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Foliations of $\mathbb{S}^3$ by cyclides

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1. Introduction

Throughout last 2-3 decades, there was great interest in extrinsic geometry of foliated Riemannian manifolds (see [As], [B-L-R] and [Ze]).

One approach is to build examples of foliations with reasonably simple singularities with leaves admitting some very restrictive geometric condition. After considering foliations of $\mathbb{S}^3$ by totally geodesic of totally umbilical leaves with isolated singularities, totally geodesic foliations of $\mathbb{H}^2$ or $\mathbb{H}^3$, [La-Si] provide families of foliations of $\mathbb{S}^3$ by Dupin cyclides with only one smooth curve of singularities. Quadrics and other families of cyclides like Darboux cyclides provide other examples. In all cases the results are obtained considering an auxiliary space associated to the geometry imposed to our leaves, the space of spheres, of lines, of circles for the examples mentioned above.

Another motivation for our construction is the use of cyclides in computer graphics, see for example [Po-Li-Sko].

The results mentioned in this conference come from a joint work with Jean-Claude Sifre.

2. The spaces of lines, spheres, and circles

2.1. The space of lines

The set of affine lines of $\mathbb{R}^3$ is a vector bundle of base $\mathbb{P}^2$ and fiber $\mathbb{R}^2$ of dimension 6. The projective space $\mathbb{RP}^3$ completes $\mathbb{R}^3$. The set of projective lines of $\mathbb{RP}^3$ is isomorphic to the Grassmann manifold $G(4,2)$ of planes of $\mathbb{R}^4$.

Let us first show how, using Plücker coordinates, $G(4,2)$ can be seen as a quadric $\Pi \subset \mathbb{RP}^5$.

The condition that a vector U of $\bigwedge^2(\mathbb{R}^4)$ is pure, that is of the form $u \wedge v$, $u \in \mathbb{R}^4$, $v \in \mathbb{R}^4$ writes $U \wedge U = 0$; this provides a quadratic form, called the *Plücker form* which defines the *Plücker cone.*

The incidence relation of two lines corresponding to the 2-vectors U and V obtained checking that the corresponding 2-planes of $\mathbb{R}^4$ generate a subspace of dimension at most 3; it writes $U \wedge V = 0$.

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A pencil of lines ℓ is the projective image of a totally isotropic plane of $\bigwedge^2(\mathbb{R}^4)$ for the Plücker form; it is of index $(3,3)$. We call the corresponding projective line a *projective light-ray*.

Geometrically, a pencil of lines of $\mathbb{P}^3$ correspond to a *contact condition*, that is a pair (m, P), $m \in P \subset \mathbb{P}^3$.

2.2. The space of spheres

It will be convenient for us to realize both our ambient space $\mathbb{S}^3$ and the set of oriented spheres as subsets of the Lorentz space $\mathbb{R}^5_1$, that is $\mathbb{R}^5$ endowed with the Lorentz quadratic form $\mathcal{L}(x) = \mathcal{L}(x_0, x_1, x_2, x_3, x_4) = -x_0^2 + \sum_{i=1}^4 x_i^2$.

The *light-cone* $\mathcal{L}i$ is the set $\mathcal{L}(x) = 0$. Its generatrices are called *light-rays*. We also call affine lines parallel to a generatrix of the light-cone light-rays.

The light-cone separates vectors of $\mathbb{R}^5 \setminus \mathcal{L}i$ in two types: space-like vectors, such that $\mathcal{L}(v) > 0$ and time-like vectors, such that $\mathcal{L}(v) < 0$. A plane will be called space-like if it contains only space-like (non-zero) vectors. It is called time-like if it contains non zero time-like vectors (then it contains vectors of the three types). It is called light-like is it contains non-zero light-like vectors but no time-like vector.

The space of oriented 2-dimensional spheres in $\mathbb{S}^3$ may be parameterized by the *de Sitter quadric* $\Lambda^4 \subset \mathbb{R}^5_1$ defined as the set of points $\sigma = (x_0, x_1, x_2, x_3, x_4)$ such that $\mathcal{L}(\sigma) = 1$, in the following way. The hyperplane $\sigma^\perp$ orthogonal to σ (for the Lorentz quadratic form $\mathcal{L}$) cuts the affine hyperplane $H_0 = \{x_0 = 1\}$ along a 3-dimensional oriented affine hyperplane, which cuts the unit sphere $\mathbb{S}^3 \subset H_0$ along a 2-dimensional sphere Σ. Let us orient the sphere Σ as boundary of the ball $B_\sigma = \mathbb{S}^3 \cap \{\mathcal{L}(x, \sigma) \geq 0\}$.

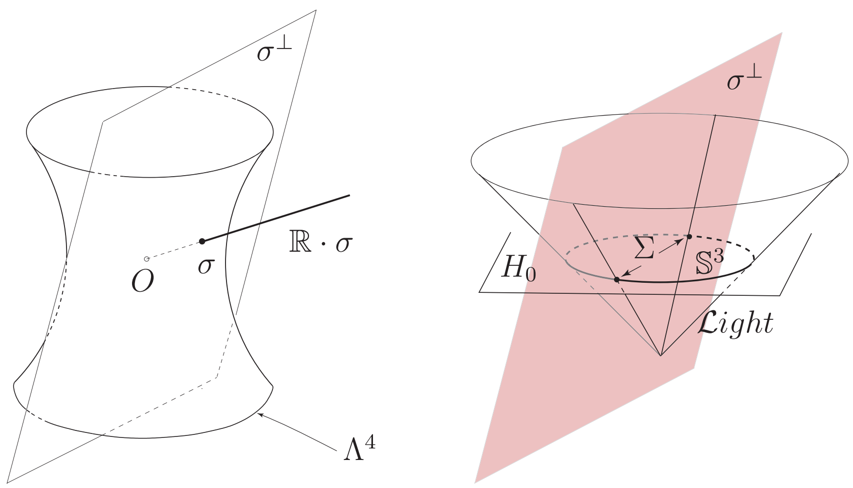

Figure 1: The correspondence between points of Λ^4 and spheres in $\mathbb{S}^3$.

This correspondence between points σ of Λ^4 and oriented spheres $\Sigma \subset \mathbb{S}^3 \subset H_0$ is bijective.

Geometric properties of spheres have a counterpart in Λ^4. For example, two oriented spheres Σ and Σ' in $\mathbb{S}^3$ are positively (i.e. respecting the orientation) tangent if and only if the corresponding points σ and σ' in Λ^4 verify: $\mathcal{L}(\sigma, \sigma') = 1$. In that case, the points σ and σ' are joined by a segment of light-ray contained in Λ^4. In fact the oriented spheres tangent to Σ correspond to the points of the 3-dimensional cone $T_\sigma\Lambda^4 \cap \Lambda^4$ which is a union of (affine) light-rays.

The tangent space $T_\sigma\Lambda^4$ is parallel to the hyperplane $(\mathbb{R} \cdot \sigma)^\perp$. It is therefore of index $(3, 1)$. This means it contains space-like, time like and light-like vectors.

A *contact element* (or simply a *contact*) in $\mathbb{S}^3$ is a pair (m, h), where $m \in \mathbb{S}^3$ and h is a vector plane $h \subset T_m\mathbb{S}^3$. The set of contact conditions is of dimension 5. To each contact element (m, h) corresponds a pencil of spheres tangent to h at m. Orienting $h \subset T_m\mathbb{S}^3$ allows to orient the spheres of the pencil, and distinguishes one of the light-rays of Λ^4 corresponding to spheres of the pencil.

Reciprocally, each light-ray contained in Λ^4 defines a *contact element* in $\mathbb{S}^3$. Precisely, the intersection of the direction of the light-ray ℓ with H_0 is a point m_ℓ of $\mathbb{S}^3$ and the spheres Σ associated to the points $\sigma \in \ell$ are the spheres having a common oriented contact $h \subset m_\ell$ at the point m_ℓ. We can now observe that the quadric Λ^4 is ruled by a 5-dimensional family of (affine) light-rays.

Pencils of spheres can be of the types: pencils of tangent spheres, pencils of spheres with a base circle and pencils of spheres with limit points. The corresponding points of Λ^4 are respectively two parallel light-rays, the intersection of Λ^4 with a space-like vectorial plane and the intersection of Λ^4 with a time-like plane.

2.3. The space of circles

A circle $\Gamma \subset \mathbb{S}^3$ is the axis of a pencil of spheres. This pencil corresponds to the points of intersection of the quadric $\Lambda^4 \subset \mathbb{R}^5_1$ and a space-like vectorial plane. Therefore the space of circles $\mathcal{C}$ can be seen as a subset of the set of lines of the cone $\mathcal{P} \subset \bigwedge^2(\mathbb{R}^5)$ given by the Plücker relations defining pure 2-vectors.

The wedge product defines a bilinear form $\mathcal{Q}_\mathcal{C} : \bigwedge^2(\mathbb{R}^5) \times \bigwedge^2(\mathbb{R}^5) \to \bigwedge^4(\mathbb{R}^5)$. The condition $\mathcal{Q}_\mathcal{C}(U, U) = 0$ gives 5 quadratic equations. They are not independent. One can prove that the equality $\mathcal{Q}_\mathcal{C}(U, U) = 0$ defines a 7-dimensional cone $\mathcal{P}$. We will soon see that the set of lines corresponding to circles is open in $\mathbb{P}(\mathcal{P})$. We could have checked directly that the set of oriented circles $\mathcal{C}$ is a 6-dimensional space.

Let now U_{Γ_1} and U_{Γ_2} be two pure vectors corresponding to the two circles Γ_1 and Γ_2. The condition $0 = \mathcal{Q}_{\mathcal{C}}(U_{\Gamma_1}, U_{\Gamma_2} = U_{\Gamma_1} \wedge U_{\Gamma_2}$ is equivalent to $dim(p_{\Gamma_1} + p_{\Gamma_2}) \leq 3$, that is to say $\exists \sigma \in p_{\gamma_1} \cap p_{\gamma_2} \cap \Lambda^4$. In other terms, the two circles Γ_1 and Γ_2 belong to the same sphere Σ if and only if corresponding 2-vector U_{Γ_1} and U_{Γ_2} satisfy $U_{\gamma_1} \wedge U_{Gamma_2} = 0$.

The condition is satisfied in particular when the two circles intersect at two distinct points or are tangent.

2.3.1. Plücker and Lorentz quadratic forms

It is natural to consider on $\bigwedge^2(\mathbb{R}^5)$ a quadratic form coming from the Lorentz quadratic form $\mathcal{L}$ on $\mathbb{R}^5$ defined by $\mathcal{L}(x) = -x_0^2 + x_1^2 + \cdots + x_4^2$. Consider on $\mathbb{R}^5$ the basis $e_0, e_1, \cdots e_4$; the 10 2-vectors $e_i \wedge e_j$, $i < j$ form a basis of $\bigwedge^2(\mathbb{R}^5)$. the quadratic form $\mathbb{L}$ on $\bigwedge^2(\mathbb{R}^5)$ is defined by $\mathbb{L}(e_1 \wedge e_j) = +1$ if $i \geq 1$, $\mathbb{L}(e_1 \wedge e_j) = -1$ if $i = 0$. The signature of $\mathbb{L}$ is therefore $(6, 4)$.

The light-cone of $\mathbb{L}$ contains the lines generated by wedge of vectors of $\mathbb{R}^5$ contained in a 2-plane tangent to the light-cone of $\mathcal{L}$.

One may visualize the set of "true" oriented circles of $\mathbb{S}^3$ as the intersection $\mathcal{C}$ of the Plücker cone of "pure" 2-vectors, defined by the equations $u \wedge u = 0$, and the quadric of equation $\mathbb{L}(x) = 1$.

Using on $\mathcal{C}$ the pseudo-metric induced from $\mathbb{L}$, we get a pseudo metric of signature $(4, 2)$. We admit that this pseudo-metric does not depend on the choice of the orthonormal basis (for $\mathcal{L}$) of $\mathbb{R}^5$. A way to visualize orthogonal directions for $\mathbb{L}$ in $T_\gamma\mathcal{C}$ is explained in [La-O'H].

3. The d'Alembert property

Cyclides are surfaces of $\mathbb{S}^3$ which are, at least in two different ways, union of one-parameter families of circles. We will here accept lines are particular circles. Many interesting examples are proposed in [Po-Li-Sko].

Definition 3.0.1. Two one-parameter families of circles $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the d'Alembert property if any two circles $\Gamma_1 \in \mathcal{C}_1$ and $\Gamma_2 \in \mathcal{C}_2$ are contained in a sphere $\Sigma_{1,2}$. We will call a cyclide, union of the circles of two families satisfying the d'Alembert property a d'Alembert cyclide.

Remark. – d'Alembert observed that ellipsoids admit two families of circles which are the intersection of the ellipsoid with planes parallel to the tangent planes at the umbilics (see [d'A] and Figure 2). Two circles, one in each family are always contained in a common sphere (this is clear, from topological reasons, when the two circles intersect).

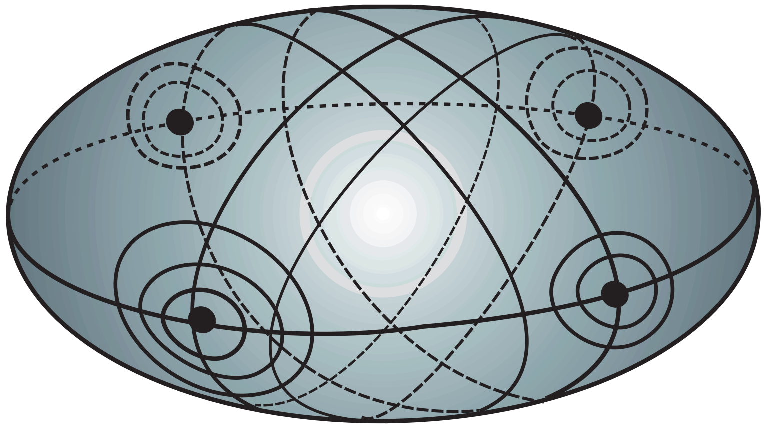

Figure 2: Two families of circles on an ellipsoid.

– Whereas a quadric can be described as the zero-set of second order polynomial in Cartesian coordinates (x_1, x_2, x_3), a large family of cyclide, called the *Darboux cyclides*, is given by the zero-set of a second order polynomial in (x_1, x_2, x_3, r^2), where $r^2 = x_1^2 + x_2^2 + x_3^2$. Thus they are quartic surfaces in Cartesian coordinates, with an equation of the form:

$$Ar^4 + 2r^2 \sum_{i=1}^{3} B_i x_i + \sum_{i,j=1}^{3} Q_{ij} x_i x_j + 2 \sum_{i=1}^{3} C_i x_i + a$$

where Q is a 3×3 matrix, B_i are a 3-dimensional vectors, and A and a are constants [Ta].

They are d'Alembert cyclides, and have been classified by Takeushi [Ta]. We hope to know soon wether all d'Alembert cyclides are Darboux or not.

Proposition 3.0.2. *The points of Λ^4 corresponding to spheres which contain a pair of circles, one in each family, of a d'Alembert cyclide, are contained in a 4 dimensional subspace of $\mathbb{R}^5_1$.*

Proof. Let us chose two circles γ_1, γ_2 of the first family, they are the axis of two pencils of spheres P_{γ_1} and P_{γ_2}. The points corresponding to the spheres of these pencils are intersection of Λ^4 with the planes p_1 and p_2. A circle τ of the second family is the axis of the pencil P_τ. The definition of a d'Alembert cyclide implies that a sphere Σ_1 of $P_{\gamma_1} \cap P_\tau$ contains γ_1 and τ and that a sphere Σ_2 of $P_{\gamma_2} \cap P_\tau$ contains γ_2 and τ. This implies that τ is the pencil generated by Σ_1 and Σ_2. The spheres of this pencil correspond to points of Λ^4 contained in the 4-dimensional subspace $p_1 \oplus p_2 \subset \mathbb{R}^5_1$. It is now enough to use the d'Alembert condition satisfied by a circle of the first family and to given circles of the second family to obtain a proof of the proposition. $\square$

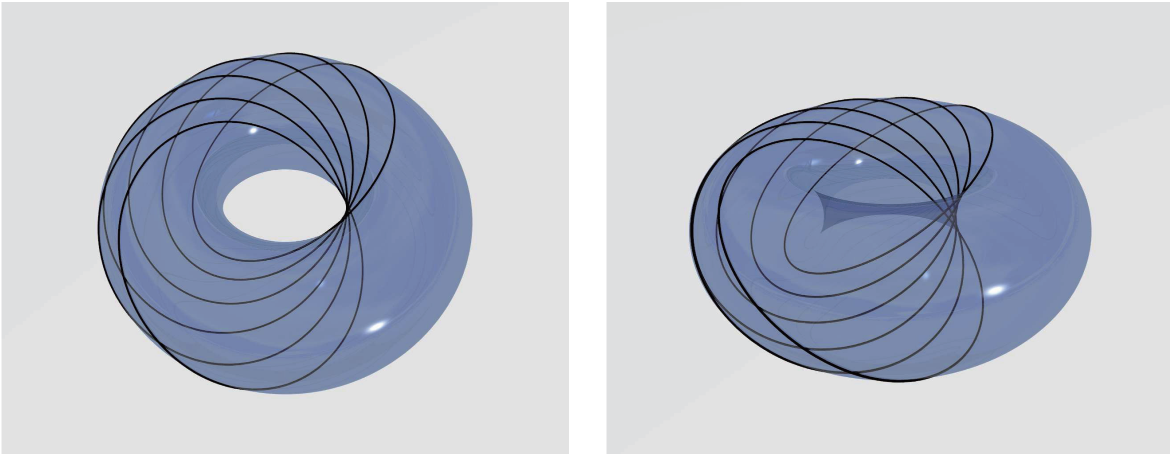

Figure 3: Villarceau circles on a torus

Remark. A regular Dupin cyclide, that is an embedded torus which is in two different ways the envelope of a one-parameter family of spheres is also a d'Alembert cyclide, as the two families of Villarceau circles satisfy the d'Alembert condition. The d'Alembert property reflects on the two curves of the space of circles corresponding to the circles of the two families.

In Proposition 3.0.3 the notions of time-like, space-like and light-like refer to the Lorentz quadratic form $\mathcal{L}(x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^4$.

Proposition 3.0.3. *To each 4-dimensional subspace $\mathcal{H} \subset \mathbb{R}^5_1$ corresponds a 9-dimensional family of d'Alembert cyclides $\mathcal{A}_{\mathcal{H}}$.*

1) *If $\mathcal{H}$ is space-like, there exist a metric of $\mathbb{S}^3$ of constant curvature* 1 *such that all the circles of the two families are geodesics.*
2) *If $\mathcal{H}$ is light-like, that is tangent to the light-cone along a light-ray $\mathbb{R} \cdot m$, then, choosing m as the point at infinity, the cyclide becomes a ruled quadric of $\mathbb{R}^3 \simeq \mathbb{S}^3 \setminus m$.*
3) *If $\mathcal{H}$ is time-like, then all the circles of the cyclide are orthogonal to the sphere Σ corresponding to the two points of $\mathcal{H}^\perp \cap \Lambda^4$.*

From now on, we will use the quadratic form defined on $\bigwedge^2(\mathcal{H})$ by $Plu(U,U) = U \wedge U$. It is of index $(3,3)$. The totally isotropic subspaces of $\bigwedge^2(\mathcal{H})$ will be called like-like subspaces. It is convenient , instead of dealing with planes, 3-dimensional subspaces and the Plücker cone of $\mathbb{R}^6 \simeq \bigwedge^2(\mathcal{H})$ to work in the projective space $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2(\mathcal{H}))$. The Plucker quadric π is the image of the Plücker cone of equation (only one in $\bigwedge^2(\mathcal{H})$) $Plu(U,U) = 0$. A projective light-ray is the image of a totally isotropic plane and two orthogonal 3-dimensional subspaces provide two conjugate projective planes.

Theorem 3.0.4. *The two families of circles of a d'Alembert cyclide form two conics, intersection of the Plücker quadric $\pi\mathbb{P}^5$ with conjugate projective planes.*

The proof is quite similar to the analogous result obtained in [La-Si-Dru-Gar-Pa] for Dupin cyclides.

4. Cyclides, contact conditions and foliations

In [La-Si-Dru-Gar-Pa] the authors studied the existence of Dupin cyclides satisfying three contact conditions, that is tangent to three planes at three points. The solutions, when they exist, form a foliation of $\mathbb{S}^3$ with a singular locus which is a curve where all the solutions are tangent (see [La-Si].

Propositions 3.0.2 and 3.0.3 let us hope for a similar result for each family of d'Alembert cyclide.

The proofs will use a dynamical construction using three projective light-rays of $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2(\mathcal{H}))$ corresponding to the three d'Alembert pairs, two circles contained in a common sphere. When cyclides containing the three d'Alembert pairs exist, they are tangent along a curve and form a foliation of $\mathbb{S}^3$ in case 1) of Proposition 3.0.3, and other wise a foliation of a simple domain of ${}_s s^3$ that can be used as a building block.

Remark. Algebraic geometers would give a proof of theorem 3.0.4 using linear families. In particular, the three contact problem in case 2) of Proposition 3.0.3 can be reduced to Brianchon theorem for conics.

5. Examples of foliations by d'Alembert cyclides and tangent d'Alembert cyclides

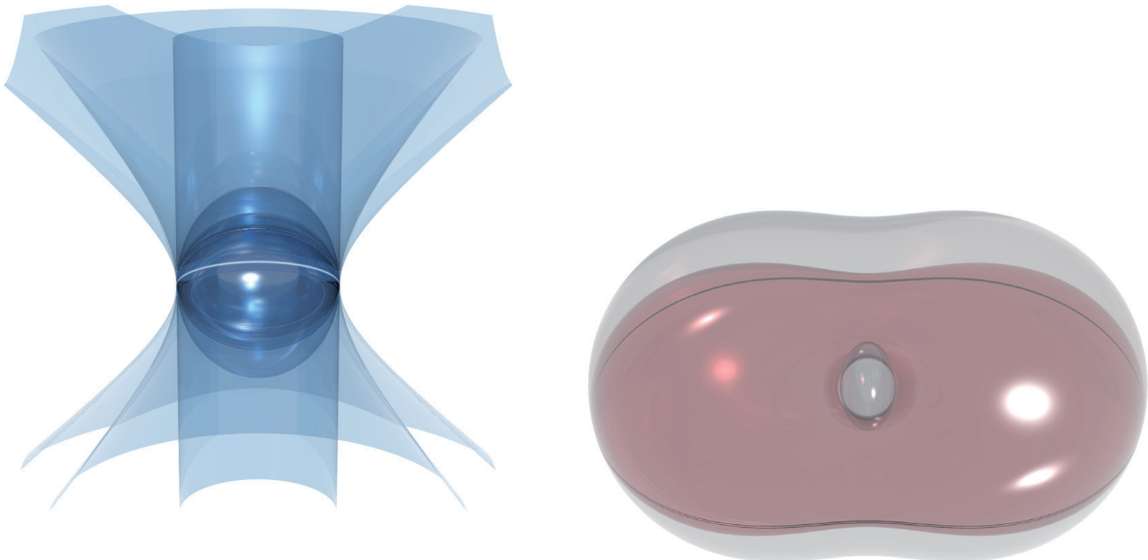

Figure 4: Foliation of $\mathbb{R}^3$ by quadrics and Darboux cyclides tangent along a curve

References

[d'A] J. d'Alembert. Opuscules mathémathiques ou Mémoires sur différens sujets de géométrie, de méchanique, d'optique, d'astronomie, Tome VII, (1761), p. 163.

[As] D. Asimov. *Average Gaussian curvature of leaves of foliations,* Bulletin of the American Math. Soc. **84** (1),(1978) pp. 131–133.

[B-L-R] F. Brito, R. Langevin and H. Rosenberg, *Intégrales de courbure sur des variétés feuilletées*, J. Diff. Geom. **16** (1981), p. 19–50.

[La-O'H] R. Langevin and J.O'Hara. *Extrinsic Conformal Geometry*, manuscript of a book.

[La-Si] R. Langevin and J-C Sifre. *Foliations of $\mathbb{S}^3$ by Dupin cyclides* Accepted for publication in Foliation 2012 (Lodz, Poland).

[La-Si-Dru-Gar-Pa] R. Langevin, J-C. Sifre, L. Druoton IMB, L. Garnier, and M. Paluszny. *Gluing Dupin cyclides along circles, finding a cyclide given three contact conditions*, Preprint IMB, Dijon (2012).

[LW] R. Langevin and P. Walczak. *Conformal geometry of foliations*, Geom. Dedicata **132** (2008), p. 135–178.

[Po-Li-Sko] H. Pottmann, Ling Shi and M. Skopenkov. *Darboux Cyclides and Webs from Circles*, Archiv, June 7^{th} 2011.

[Ta] N. Takeuchi, *Cyclides*, Hokaido Math Journal, Vol 29 (2000), p 119-148.

[Ze] A. Zeghib, *Sur les feuilletages géodésiques continus des variétés hyperboliques*, Invent. Math. 114 (1993), pp. 193–206.

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