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# Codimension one foliations calibrated by non-degenerate closed 2-forms

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## 1. Introduction

In this talk we will be presenting some fundamental results about a class of codimension one foliations which generalises 3-dimensional taut foliations.

Definition 1.1. A codimension one foliation $\mathcal{F}$ of $M^{2n+1}$ is said to be 2-calibrated, if there exists a closed 2-form $\omega$ such that the restriction of $\omega^n$ to the leaves of $\mathcal{F}$ is no-where vanishing.

A 2-calibrated foliation $(M, \mathcal{F}, \omega)$ ($M$ always closed) is an object which essentially belongs to symplectic geometry, as the following fact illustrates: the flow along the kernel of $\omega$ induces on each small open subset of each leaf a Poincaré return map which is a symplectomorphism.

### 1.1. Examples

There are three elementary families of 2-calibrated foliations: products, cosymplectic foliations and symplectic bundle foliations.

A **product** is the result of crossing a 2-calibrated foliation, typically a 3-dimensional taut foliation, with a (non-trivial) symplectic manifold, and putting the product foliation and the obvious closed 2-form.

A **cosymplectic foliation** is a triple $(M, \alpha, \omega)$, where $\alpha$ is a no-where vanishing closed 1-form and $(M, \ker\alpha, \omega)$ is a 2-calibrated foliation. If $\alpha$ has rational periods then the cosymplectic foliation is a symplectic mapping torus, i.e., a mapping torus with fibre a symplectic manifold and return map a symplectomorphisms.

A bundle foliation with fibre $S^1$ is by definition an $S^1$-fibre bundle $\pi\colon M \to X$ endowed with a codimension one foliation $\mathcal{F}$ transverse to the fibres. If the base space admits a symplectic form $\sigma$, then $(M, \mathcal{F}, \pi^*\sigma)$ is a 2-calibrated foliation which we refer to as a **symplectic bundle foliation**.

## 2. Surgeries

It is possible to build new examples out of old ones: via a surgery with generalises the **normal connected sum** of symplectic manifolds [2], one can construct 2-calibrated foliations which belong to none of the three elementary classes [4].

There exist a second surgery in which a neighbourhood of a Lagrangian sphere is separated into 2 copies, and glued back by a generalised Dehn twist (a symplectic generalisation to any dimension of a 2-dimensional Dehn twist around a closed curve). **Generalised Dehn surgery** has an alternative presentation, in which the original and the resulting 2-calibrated foliations are the boundary of certain elementary symplectic cobordism [4]; this is related to the fact that a Lagrangian sphere in a leaf of a 2-calibrated foliation determined a canonical framing, and therefore an elementary cobordism.

## 3. Submanifolds and transverse geometry

Up to date, there are no differential geometric conditions on a foliation which guarantee the existence of submanifolds everywhere transverse to the leaves.

A **2-calibrated submanifold** is an embedded submanifold $j\colon W \hookrightarrow M$ everywhere transverse to $\mathcal{F}$ and intersecting each leaf in a symplectic submanifold. This means that $W$ inherits a 2-calibrated foliation $(\mathcal{F}_W, \omega_W)$.

Methods of symplectic geometry developed by Donaldson [1] can be used to prove the following essential property:

**Theorem 3.1.** *[3] A 2-calibrated foliation* $(M^{2n+1}, \mathcal{F}, \omega)$ *has 2-calibrated* submanifolds *of any even codimension. In particular* $(M^{2n+1}, \mathcal{F}, \omega)$ *(*$2n+1 \geq 5$*) contains* $W^3$ *a 3-dimensional manifold inheriting a taut foliation.*

### 3.1. Transverse geometry

Following Haefliger's viewpoint, the transverse geometry of a foliation $(M, \mathcal{F})$ is captured by the group-like structures in which the holonomy parallel transport is encoded. These are either the holonomy pseudogroup or the holonomy groupoid.

Here is our main result formulated in the framework of holonomy groupoids.

**Theorem 3.2.** *[5] Let* $(M, \mathcal{F}, \omega)$ *be a 2-calibrated foliation. There exist* $W \hookrightarrow M$ *3-dimensional submanifold* $W \pitchfork \mathcal{F}$*, which inherits a taut foliation* $\mathcal{F}_W$ *from* $\mathcal{F}$ *with the following property: the map induced by the inclusion* *between holonomy groupoids*

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F}). \tag{3.3}$$

*is an essential equivalence.*

The interpretation of theorem 3.2 is as follows: the taut foliation $(W, \mathcal{F}_W)$ and the 2-calibrated foliation $(M, \mathcal{F}, \omega)$ have equivalent transverse geometry.

## References


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