldots, v_n)$ of $\mathbf{R}^n$, we define a smooth $\Gamma_{n,k}$-action ρ_B on S^n (*i.e.* a homomorphism from $\Gamma_{n,k}$ to $\mathrm{Diff}(S^n)$) by $\rho_B(a) = f_k$ and $\rho_B(b_i) = g_{v_i}$. Let $\phi : S^n \setminus \{0\} \to \mathbf{R}^n$ be a coordinate at ∞ given by $\phi(x) = x/\|x\|^2$. We define a local $\Gamma_{n,k}$-action P_B (*i.e.* a homomorphism from $\Gamma_{n,k}$ to $\mathrm{Diff}(\mathbf{R}^n, 0)$) by $P_B(\gamma) = \phi \cdot \rho_B(\gamma) \cdot \phi^{-1}$. Remark that the local action P_B preserves the standard conformal structure on $\mathbf{R}^n$.

Theorem 2.1 ([1]). *If a local action $P : \Gamma_{n,k} \to \mathrm{Diff}(\mathbf{R}^n, 0)$ is sufficiently close to P_B, then there exists a basis B' of $\mathbf{R}^n$ and a local diffeomorphism $H \in \mathrm{Diff}(\mathbf{R}^n, 0)$ such that $P(\gamma) = H \cdot P_{B'}(\gamma) \cdot H^{-1}$ for any $\gamma \in \Gamma_{n,k}$. In particular, the local $\Gamma_{n,k}$-action P preserves a smooth conformal structure on $\mathbf{R}^n$.*

Outline of Proof. Notice that a local action is a $\mathrm{Diff}(\mathbf{R}^n, 0)$-cocycle over the trivial action on a point. We can check that the sub-action generated by a and b_i satisfies the assumptions of the rigidity lemma. So, it is sufficient to show that P coincides with a conjugate of some $P_{B'}$ up to 2-jet. Finding the basis B' can be done using a variant of Weil's rigidity theorem of homomorphisms between Lie groups [5]. □

Using the persistence of global fixed point ∞, we can derive a global rigidity theorem from the above theorem.

Theorem 2.2 ([1]). *If a smooth $\Gamma_{n,k}$-action ρ is sufficiently close to ρ_B, then there exists a basis B' of $\mathbf{R}^n$ and a diffeomorphism h of S^n such that $\rho(\gamma) = h \cdot \rho_{B'}(\gamma) \cdot h^{-1}$.*

A similar local or global rigidity theorem can be shown for a $\Gamma_{n,k}$-action on the n-dimensional torus $\mathbf{T}^n$. We identify $\mathbf{T}^n$ with $(\mathbf{R} \cup \{\infty\})^n$. For a basis $B = (v_1, \ldots, v_n)$ of $\mathbf{R}^n$, we define a $\Gamma_{n,k}$-action σ_B on $\mathbf{T}^n$ by

$$\sigma_B(a)(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n)$$
$$\sigma_B(b_i)(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + v_i, x_{i+1}, \ldots, x_n).$$

By the same way as above, we can show a rigidity result for this action.

Theorem 2.3. *If a smooth $\Gamma_{n,k}$-action σ is sufficiently close to σ_B, then there exists a basis B' of $\mathbf{R}^n$ and a diffeomorphism h of $\mathbf{T}^n$ such that $\sigma(\gamma) = h \cdot \sigma_{B'}(\gamma) \cdot h^{-1}$.*

The second application is another proof of Ghys's local rigidity theorem on Fuchsian action on $\mathbf{R}P^1$. Let Γ be a cocompact lattice of $\mathrm{PSL}(2, \mathbf{R})$ Since $\mathrm{PSL}(2, \mathbf{R})$ acts on $\mathbf{R}P^1$ naturally, Γ acts on $\mathbf{R}P^1$ as a subgroup of $\mathrm{PSL}(2, \mathbf{R})$. We denote this action by ρ_Γ. More generally, when a homomorphism $\pi : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ is given, we can define a Γ-action ρ_π on $\mathbf{R}P^1$ by $\rho_\pi(\gamma)(x) = \pi(\gamma) \cdot x$.

Theorem 2.4 (Ghys [2]). *If a Γ-action ρ on $\mathbf{R}P^1$ is sufficiently close to ρ_Γ, then there exists an homomorphism $\pi : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ and a diffeomorphism h of S^1 such that $\rho(\gamma) = h \cdot \rho_\pi(\rho) \cdot h^{-1}$ for any $\gamma \in \Gamma$.*

All known proofs ([2, 3, 4]) use the Schwarzian derivative, but our proof does not. We use that fact that any $j^2 \operatorname{Diff}(\mathbf{R}, 0)$-cocycle can be extended to a cocycle valued in projective transformations of $(\mathbf{R}, 0)$

Outline of our proof. Let P be the subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that consists of lower triangular elements. It is generated by one-parameter subgroups $A = (a^t)_{t \in \mathbf{R}}$ and $N = (b^s)_{s \in \mathbf{R}}$ with a relation $a^t b^s a^{-t} = b^{s \exp t}$. Define a smooth right P-action ρ_P on $\Gamma \backslash \mathrm{PSL}(2, \mathbf{R})$ by $\rho_P(\Gamma g, p) = \Gamma(gp)$, and denote the orbit foliation of ρ_P by $\mathcal{F}_P$. As Ghys proved, it is sufficient to show that any foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_P$ admits a smooth transversely projective structure. Since the restriction of ρ_P to A is an Anosov flow and $\mathcal{F}_P$ is its unstable foliation, we can find a homeomorphism of M which sends each leaf of $\mathcal{F}_P$ to that of $\mathcal{F}$. This homeomorphism induces a continuous P-action ρ whose orbit foliation is $\mathcal{F}$. The holonomy map of $\mathcal{F}$ gives a $\mathrm{Diff}(\mathbf{R}, 0)$-valued cocycle $\bar{\alpha}$ over ρ.

The group P naturally contains $BS(1, k)$ as a subgroup. Let α be the restriction of the cocycle $\bar{\alpha}$ to $BS(1, k)$. To show that $\mathcal{F}$ is transversely projective, it is enough to see that $\bar{\alpha}$ is $\mathrm{Diff}(\mathbf{R}, 0)$-equivalent to a cocycle whose values are projective transformation. But, it is an easy consequence of the rigidity lemma. In fact, as mentioned above. any $j^2\,\mathrm{Diff}(\mathbf{R}, 0)$-valued cocycle can be extended to a cocycle whose values are projective transformations in one-dimension (it is not true for higher dimension). So, the rigidity lemma implies that α is $G^{(2)}$-equivalent to such a cocycle. □

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