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On Fatou-Julia decompositions of complex dynamical systems

Taro ASUKE

1. Introduction

A Fatou-Julia decomposition for transversely holomorphic, complex codimension-one foliations is introduced by Ghys, Gomez-Mont and Saludes [4] (and in [6]) in terms of deformations of holomorphic structures. Another decomposition is introduced in [2] in terms of normal families. These decompositions enjoy some properties similar to those of classical Fatou-Julia decomposition and also to the decomposition of the sphere into the domains of discontinuity and the limit sets (of Kleinian groups). In [3], a Fatou-Julia decomposition is introduced for pseudosemigroups. The decomposition is still difficult to study, however, it provides a natural unification of the notions of Fatou-Julia decomposition of mapping iterations, foliations and the decomposition of sphere with respect to the action of Kleinian groups. In this article, we will introduce pseudosemigroups and the Fatou-Julia decomposition, and explain how decompositions are unified (Theorem 2.16) after [2] and [3].

2. Pseudosemigroups and Fatou-Julia decompositions

We first introduce notions of pseudosemigroups and their Fatou-Julia decompositions. The notion of pseudosemigroups has already appeared (cf. [8], [11] and [7]). We will make use of a similar but different one.

In what follows, we consider holomorphic mappings unless otherwise mentioned, although pseudosemigroups can be considered in much more generalities.

In short, a pseudosemigroup is a pseudogroup but the inverse is not necessarily defined.

Definition 2.1. Let T be an open subset (not necessarily connected) of $\mathbb{C}^n$ and Γ be a family of mappings from open subsets of T into T (we call such mappings *local mappings*). Then, Γ is a (holomorphic) *pseudosemigroup* (psg for short) if the following conditions are satisfied.

1) $\mathrm{id}_T \in \Gamma$, where id_T denotes the identity map of T.

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2) If $\gamma \in \varGamma$, then $\gamma|_U \in \varGamma$ for any open subset U of $\operatorname{dom}\gamma$.
3) If $\gamma_1, \gamma_2 \in \varGamma$ and $\operatorname{range}\gamma_1 \subset \operatorname{dom}\gamma_2$, then $\gamma_2 \circ \gamma_1 \in \varGamma$, where $\operatorname{dom}\gamma$ and $\operatorname{range}\gamma$ denotes the domain and the range of γ, respectively.
4) Let U be an open subset of T and γ continuous mapping defined on U. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to $\varGamma$, then $\gamma \in \varGamma$.

If in addition $\varGamma$ consists of local homeomorphisms, namely, homeomorphisms from domains to ranges, then $\varGamma$ is a *pseudogroup* (pg for short) if $\varGamma$ satisfies 1), 2), 3) and the following conditions.

4′) Let U be an open subset of T and γ a homeomorphism from U to $\gamma(U)$. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to $\varGamma$, then $\gamma \in \varGamma$.
5) If $\gamma \in \varGamma$, then $\gamma^{-1} \in \varGamma$.

If $\varGamma$ is either a psg or pg, then we set for $x \in T$

$$\varGamma_x = \{\gamma_x \mid x \in \operatorname{dom}\gamma\}.$$

By abuse of notation, an element of $\varGamma_x$ is considered as an element of $\varGamma$ defined on a neighborhood of x.

One might expect that a pg is a psg but it is not always the case.

Example 2.2. Let $T = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and define an automorphism f of $\mathbb{C}P^1$ by $f(z) = -z$. We denote by $\varGamma$ the pg generated by f, that is, the smallest pg which contains f. Let $U = \{z \in \mathbb{C} \mid |z-2| < 1\}$ and $V = f(U)$. If we set $\gamma = f|_V$, then $\gamma \cup \mathrm{id}\colon V \cup U \to U$ is not an element of $\varGamma$, because $\gamma \cup \mathrm{id}$ is not a homeomorphism. If $\varGamma$ were a psg, then $\gamma \cup \mathrm{id} \in \varGamma$ by the condition 4).

Definition 2.3. We denote by $\varGamma_0^\times$ the subset of $\varGamma$ which consists of invertible elements, namely,

$$\varGamma_0^\times = \{\gamma \in \varGamma \mid \gamma^{-1} \in \varGamma\}.$$

We denote by $\varGamma^\times$ the subset of $\varGamma$ which consists of locally invertible elements, namely,

$$\varGamma^\times = \left\{\gamma \in \varGamma \,\middle|\, \begin{array}{l}\exists \text{ an open covering } \{U_\lambda\}_{\lambda\in\Lambda} \text{ of } \operatorname{dom} f\\ \text{such that } (\gamma|_{U_\lambda})^{-1} \in \varGamma\end{array}\right\}.$$

Note that $\varGamma_0^\times$ is a pseudogroup.

Definition 2.4. Let $(\varGamma, T)$ be a psg. We denote by $\mathscr{T}$ the family of relatively compact open subsets of T. If $T' \in \mathscr{T}$, then the *restriction* of $\varGamma$

to T' is defined by

$$\Gamma_{T'} = \{\gamma \in \Gamma \mid \operatorname{dom}\gamma \subset T' \text{ and } \operatorname{range}\gamma \subset T'\}.$$

The notion of *compact generation* [6] is also significant for psg's. The notions of morphisms and equivalences are given as follows.

DEFINITION 2.5. Let (Γ, T) and (Δ, S) be psg's. A (holomorphic) *morphism* $\Phi\colon \Gamma \to \Delta$ is a collection Φ of local mappings from T to S with the following properties.

1) $\{\operatorname{dom}\phi \mid \phi \in \Phi\}$ is an open covering of T.
2) If $\phi \in \Phi$, then any restriction of ϕ to an open set of $\operatorname{dom}\phi$ also belongs to Φ.
3) Let U be an open subset of T and ϕ a mapping from U to S. If for any $x \in U$, there exists an open neighborhood U_x of x such that $\phi|_{U_x} \in \Phi$, then $\phi \in \Phi$.
4) If $\phi \in \Phi$, $\gamma \in \Gamma^\times$ and $\delta \in \Delta^\times$, then $\delta \circ \phi \circ \gamma \in \Phi$.
5) Suppose that $\gamma \in \Gamma$ and $x \in \operatorname{dom}\gamma$. If $x \in \operatorname{dom}\phi$ and $\gamma(x) \in \operatorname{dom}\phi'$, where $\phi, \phi' \in \Phi$, then there is an element $\delta \in \Delta$ such that $\phi(x) \in \operatorname{dom}\delta$, and $\delta \circ \phi = \phi' \circ \gamma$ on a neighborhood of x.

A morphism from (Γ, T) to itself is called an *endomorphism* of (Γ, T).

DEFINITION 2.6. Let (Γ, T) and (Δ, S) be psg's and Φ a morphism from Γ to Δ.

1) Φ is called an *étale morphism* if Φ consists of étale mappings, namely, mappings of which the restriction to sufficiently small open sets are homeomorphisms.
2) Suppose that Γ and Δ are psg's on complex one-dimensional manifolds. A morphism is said to be *ramified* if $\phi \in \Phi$ and $x \in \operatorname{dom}\phi$, then there exists an open neighborhood U_x of x such that $\phi|_{U_x}$ is the restriction of the composite of ramified coverings and holomorphic étale mappings.

DEFINITION 2.7. Let (Γ, T) and (Δ, S) be psg's. A collection Φ of local homeomorphisms from T to S is an *étale morphism* of pg's if Φ satisfies the conditions in Definition 2.5 but 'a continuous map from U to S' in 3) is replaced by 'a local homeomorphism from T to S'.

DEFINITION 2.8. Let A be a set which consists of local mappings on T. A psg Γ is said to be *generated* by A if Γ contains A and is the smallest with respect to inclusions. The psg generated by A is denoted by $\langle A\rangle$. Similarly, we consider morphisms generated by local mappings from T to S.

Definition 2.9. If $\Phi_1\colon \mathit{\Gamma}_1 \to \mathit{\Gamma}_2$ and $\Phi_2\colon \mathit{\Gamma}_2 \to \mathit{\Gamma}_3$ are morphisms of psg's, then the *composite* $\Phi_2 \circ \Phi_1$ is defined by

$$\Phi_2 \circ \Phi_1 = \langle \phi_2 \circ \phi_1 \mid \phi_1 \in \Phi_1,\ \phi_2 \in \Phi_2,\ \operatorname{range}\phi_1 \subset \operatorname{dom}\phi_2 \rangle.$$

Definition 2.10. An étale morphism $\Phi\colon \mathit{\Gamma} \to \mathit{\Delta}$ is an *equivalence* if there is an étale morphism $\Psi\colon \mathit{\Delta} \to \mathit{\Gamma}$ such that $\Psi \circ \Phi = \mathit{\Gamma}^\times$ and $\Phi \circ \Psi = \mathit{\Delta}^\times$. Such a Ψ is unique so that it is denoted by Φ^{-1}. We call Φ^{-1} the *inverse morphism* of Φ. An equivalence from $(\mathit{\Gamma}, T)$ to itself is called *automorphism*.

If Φ_1 and Φ_2 are equivalences, then $\Phi_2 \circ \Phi_1$ is also an equivalence.

Definition 2.11. A psg $(\mathit{\Gamma}, T)$ is *compactly generated* if there is a relatively compact open set T' in T, and a finite subset $\{\gamma_1, \dots, \gamma_r\}$ of $\mathit{\Gamma}$ such that the domains and the ranges are contained in T' and that

1) if we denote by $\mathit{\Gamma}_{T'}$ the restriction of $\mathit{\Gamma}$ to T', then $\mathit{\Gamma}_{T'}$ is generated by $\{\gamma_1, \dots, \gamma_r\}$,
2) for each γ_i, there exists an element $\widetilde{\gamma}_i$ of $\mathit{\Gamma}$ such that $\operatorname{dom}\widetilde{\gamma}_i$ contains the closure of $\operatorname{dom}\gamma_i$, $\widetilde{\gamma}_i|_{\operatorname{dom}\gamma_i} = \gamma_i$ and that $\widetilde{\gamma}_i$ is étale on a neighborhood of $\operatorname{dom}\widetilde{\gamma}_i \setminus \operatorname{dom}\gamma_i$,
3) the inclusion of T' into T induces an equivalence from $\mathit{\Gamma}_{T'}$ to $\mathit{\Gamma}$.

Such a $(\mathit{\Gamma}_{T'}, T')$ is called a *reduction* of $(\mathit{\Gamma}, T)$.

Remark 2.12. If $\mathit{\Gamma}$ is a compactly generated psg on a one-dimensional complex manifold, then $\mathit{\Gamma}$ is étale or ramified. In addition, the last condition in 2) is equivalent to $\operatorname{Sing}\widetilde{\gamma}_i = \operatorname{Sing}\gamma_i$.

For example, if $(\mathit{\Gamma}, T)$ is generated by a holonomy pseudogroup of a foliation of a closed manifold, then $(\mathit{\Gamma}, T)$ is compactly generated. We need to choose a complete transversal in order to define a holonomy pseudogroup. If we change the choice of complete transversals, then we obtain pseudogroups which are equivalent. Another source of compactly generated psg's are rational mappings on $\mathbb{C}P^1$. $(\mathit{\Gamma}, T)$ is also compactly generated if $T = \mathbb{C}P^1$ and $\mathit{\Gamma}$ is generated by a rational semigroup [10] which acts on $\mathbb{C}P^1$. See [3] for details.

Assumption 2.13. We assume that $\mathit{\Gamma}$ is generated by local biholomorphic diffeomorphism of $\mathbb{C}^q$, $q > 1$, or by local biholomorphic diffeomorphisms of $\mathbb{C}$ or ramified coverings, where a holomorphic map, say f, from an open set of $\mathbb{C}$ to $\mathbb{C}$ is said to be ramified covering if there exist biholomorphic diffeomorphisms φ from $\operatorname{dom} f$ to a domain in $\mathbb{C}$ and ψ from $\operatorname{range} f$ to a domain in $\mathbb{C}$ such that $\psi \circ f \circ \varphi^{-1}(z) = z^n$ holds for some positive integer n, where $z \in \operatorname{range}\varphi$.

Note that under our assumption, $\mathit{\Gamma}$ consists of holomorphic open mappings.

DEFINITION 2.14. Let $T' \in \mathscr{T}$.

1) A connected open subset U of T' is a wF-open set (weak 'Fatou'-open set) if the following conditions are satisfied:
   i) If γ_x is the germ of an element of $\mathit{\Gamma}_{T'}$ at x, γ is defined on U as an element of $\mathit{\Gamma}$, where $(\mathit{\Gamma}_{T'}, T')$ is the restriction of $\mathit{\Gamma}$ to T'.
   ii) Let $\mathit{\Gamma}^U$ be the subset of $\mathit{\Gamma}$ which consists of elements of $\mathit{\Gamma}$ obtained as in (a). Then $\mathit{\Gamma}^U$ is a normal family.
2) A connected open subset V of T' is an F-open set ('Fatou'-open set) if $\gamma \in \mathit{\Gamma}'$ and if $\operatorname{dom}\gamma \subset V$, then $\operatorname{range}\gamma$ is a union of wF-open sets.

DEFINITION 2.15. Let $(\mathit{\Gamma}, T)$ be a psg which fulfills Assumption 2.13. If $T' \in \mathscr{T}$, then let $F(\mathit{\Gamma}_{T'})$ be the union of F-open subsets of T'. Let $J(\mathit{\Gamma}_{T'}) = T' \setminus F(\mathit{\Gamma}_{T'})$, and $J_0(\mathit{\Gamma}) = \bigcup_{T' \in \mathscr{T}} J(\mathit{\Gamma}_{T'})$. Let $J(\mathit{\Gamma})$ be the closure of $J_0(\mathit{\Gamma})$ and $F(\mathit{\Gamma}) = T \setminus J(\mathit{\Gamma})$. We call $F(\mathit{\Gamma})$ and $J(\mathit{\Gamma})$ the Fatou set and the Julia set of $(\mathit{\Gamma}, T)$, respectively.

Roughly speaking, $J(\mathit{\Gamma})$ is defined as follows. We regard $(\mathit{\Gamma}_{T'}, T')$ as an approximation of $(\mathit{\Gamma}, T)$, and define $J(\mathit{\Gamma}_{T'})$. Indeed, it can be shown that if $(\mathit{\Gamma}, T)$ is compactly generated, then $J(\mathit{\Gamma}_{T'}) = J(\mathit{\Gamma}) \cap T'$ holds for sufficiently large T'. If $T' \subset T''$, then $J(\mathit{\Gamma}_{T'}) \subset J(\mathit{\Gamma}_{T''}) \cap T'$ so that we take the union. Finally by taking the closure, we will obtain a set which consists of points where some 'complicated dynamics' occur in every neighborhood of that point.

Thus defined Julia sets have the following properties.

Theorem 2.16. *If $\mathit{\Gamma}$ is a psg, then we denote by $J_{\mathrm{psg}}(\mathit{\Gamma})$ its Julia set in the sense of Definition 2.15. Then we have the following.*

1) *If f is a rational mapping on $\mathbb{C}P^1$, then $J(f) = J_{\mathrm{psg}}(\langle f \rangle)$, where $\langle f \rangle$ denotes the pseudosemigroup generated by f. More generally, if $f_1, \dots, f_r$ are rational mappings on $\mathbb{C}P^1$ and if G is the semigroup generated by $f_1, \dots, f_r$, then $J(G) = J_{\mathrm{psg}}(\langle f_1, \dots, f_r \rangle)$, where $\langle f_1, \dots, f_r \rangle$ denotes the pseudosemigroup generated by $f_1, \dots, f_r$ (or by G).*
2) *If f is an entire function, then let $\langle f \rangle$ be the pseudosemigroup generated by f which acts on $\mathbb{C}P^1$, where $\operatorname{dom} f$ is considered to be $\mathbb{C}$. Then, $J(f) \cup \{\infty\} = J_{\mathrm{psg}}(\langle f \rangle)$.*
3) *If G is a finitely generated Kleinian group, then $\Lambda(G) = J_{\mathrm{psg}}(\mathit{\Gamma})$, where $\mathit{\Gamma}$ is the pseudosemigroup generated by G and $\Lambda(G)$ denotes the limit set of G.*

4) *If Γ is the holonomy pseudogroup of a complex codimension-one foliation of a closed manifold with respect to a complete transversal (it suffices to assume that Γ is a compactly generated pseudogroup of local biholomorphic diffeomorphisms on $\mathbb{C}$). If we denote by Γ_{psg} the smallest pseudosemigroup which contains Γ, then $J(\Gamma) = J_{\mathrm{psg}}(\Gamma_{\mathrm{psg}})$, where $J(\Gamma)$ is the Julia set of compactly generated pseudogroup in the sense of* [2].

Theorem 2.16 can be seen as a partial refinement of Sullivan's dictionary [9].

In the 4) of Theorem 2.16 the Julia set in the sense of Ghys, Gomez-Mont and Saludes is also defined [4]. The following is known.

Theorem 2.17. *Let Γ be a compactly generated pseudogroup of local biholomorphic diffeomorphisms on $\mathbb{C}$. If we denote by $J_{\mathrm{GGS}}(\Gamma)$ the Julia set of Γ in the sense of Ghys, Gomez-Mont and Saludes, then $J(\Gamma) \subset J_{\mathrm{GGS}}(\Gamma)$.*

There are examples where the inclusion is strict.

Remark 2.18. If we denote by $F_{\mathrm{GGS}}(\Gamma)$ the Fatou set of Γ in the sense of [4], there is a classification of the connected components of $F_{\mathrm{GGS}}(\Gamma)$. We have also a classification of $F(\Gamma)(\supset F_{\mathrm{GGS}}(\Gamma))$ of the same kind. We refer [2] and [1] for more properties of Fatou-Julia decompositions of compactly generated pg's.

Pseudosemigroups in Theorem 2.16 are compactly generated except the case 3). Other psg's which are not necessarily compactly generated are obtained by studying (transversely) holomorphic foliations of open manifolds, or singular holomorphic foliations. A Fatou–Julia decomposition of these foliations can be introduced by using the decomposition in the sense of Definition 2.15. In [3], some properties of such decompositions are studied.

Some of common properties of the Julia sets and the limit sets can be regarded as properties of Julia sets of compactly generated pseudosemigroups. For example, we have the following.

Lemma 2.19. *Let Γ be a compactly generated pseudosemigroup. If we denote by $F(\Gamma)$ and $J(\Gamma)$ Fatou and Julia sets of Γ, then we have the following.*

1) *$F(\Gamma)$ is forward Γ-invariant, i.e., $\Gamma(F(\Gamma)) = \Gamma$, where $\Gamma(F(\Gamma)) = \{x \in T \mid \exists\, \gamma \in \Gamma,\ \exists\, y \in F(\Gamma)\ \text{s.t.}\ x = \gamma(y)\}$.*
2) *$J(\Gamma)$ is backward Γ-invariant, i.e., $\Gamma^{-1}(J(\Gamma)) = J(\Gamma) = \{x \in T \mid \exists\, \gamma \in \Gamma\ \text{s.t.}\ \gamma(x) \in J(\Gamma)\}$.*

If (Γ, T) is a compactly generated pg, then there is a Hermitian metric on $F(\Gamma)$ invariant under Γ [2]. In this sense, the action of Γ is not quite wild on $F(\Gamma)$. If (Γ, T) is a psg, then invariant metrics need not exist in general. Indeed, if $z \in F(\Gamma)$, $\gamma \in \Gamma$ and $\gamma'_z = 0$, then $(\gamma^* g)_z = 0$ so that there is no Γ-invariant metric on $F(\Gamma)$, where γ'_z denotes the derivative of γ at z. For example, let $T = \mathbb{C}$ and define $f \colon T \to T$ by $f(z) = z^2$. Then, the open unit disc is a connected component of the Fatou set, however, f cannot be an isometry for any metric.

Inspired by the Schwarz lemma on the Poincaré disc, we introduce the notion of semi-invariant metrics as follows.

Definition 2.20. Let g_1 and g_2 be Hermitian metrics on $F(\Gamma)$. If $z \in F(\Gamma)$, then we denote by $(g_1)_z$ the metric on $T_z F(\Gamma)$. Suppose that we have $g_1 = f_1^2 g_0$ and $g_2 = f_2^2 g_0$ on a neighborhood of z, where g_0 denotes the standard Hermitian metric on $\mathbb{C}$. If $f_1(z) \leq f_2(z)$, then we write $(g_1)_z \leq (g_2)_z$. Note that this condition is independent of the choice of charts about z. If $(g_1)_z \leq (g_2)_z$ holds on $F(\Gamma)$, then we write $g_1 \leq g_2$.

Definition 2.21. Let g be a Hermitian metric on $F(\Gamma)$. The metric g is said to be *semi-invariant* if $z \in F(\Gamma)$ and if $\gamma \in \Gamma$ is defined on a neighborhood of z, then $\gamma^* g \leq g$ holds on $\operatorname{dom} \gamma$.

The following is known. See [3] and [2] for details.

Theorem 2.22.
1) *Suppose that (Γ, T) is compactly generated, then the metric g is finite and locally Lipschitz continuous on $F(\Gamma)$.*
2) *If $\Gamma^\times = \Gamma$, then $F(\Gamma)$ admits a Hermitian metric which is locally Lipschitz continuous and Γ-invariant.*
3) *If (Γ, T) is generated by a compactly generated pg, then $F(\Gamma)$ admits a Hermitian metric which is of class C^ω and Γ-semiinvariant.*

Example 2.23 ([3], Example 4.21). We define $\gamma \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = z^2$. Then, $J(\gamma) = \{|z| = 1\}$. If we set

$$f(z) = \begin{cases} 1 & \text{if } |z| \leq \frac{1}{2}, \\ 2^k |z|^{2^k - 1} & \text{if } 2^{-\frac{1}{2^{k-1}}} \leq |z| \leq 2^{-\frac{1}{2^k}}, \\ 2^k |z|^{-2^k - 1} & \text{if } 2^{\frac{1}{2^k}} \leq |z| \leq 2^{\frac{1}{2^{k-1}}}, \\ \frac{1}{|z|^2} & \text{if } |z| \geq 2, \end{cases}$$

then $g = f^2 |dz|^2$ gives a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is locally Lipschitz continuous and semi-invariant under the action of Γ, where $\Gamma = \langle \gamma \rangle$. On the other hand, if we consider the Poincaré metric on the unit disc, then γ is contracting by the Schwarz lemma. Hence the Poincaré metrics on the unit disc and $\mathbb{C}P^1 \setminus \{|z| \leq 1\}$ give rise to a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is of class C^ω and semi-invariant under the action of Γ. On the other hand, there is no Γ-invariant metric on $F(\Gamma)$. Indeed, $0 \in F(\Gamma)$ but $(\gamma^* g)_0 = 0$ for any metric g on $F(\Gamma)$.

Let $\widehat{\Gamma}$ be the psg generated by $\gamma|_{\mathbb{C}P^1 \setminus \{0,\infty\}}$ and its local inverses. Then $F(\widehat{\Gamma}) = \mathbb{C} \setminus (S^1 \cup \{0\})$. An invariant metric on $F(\widehat{\Gamma})$ is given by $|dz|^2 / (|z| \log |z|)^2$ on $\{0 < |z| < 1\}$. We can find on $\{1 < |z|\}$ a metric of the same kind.

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Graduate School of Mathematical Sciences, University of Tokyo
Komaba, Meguro-ku, Tokyo, 153–8914, Japan
E-mail: asuke@ms.u-tokyo.ac.jp