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Hyperbolic Geometry and Homeomorphisms of Surfaces

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1. Introduction

Let L be an arbitrary connected surface, compact or noncompact, with or without boundary and orientable or nonorientable. Let $f : L \to L$ be a homeomorphism. We discuss two topics which are related but perhaps, at first, not obviously so.

The first topic is the Handel-Miller theory of endperiodic maps of surfaces, never published even as an announcement, although it has been used by various authors in the study of foliated 3-manifolds. The second topic concerns the Epstein-Baer theorem that homotopic homeomorphisms of surfaces are isotopic. Both of these topics will be studied via a suitable hyperbolic metric on L (Definition 1.1).

For endperiodic maps, we will sketch the main points of the theory and announce new results. For homotopic homeomorphisms, we will outline a new line of proof of Epstein-Baer using hyperbolic geometry. This involves extending classical results about complete hyperbolic surfaces with finite area to complete hyperbolic surfaces with geodesic boundary and infinite Euler characteristic.

The Handel-Miller theory determines an endperiodic map $h : L \to L$, in the same isotopy class as f, which preserves a pair of transverse geodesic laminations and has, in a certain sense, the "tightest" dynamics in its isotopy class. This has obvious analogies with the Nielsen-Thurston theory of automorphisms of compact surfaces, but there are remarkable differences also. In proving that h is in the isotopy class of f, one is led to the second topic of this talk.

DEFINITION 1.1. A hyperbolic metric on a surface L is "standard" if it is complete, makes ∂L geodesic and admits no isometrically imbedded hyperbolic half planes. A surface equipped with such a metric is called a standard hyperbolic surface. A surface which is homeomorphic to a standard hyperbolic surface will simply be called standard.

This is not a serious restriction topologically. Up to homeomorphism, there are exactly 13 nonstandard surfaces, none of them interesting for Handel-Miller theory.

2. Endperiodic Homeomorphisms

Let $\mathcal{E}(L)$ denote the set of ends of L, a compact, totally disconnected, metrizable space which compactifies L.

Definition 2.1. An end $e \in \mathcal{E}(L)$ is an attracting end if it admits a neighborhood $U_e \subset L$ such that, for a least integer $p_e \geq 0$,

$$U_e \supset f^{p_e}(U_e) \supset \cdots \supset f^{np_e}(U_e) \supset \cdots$$

and $\bigcap_{n=0}^{\infty} f^{np_e}(U_e) = \emptyset$. Repelling ends are defined similarly, using iterates of f^{-1}. The integer p_e is called the period of e.

Definition 2.2. A homeomorphism $f : L \to L$ is endperiodic if $\mathcal{E}(L)$ is finite and each end is either attracting or repelling..

Examples will be pictured in the talk. The definition of "endperiodic" can be extended to surfaces with infinite endset, even a Cantor set of ends, and this has important applications to foliations. But in this generalization, there will only be finitely many attracting and repelling ends, and one passes to the "soul" of L, an f-invariant subsurface with finitely many ends on which all of the interesting dynamics takes place. This effectively reduces us to the case considered by Handel and Miller.

Definition 2.3. An end e is simple if it is isolated and either annular or simply connected. Standard hyperbolic surfaces without simple ends are called "admissible" surfaces.

In the rest of this section we consider admissible surfaces L with finitely many ends and endperiodic homeomorphisms $f : L \to L$.

An attracting end e of period p_e has compact fundamental domains B_i such that $U_e = B_0 \cup B_1 \cup \cdots$ and $f^{p_e}(B_i) = B_{i+1}$, $0 \leq i < \infty$. There is a similar notion of fundamental domain for repelling ends. The intersection $B_i \cap B_{i+1}$ is called a positive juncture. It is a compact 1-manifold. The negative junctures are defined similarly in neighborhoods of repelling ends. Each juncture is the union of finitely many 2-sided, essential closed curves and/or properly imbedded arcs.

The Handel-Miller construction. Start applying powers of f^{-1} to positive junctures. The result is an infinite family of ultimately "distorted" junctures. Generally the distortions get enormous and the distorted junctures wrap around in L in increasingly complex ways. (In the talk, examples will be pictured to illustrate this.) It will be convenient also to call a distorted juncture by the name "juncture". In the homotopy class of each component of a juncture (endpoint preserving homotopy for properly imbedded arcs) there is a unique geodesic. This infinite family of geodesics accumulates exactly on a closed geodesic lamination Λ_- with complete, noncompact leaves (the absence of half planes is critical here). Every leaf of this lamination penetrates arbitrarily deeply into the neighborhoods of repelling ends, but the lamination is uniformly bounded away from the attracting ends. Using the junctures of negative ends, one similarly defines the geodesic lamination Λ_+, transverse to Λ_-, which penetrates arbitrarily deeply into the attracting ends but is uniformly bounded away from the repelling ends. The final step is to define an endperiodic homeomorphism $h : L \to L$ which preserves these laminations and is isotopic to f. The dynamics of h is "tightest possible" in its isotopy class, in the sense that h has the smallest possible invariant set $\mathcal{I}$ and the dynamics of $h|\mathcal{I}$ is Markov.

Definition 2.4. A pseudo-geodesic σ in L is a continuous, imbedded curve, any lift of which to the universal cover $\widetilde{L}$ (viewed as a surface in the Poincaré disk) has well defined endpoints on the circle at infinity.

We have axiomatized the Handel-Miller theory to allow the laminations to be pseudo-geodesic. Again the endperiodic homeomorphism preserving the laminations is isotopic to f. This generalization is quite useful in applications to foliation theory. We have developed an extensive structure theory for the laminations, based on the axioms, which reveals many surprising features.

Here are two new results.

Theorem 2.5. *The pseudo-geodesic laminations of the axiomatized Handel-Miller theory are simultaneously ambient isotopic to the geodesic laminations described above.*

Generally, h is not smooth, even in the case that the laminations are geodesic. One advantage to relaxing the geodesic condition is the following.

Theorem 2.6. *There is a choice of Handel-Miller map h, corresponding to pseudo-geodesic laminations, which is a diffeomorphism except, perhaps, at finitely many p-pronged singularities.*

3. Homotopic Homeomorphisms

The fact that the Handel-Miller endperiodic map h is isotopic to the original endperiodic map f from which it is derived needs to be proven using the ideas in this section.

The surface L is now any standard one. Give it a standard hyperbolic metric. Denote by Δ the open unit disk with the Poincaré metric. Then either $\partial L = \emptyset$ and the universal covering space is $\widetilde{L} = \Delta$, or $\widetilde{L} \subsetneq \Delta$. The completion $\widehat{L}$ is the closure of $\widetilde{L}$ in the closed disk $\overline{\Delta}$. We denote by E the "ideal boundary" of $\widetilde{L}$, namely $E = S^1 \cap \widehat{L}$. The following is well known for complete hyperbolic surfaces of finite area. For standard hyperbolic surfaces, we can find no proof in the literature.

Theorem 3.1. *Any lift $\widetilde{h} : \widetilde{L} \to \widetilde{L}$ of a homeomorphism $h : L \to L$ extends canonically to a homeomorphism $\widehat{h} : \widehat{L} \to \widehat{L}$.*

The following is also known for compact hyperbolic surfaces.

Theorem 3.2. *If $h : L \to L$ is a homeomorphism having a lift such that $\widehat{h}|E = \mathrm{id}_E$, then h is isotopic to id_L.*

In particular, if f, g are two homeomorphisms of L with lifts such that $\widehat{f}|E = \widehat{g}|E$, then f and g are isotopic. In the Handel-Miller theory, one easily verifies this condition for f and h, hence h represents the isotopy class of f.

The following is an easy corollary of Theorem 3.2.

Theorem 3.3 (Epstein-Baer)**.** *If L is a standard surface, then homotopic homeomorphisms of L are isotopic.*

In proving this, Epstein put no restriction on L, but required the homotopy to respect ∂L and, if there were noncompact boundary components, required the homotopy to be proper. The first requirement is only needed for 4 of the 13 nonstandard surfaces. The second requirement is inconvenient in applications and is not needed at all in our proof.

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