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Towards the classification theorem for one-dimensional pseudogroups

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My talk will be devoted to a joint project with B. Deroin, D. Filimonov and A. Navas, that has as its ultimate goal to understand and classify real one-dimensional finitely generated pseudo-group actions. This naturally includes group actions on the circle and real codimension one foliations of compact manifolds.

Our study was motivated by following questions, going back to 1980's, that were asked by D. Sullivan, E. Ghys, and G. Hector; we're stating them both in the group actions and foliations setting:

Question 0.1 (Ghys, Sullivan). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle minimally. Is this action necessarily ergodic with respect to the Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold, which is minimal. Is it necessarily ergodic with respect to the Lebesgue measure?

Question 0.2 (Ghys, Sullivan). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Is K necessarily of zero Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold, having an exceptional minimal set $\mathcal{K}$. Is it necessarily of zero Lebesgue measure?

Question 0.3 (Hector). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Does the action of G on the connected components of $S^1 \setminus K$ necessarily have but a finite number of orbits?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold M, having an exceptional minimal set $\mathcal{K}$. Does the complement $M \setminus \mathcal{K}$ necessarily have at most finite number of connected components?

Our results partially answer these questions; what is even more important, some general paradigm seems to turn up. Namely, it *seems* that (up

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Talk on a joint project with Bertrand Deroin, Dmitry Filimonov, Andrés Navas

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to some modifications[1]) the following general paradigm takes place:

Paradigm. *For a finitely-generated pseudogroup of transformations of a real line, the following dichotomy holds:*

- *Either it has local flows in its local closure,*
- *Or it admits a Markov partition (of the minimal set).*

This is closely related to what was done and suggested as a generic behavior for the case of an exceptional minimal set by Cantwell and Conlon in [1, 2]. Though, for the case of an exceptional minimal set we expect that Markov partition always exists, as the sense in which we understand the Markov partition is slightly weaker than the one of Cantwell–Conlon (and this covers also the type of behavior mentioned in [2, §7] that did not fit in their definition).

A road towards this paradigm lies through the local discreteness, which makes the dichotomy between the two cases above. Namely, *if* a (pseudo)group is not locally discrete and its action does not preserve a measure, the arguments of Scherbakov-Nakai-Loray-Rebelo ([5, 12, 11, 14]) imply that it contains local flows in its local closure. Roughly speaking, due to the absence of a preserved measure there is a map with a hyperbolic fixed point; expanding the sequence of maps (C^1-)convergent to the identity with help of this map, one finds local flows.

The case to consider is then the one of locally discrete groups, and here the *non-expandable points* come into play. Namely, there Sullivan's exponential expansion strategy allows to expand arbitrarily small neighborhoods of points of the minimal set with a uniform control on the distortion, provided that for any point of the minimal set there is a map that expands linearly at this point. This allow to obtain, under this assumption, positive answers to the Questions 0.1–0.3, and it is quite likely to provide a Markov partition (with the same mechanism as the one used in [6]: under the expansion the maps stay uniformly close to the identity, and thus there should be a repetition between the expanded images).

The non-expandable points that we mentioned earlier are the obstacles to the application of Sullivan's expansion strategy:

Definition 0.4. A point x of a minimal set is *non-expandable* for the action of a (pseudo)group $\mathcal{G}$, if for any $g \in \mathcal{G}$ (defined in x) one has $|g'(x)| \leq 1$.

Note, that their presence in a minimal set immediately implies the local discreteness of the group: otherwise, local vector flows would allow minimal-

[1] For instance, for the local flows one shouldn't consider glueing or too large domains of definition: otherwise, for the standard Thomson group action, generated by the doubling map, one would both have a Markov partition and a flow.

ity with a uniform control on the derivative, and bringing a non-expandable point close to a hyperbolic repelling one would imply a contradiction. Thus, in view of the paradigm above, we should try showing that in this case there exists a Markov partition.

Though the non-expandable points are an obstacle to the "fast" expansion procedure, an additional assumption allows to handle this difficulty:

DEFINITION 0.5 ([3]). A minimal action of a finitely generated (pseudo)group $\mathcal{G}$ has property $(\star)$, if any non-expandable fixed point is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.

An action of a finitely generated (pseudo)group $\mathcal{G}$ with a Cantor minimal set K has property $(\Lambda\star)$, if any non-expandable fixed point $x \in K$ is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.

When this assumption is satisfied, one can modify the Sullivan's exponential expansion strategy by a "slow" expansion near the non-expandable points, by iterating the $g_\pm$ (or their inverses) till the point leaves the neighborhood of a non-expandable point. Such a modification have allowed us in [3] to obtain *under this assumption* the positive answers to Questions 0.1–0.3.

This also allows to describe the structure of a (pseudo)group: it turns out (see [6, 7]) that if this assumption is satisfied, and there actually is at least one non-expandable point, then the dynamics indeed admits some kind of Markov partition. Also, the (pseudo)group is in a sense Thomson-like: for the piecewise-nonstrictly expanding map R, associated to this partition, the maps from the (pseudo)group locally are composition of its iterations, the branches of its inverse, and an intermediate map chosen from a finite set.

What is left for establishing the paradigm and for answering Questions 0.1–0.3 is thus to prove that the property $(\star)$ (or $(\Lambda\star)$) always holds. And two our recent works make an advancement towards it:

Finally, for some cases, the property $(\star)$ (or $(\Lambda\star)$) can be shown to hold:

Theorem 0.6 ([4]). *Let G be a (virtually) free finitely generated subgroup of the group of analytic circle diffeomorphisms, such that the action G does not have finite orbits. Then, G satisfies property $(\star)$ or $(\Lambda\star)$ (depending on whether the action is minimal or possesses an exceptional minimal set).*

REMARK 0.7. Recall, that due to a result by Ghys [10], a finitely generated group of analytic circle diffeomorphisms, acting with a Cantor minimal set, is *always* virtually free. Hence, Theorem 0.6 implies positive answers for Questions 0.2 and 0.3 for the case of an analytic group action on the circle.

Theorem 0.8 ([8]). *Let G be a finitely generated subgroup of the group of analytic circle diffeomorphisms, acting minimally, that has one end, is finitely presented, and in which one cannot find elements of arbitrarily large finite order. Then, G satisfies property ($\star$).*

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