Geometry and Foliations 2013
Komaba, Tokyo, Japan

Classification and rigidity of totally periodic pseudo-Anosov flows in graph manifolds

Sergio FENLEY

This is joint work with Thierry Barbot.

Pseudo-Anosov flows are extremely common amongst three manifolds, for example 1) Suspension pseudo-Anosov flows [Th1, Th2, Th3]; 2) Geodesic flows in the unit tangent bundle of negatively curved surfaces [An]; 3) Certain flows transverse to foliations in closed atoroidal manifolds [Mo, Cal1, Cal2, Cal3, Fe]; flows obtained from these by either 4) Dehn surgery on a closed orbit of the pseudo-Anosov flow [Go, Fr]; or 5) Shearing along tori [Ha-Th]; 6) Non transitive Anosov flows [Fr-Wi] and flows with transverse tori [Bo-La].

We consider the following question: how many different pseudo-Anosov flows are there in a manifold up to topological conjugacy? *Topological conjugacy* means that there is a homeomorphism between the manifolds which sends orbits of the first flow to orbits of the second flow. We also consider the notion of *isotopic equivalence*, i.e. a topological conjugacy induced by an isotopy, that is, a homeomorphism isotopic to the identity.

Here we consider only closed, orientable, toroidal manifolds. They have incompressible tori and and also since they support a pseudo-Anosov flow they are irreducible. Therefore the manifolds are Haken manifolds. We recently proved that if M is Seifert fibered, then the flow is up to finite covers topologically conjugate to a geodesic flow in the unit tangent bundle of a closed hyperbolic surface [Ba-Fe1]. We also proved that if the ambient manifold is a solvable three manifold, then the flow is topologically conjugate to a suspension Anosov flow [Ba-Fe1]. We stress that in both cases the results imply that the flow does not have singularities, that is, the type of the manifold strongly restricts the type of pseudo-Anosov that it can admit. This is in contrast with the strong flexibility in the construction of pseudo-Anosov flows – that is because many flows are constructed in atoroidal manifolds or are obtained by flow Dehn surgery on the pseudo-Anosov flow, which changes the topological type of the manifold. Therefore in many constructions one cannot expect the underlying manifold to be toroidal.

Here we consider pseudo-Anosov flows in graph manifolds. A *graph manifold* is an irreducible three manifold which is a union of Seifert fibered pieces. In [Ba-Fe1] we produced a large new class of examples in graph manifolds. These flows are *totally periodic*. This means that each Seifert piece of the torus decomposition of the graph manifold is *periodic*, that is, up to finite powers, a regular fiber is freely homotopic to a closed orbit of the flow. More recently, Russ Waller [Wa] has been studying how common these examples are, that is, the existence question for these type of flows. He showed that these flows are as common as they could be (modulo the necessary conditions).

Here we analyse the question of the classification and rigidity of such flows. To do that we introduce *Birkhoff annuli.* A Birkhoff annulus is an a priori only immersed annulus, so that the boundary is a union of closed orbits of the flow and the interior of the annulus is transverse to the flow. For example consider the geodesic flow of a closed, orientable hyperbolic surface. The ambient manifold is the unit tangent bundle of the surface. Let α be an oriented closed geodesic - a closed orbit of the flow - and consider a homotopy that turns the angle along α by π. The image of the homotopy from α to the same geodesic with opposite orientation is a Birkhoff annulus for the flow in the unit tangent bundle. If α is not embedded then the Birkhoff annulus is not embedded. In general Birkhoff annuli are not embedded, particularly in the boundary.

In [Ba-Fe1] we proved the following basic result about the relationship of a pseudo-Anosov flow and a periodic Seifert piece P: there is a spine Z for P which is a connected union of finitely many elementary Birkhoff annuli. In addition the union of the interiors of the Birkhoff annuli is embedded and also disjoint from the closed orbits in Z. These closed orbits, boundaries of the Birkhoff annuli in Z, are called *vertical periodic orbits.* The set Z is a deformation retract of P, so P is isotopic to a small compact neighborhood $N(Z)$ of Z.

The first theorems (Theorem A and B) are valid for general Seifert fibered pieces in any closed orientable manifold M, not necessarily a graph manifold.

Theorem A ([Ba-Fe2]). *Let Φ be a pseudo-Anosov flow in M^3. If $\{P_i\}$ is the (possibly empty) collection of periodic Seifert pieces of the torus decomposition of M, then the spines Z_i and neighborhoods $N(Z_i)$ can be chosen to be pairwise disjoint.*

The next result shows that the boundary of the pieces can be put in good position with respect to the flow:

Theorem B ([Ba-Fe2]). *Let Φ be a pseudo-Anosov flow and P_i, P_j be periodic Seifert pieces with a common boundary torus T. Then T can be isotoped to a torus transverse to the flow.*

Theorem C ([Ba-Fe2]). *Let Φ be a totally periodic pseudo-Anosov flow with periodic Seifert pieces $\{P_i\}$. Then neighborhoods $\{N(Z_i)\}$ of the spines*

$\{Z_i\}$ can be chosen so that their union is M and they have pairwise disjoint interiors. In addition each boundary component of every $N(Z_i)$ is transverse to the flow. Each $N(Z_i)$ is flow isotopic to an arbitrarily small neighborhood of Z_i.

We stress that for general periodic pieces it is not true that the boundary of $N(Z_i)$ can be isotoped to be transverse to the flow. There are some simple examples as constructed in [Ba-Fe1]. The point here is that we assume that *all* pieces of the JSJ decomposition are periodic Seifert pieces.

Hence, according to Theorem C, totally periodic pseudo-Anosov flow are obtained by glueing along the bondary a collection of small neighborhoods $N(Z_i)$ of the spines. There are several ways to perform this glueing which lead to pseudo-Anosov flows. The main result is that the resulting pseudo-Anosov flows are all topologically conjugate to each other:

Theorem D ([Ba-Fe2])**.** *Let Φ, Ψ be two totally periodic pseudo-Anosov flows on the same orientable graph manifold M. Let P_i be the Seifert pieces of M, and let $Z_i(\Phi)$, $Z_i(\Psi)$ be spines of Φ, Ψ in P_i. Then, Φ and Ψ are topologically conjugate if and only if there is a homeomorphism of M mapping the collection of spines $\{Z_i(\Phi)\}$ onto the collection $\{Z_i(\Psi)\}$ and preserving the orientations of the vertical periodic orbits induced by the flows.*

Finally we show that for any totally periodic pseudo-Anosov flow Φ there is a model pseudo-Anosov flow as constructed in [Ba-Fe1] which has precisely the same data Z_i, $N(Z_i)$ that Φ has. This proves the following:

Main theorem ([Ba-Fe2])**.** *Let Φ be a totally periodic pseudo-Anosov flow in a graph manifold M. Then Φ is topologically equivalent to a model pseudo-Anosov flow.*

Model pseudo-Anosov flows are defined by some combinatorial data (essentially, the data of some fat graphs and Dehn surgery coefficients) and some parameter λ. A nice corollary of these results is that, up to topological conjugation the model flows actually do not depend on the choice of λ, nor on the choice of the selection of the particular glueing map between the model periodic pieces.

References

[An] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1969).

[Ba-Fe1] T. Barbot and S. Fenley, *Pseudo-Anosov flows in toroidal 3-manifolds*, Geometry and Topology, **17** (2013) 1877-1954.

[Ba-Fe2] T. Barbot and S. Fenley, *Classification and rigidity of totally periodic pseudo-Anosov flows in graph manifolds*, Arxiv.math.GT (math.DS) 1211.7327, submitted for publication.

[Bo-La] C. Bonatti and R. Langevin, *Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension*, Erg. Th. Dyn. Sys. **14** (1994) 633-643.

[Cal1] D. Calegari, *The geometry of* $\mathbf{R}$*-covered foliations*, Geometry and Topology **4** (2000) 457-515.

[Cal2] D. Calegari, *Foliations with one sided branching*, Geom. Ded. **96** (2003) 1-53.

[Cal3] D. Calegari, *Promoting essential laminations*, Inven. Math. **166** (2006) 583-643.

[Fe] S. Fenley, *Foliations and the topology of* 3*-manifolds I:* $\mathbf{R}$*-covered foliations and transverse pseudo-Anosov flows*, Comm. Math. Helv. **77** (2002) 415-490.

[Fr-Wi] J. Franks and R. Williams, *Anomalous Anosov flows*, in Global theory of Dyn. Systems, Lecture Notes in Math. **819** Springer (1980).

[Fr] D. Fried, *Transitive Anosov flows and pseudo-Anosov maps*, Topology **22** (1983) 299-303.

[Go] S. Goodman, *Dehn surgery on Anosov flows*, Geometric dynamics, (Rio de Janeiro, 1981), 300-307, Lec.Notes in Math. **1007**, Springer, Berlin, 1983.

[Ha-Th] M. Handel and W. Thurston, *Anosov flows on new three manifolds*, Inv. Math. **59** (1980) 95-103.

[Mo] L. Mosher, *Laminations and flows transverse to finite depth foliations*, manuscript available in the web from http://newark.rutgers.edu:80/~mosher/, Part I: Branched surfaces and dynamics, Part II in preparation.

[Th1] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University Lecture Notes, 1982.

[Th2] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. A.M.S **19** (1988) 417-431.

[Th3] W. Thurston, *Hyperbolic structures on* 3*-manifolds II, Surface groups and* 3*-manifolds that fiber over the circle*, preprint.

[Wa] R. Waller, *Surfaces which are fat graphs*, preliminary version.

Florida State University
Tallahasse, FL 32312
USA