Geometry and Foliations 2013
Komaba, Tokyo, Japan

Connectedness of the space of smooth $\mathbb{Z}^2$ actions on $[0, 1]$

Hélène EYNARD-BONTEMPS

(joint work with C. Bonatti)

1. Introduction

Our interest in $\mathbb{Z}^2$-actions on $[0, 1]$ stems from the general study of the set $\mathcal{F}ol(M)$ of all smooth codimension one cooriented foliations on a given closed oriented 3-manifold. By identifying every such foliation with its tangent plane field, one can think of $\mathcal{F}ol(M)$ as a subspace of the space $\mathcal{P}(M)$ of smooth plane fields on M, endowed with the usual C^∞ topology.

The inclusion $\mathcal{F}ol(M) \overset{\iota}{\hookrightarrow} \mathcal{P}(M)$ is strict. In fact, most plane fields are *not* tangent to foliations (or, in other words, are not *integrable*): one can easily see that $\mathcal{F}ol(M)$ is a closed subset of $\mathcal{P}(M)$ *with empty interior.* On the other hand, it has been known since the late sixties that $\mathcal{F}ol(M)$ is nonempty, and even that every plane field can be deformed into a (plane field tangent to a) foliation (see [13]). In other words, the map $\pi_0\mathcal{F}ol(M) \overset{\iota_*}{\to} \pi_0\mathcal{P}(M)$ induced by the inclusion is *surjective.* It is then natural to wonder whether this map is also injective, i.e:

Question 1.1. If two foliations have homotopic tangent plane fields, are they connected by a path of foliations?

Larcanché [7] gave a positive answer for foliations transverse to the fibers of a circle bundle over a closed surface, and for pairs of taut foliations sufficiently close to each other. In [4], we extended this result to any pair of taut foliations homotopic as plane fields. Note that the foliations of the connecting path are not necessarily taut, and recent works by T. Vogel [12] and J. Bowden [3] actually show that the space of taut foliations in a given homotopy class is in general not path-connected. As for non-taut foliations, we reduced Question 1.1 to the particular case of "horizontal" foliations on the thick torus:

Question 1.2. Consider a foliation τ on $\mathbb{T}^2 \times [0, 1]$ tangent to the boundary and transverse to the direction $[0, 1]$. As a plane field, τ is homotopic to the trivial foliation by $\mathbb{T}^2 \times \{.\}$ (rel. to the boundary) since both are transverse to $[0, 1]$. But are they connected by a path of smooth foliations transverse to $[0, 1]$?

This question has a translation in terms of holonomy. A foliation τ as above has a so-called *holonomy representation* which is a homomorphism $\rho(\tau) : \pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2 \to \mathrm{Diff}^{\infty}_{+}[0, 1]$. Let us denote by $\mathcal{R}$ the set of all such homomorphisms. Since such a map is completely determined by the images of the standard generators of $\mathbb{Z}^2$, $\mathcal{R}$ can be thought of as the space of pairs of commuting elements of $\mathrm{Diff}^{\infty}_{+}[0, 1]$, endowed with the usual C^{∞} topology. One can then show that question 1.2 is equivalent to:

QUESTION 1.3. Is the space $\mathcal{R}$ path-connected?

Our aim here is to present the following partial answer obtained in collaboration with C. Bonatti in [2].

2. Main result

Theorem 2.1 (Bonatti, E-B.). *The space $\mathcal{R}$ of smooth orientation preserving $\mathbb{Z}^2$-actions on $[0, 1]$ is connected. More precisely, the path-connected component $\mathcal{C}_{\mathrm{id}}$ of $(\mathrm{id}, \mathrm{id})$ is dense in $\mathcal{R}$.*

Combined with [4], this yields the following:

Theorem 2.2. *For any closed 3-manifold M, the inclusion of $\mathcal{F}ol(M)$ into $\mathcal{P}(M)$ induces a bijection between the connected components of those two spaces.*

The analogous question for *path*-connected components however remains open (for foliations as well as for $\mathbb{Z}^2$ actions). One of our aims here will be to highlight the gap between connectedness and path-connectedness. But let us make a few remarks beforehand.

First of all, why isn't the answer to Question 1.2 obvious? Indeed, the space $\mathrm{Diff}^{\infty}_{+}[0, 1]$ is contractible, so one can easily deform any given pair $(f, g) \in (\mathrm{Diff}^{\infty}_{+}[0, 1])^2$ into any other. But this forgets about the commutativity condition, which is a huge constraint. It is not the only source of trouble though. Regularity is another. Indeed if we consider the same question for homeomorphisms of $[0, 1]$ instead of smooth diffeomorphisms, we can easily see using some kind of "Alexander trick" that the space of orientation preserving C^0-actions of $\mathbb{Z}^2$ on $[0, 1]$ is *contractible*. But such a "brutal" method is bound to fail in the C^{∞} setting. The C^1 case is still different and was solved by A. Navas in [8] using completely different tools.

Outline of proof. As we already mentioned, deforming a given pair of diffeomorphisms (f, g) into another one is not difficult if one forgets about the commutativity condition, but this constraint adds a lot of rigidity to the problem. Namely, if we restrict to the case of diffeomorphisms f, g which are nowhere infinitely tangent to the identity in $(0, 1)$ (such pairs will be referred to as "nondegenerate"), classical results by N. Kopell [6], G. Szekeres [10] and F. Takens [11] imply that f and g belong either to a common infinite cyclic group generated by some C^∞ diffeomorphism h of $[0, 1]$ or to a common C^1 flow (C^∞ on $(0, 1)$ but in general not C^2 on $[0, 1]$). Then, our strategy is as follows.

- In the first case, any isotopy $t \in [0, 1] \mapsto h_t$ from id to h yields a path $t \mapsto (h_t^p, h_t^q)$ of commuting C^∞-diffeomorphisms from $(\mathrm{id}, \mathrm{id})$ to $(f = h^p, g = h^q)$, so (f, g) is actually *in* the path-connected component of (id, id) ($\mathcal{C}_{\mathrm{id}}$) and we have nothing to do.
- In the second case, however, extra-work is called for. If f and g are the time-α and β maps of a C^1 vector field ξ (C^∞ on $(0, 1)$), the idea is to construct a C^∞ vector field $\tilde{\xi}$ whose time-α and β maps φ^α and φ^β are arbitrarily C^∞ close to f and g respectively. The pair $(\varphi^\alpha, \varphi^\beta)$ is then easily connected to $(\mathrm{id}, \mathrm{id})$ by a continuous path of pairs of commuting C^∞ diffeomorphisms $t \in [0, 1] \mapsto (\varphi^{t\alpha}, \varphi^{t\beta})$. One can then conclude that (f, g) belongs to the closure of $\mathcal{C}_{\mathrm{id}}$.

In other words, what we show is that, among "nondegenerate" pairs, those made of iterates of the same smooth diffeomorphism or of elements of the same smooth flow form a *dense* and path-connected subset. Then, deriving the general result from the restricted ("nondegenerate") one we just mentioned is elementary.

The strategy seems very simple. But let us stress that, in the second case above, a random smoothing of the vector field ξ near the boundary won't do in general, for the resulting flow would be no more than C^1 close to that of ξ. So first, one needs to derive some nice estimates on ξ from the knowledge that *some* times of its flow are C^∞. More precisely, if ξ is not C^∞ near a point of the boundary, say 0, according to Takens [11], f and g are necessarily infinitely tangent to the identity at that point. What we show in that case is that, though the derivatives of ξ of order ≥ 2 globally diverge when one approaches 0, arbitrarily close to 0, one can find whole fundamental intervals of f and g where these derivatives are arbitrarily small. These estimates are a generalization of those obtained by F. Sergeraert in [9] for diffeomorphisms without fixed points in $(0, 1)$. Then the rough idea to construct $\tilde{\xi}$ is simply to replace ξ between 0 and such a "nice interval" by something smooth *and* "C^∞-small" (the latter being made possible precisely by the estimates on ξ in the "nice interval"), leaving it unchanged outside this small region. Then the time-α and β

maps of the new vector field $\tilde{\xi}$ basically coincide with f and g away from the boundary and are very close to the identity there, as are f and g !

Note, to conclude, that what our strategy provides in the situation above is an *approximation* of (f, g) by elements of $\mathcal{C}_{\text{id}}$, not a *continuous deformation*, simply because between the "nice intervals" which are essential to our construction lie "nasty" ones. Precisely there lies the gap between connectedness and *path*-connectedness. □

3. (Other) questions

QUESTION 3.1. It has been a longstanding open question whether the space of smooth orientation preserving $\mathbb{Z}^2$ actions on the *circle* is (locally) connected. It follows from Theorem 2.1 that the subspace made of non-free actions (or equivalently, of pairs of commuting diffeomorphisms with rationally dependent rotation numbers) is connected. For commuting diffeomorphisms f and g with rationally independant rotation numbers $\rho(f)$ and $\rho(g)$, on the other hand, here is what is known:

- if $\rho(f)$ and $\rho(g)$ satisfy a joint diophantine condition, B. Fayad and K. Khanin [5] proved that f and g are simultaneously conjugate to the rotations of angle $\rho(f)$ and $\rho(g)$, denoted by $R_{\rho(f)}$ and $R_{\rho(g)}$ respectively, by an element φ of $\text{Diff}^\infty_+\mathbb{S}^1$. The pair (f, g) is thus connected to (id, id) by the path $t \in [0,1] \mapsto (\varphi^{-1} \circ R_{t\rho(f)} \circ \varphi, \varphi^{-1} \circ R_{t\rho(g)} \circ \varphi)$ of smooth commuting diffeomorphisms.
- if $\rho(f)$ and $\rho(g)$ do not satisfy such a condition, (f, g) is not necessarily smoothly conjugate to $(R_{\rho(f)}, R_{\rho(g)})$. Nevertheless, according to M. Benhenda [1], there exists a Baire-dense subset B of $\mathbb{S}^1$ such that, if $\rho(f)$ *or* $\rho(g)$ belongs to B, (f, g) can be *approached* by pairs which *are* smoothly conjugate to $(R_{\rho(f)}, R_{\rho(g)})$. Thus (f, g) belongs to the *closure* of the path-connected component of (id, id).

It is not known, however, whether this last fact holds for *any* pair $(\rho(f), \rho(g)) \in (\mathbb{R} \setminus \mathbb{Q})^2$. A positive answer would imply the connectedness of the whole space of $\mathbb{Z}^2$-actions on the circle.

QUESTION 3.2. How about smooth actions of other surface groups on $[0, 1]$? On the circle? Some progress has recently been made by J. Bowden [3] on this last subject.

References

[1] M. Benhenda, Circle diffomorphisms: quasi-reducibility and commuting diffeomorphisms. preprint HAL

[2] C. Bonatti ; H. Eynard-Bontemps, Connectedness of the space of smooth actions of $\mathbb{Z}^n$ on the interval, preprint, arXiv:1209.1601.

[3] J. Bowden, Contact structures, deformations and taut foliations, preprint: arXiv:1304.3833,2013.

[4] H. Eynard, Sur deux questions connexes de connexité concernant les feuilletages et leurs holonomies, Ph. D. dissertation. http://tel.archives-ouvertes.fr/tel-00436304/fr/.

[5] B. Fayad ; K. Khanin, Smooth linearization of commuting circle diffeomorphisms. *Ann. of Math.* (2) **170** (2009), no. 2, 961–980.

[6] N. Kopell, Commuting diffeomorphisms, *Global Analysis*, *Proc. Sympos. Pure Math.* **XIV**, Amer. Math. Soc. (1968), 165–184.

[7] A. Larcanché, Topologie locale des espaces de feuilletages en surfaces des variétés fermées de dimension 3. *Comment. Math. Helvetici* **82** (2007), 385–411.

[8] A. Navas, Sur les rapprochements par conjugaison en dimension 1 et classe C^1, arXiv:1208.4815.

[9] F. Sergeraert, Feuilletages et difféomorphismes infiniment tangents à l'identité, *Invent. Math.* **39** (1977), 253–275.

[10] G. Szekeres, Regular iteration of real and complex functions. *Acta Math.* **100** (1958), 203–258.

[11] F. Takens, Normal forms for certain singularities of vector fields. *Ann. Inst. Fourier* **23** (1973), 163–195.

[12] T. Vogel, Uniqueness of the contact structure approximating a foliation, preprint: arXiv:1302.5672,2013

[13] J. Wood, Foliations on 3-manifolds. *Ann. of Math.* **89** (1969), 336–358.

Institut de Mathématiques de Jussieu
Université Pierre et Marie Curie
4 place Jussieu, Paris, France
E-mail: heynardb@math.jussieu.fr