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Conitinuous leafwise harmonic functions on codimension one transversely isometric foliations

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1. Introduction

Let M be a closed C^2 manifold, and let $\mathcal{F}$ be a continuous leafwise C^2 foliation on M. This means that M is covered by a finite union of continuous foliation charts and the transition functions are continuous, together with their leafwise partial derivatives up to order 2. Let g be a continuous leafwise[1] C^2 leafwise Riemannian metric. In this talk, such a triplet $(M, \mathcal{F}, g)$ is simply refered to as a *leafwise C^2 foliations.* For simplicity, we assume throughout that the manifold M and the foliation $\mathcal{F}$ are oriented. For a continuous leafwise[2] C^2 real valued function h on M, the leafwise Laplacian Δh is defined by $\Delta h = *d * d\,h$, where $*$ is the leafwise Hodge operator induced by the leafwise metric g.

Definition 1.1. A continuous leafwise C^2 function h is called *leafwise harmonic* if $\Delta h = 0$.

Definition 1.2. A leafwise C^2 foliation $(M, \mathcal{F}, g)$ is called *Liouville* if any continuous leafwise harmonic function is leafwise constant.

As an example, if $\mathcal{F}$ is a foliation by compact leaves, then $(M, \mathcal{F}, g)$ is Liouville. Moreover there is an easy observation:

Proposition 1.3. *If $\mathcal{F}$ admits a unique minimal set, then $(M, \mathcal{F}, g)$ is Liouville.*

This can be seen as follows. Let m_1 (resp. m_2) be the maximum (resp. minimum) value of the continuous leafwise harmonic function h on M. Assume h takes the maximum value m_1 at $x \in M$. Then by the maximum principle, $h = m_1$ on the leaf F_x which passes through x. Now the closure of F_x contains the unique minimal set X. Therefore $h = m_1$ on X. The

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[1]This means that the leafwise partial derivatives up to order 2 of the components of g in each foliation chart are continuous in the chart.
[2]The leafwise partial derivatives of h up to order 2 in each foliation chart are continuous in the chart.

same argument shows that $h = m_2$ on X. That is, $m_1 = m_2$, showing that h is constant on M.

A first example of non-Liouville foliations is obtained by R. Feres and A. Zeghib in a simple and beautiful construction [FZ]. It is a foliated S^2-bundle over a hyperbolic surface, with two compact leaves. There are also examples in codimension one. B. Deroin and V. Kleptsyn [DK] have shown that a codimension one foliation $\mathcal{F}$ is non-Louville if $\mathcal{F}$ is transversely C^1, admits no transverse invariant measure and possesses more than one minimal sets, and they have constructed such a foliation.

A codimension one foliation $\mathcal{F}$ is called $\mathbb{R}$-*covered* if the leaf space of its lift to the universal covering space is homeomorphic to $\mathbb{R}$. See [F] or [FFP]. It is shown in [F] and [DKNP] that an $\mathbb{R}$-covered foliation without compact leaves admits a unique minimal set. Therefore the above example of a codimension one non-Liouville foliation is not $\mathbb{R}$-covered. This led the authors of [FFP] to the study of Liouville property for $\mathbb{R}$-covered foliations. The main purpose of the present talk is to generalize a result of [FFP].

DEFINITION 1.4. A codimension one leafwise C^2 foliation $(M, \mathcal{F}, g)$ is called *transversely isometric* if there is a continuous dimension one foliation ϕ transverse to $\mathcal{F}$ such that the holonomy map of ϕ sending a (part of a) leaf of $\mathcal{F}$ to another leaf is C^2 and preserves the leafwise metric g.

Notice that a transversely isometric foliation is $\mathbb{R}$-covered. Our main result is the following.

Theorem 1.5. *A leafwise C^2 transversely isometric codimension one foliation is Liouville.*

In [FFP], the above theorem is proved in the case where the leafwise Riemannian metric is negatively curved. Undoubtedly this is the most important case. But the general case may equally be of interest.

If a transversely isometric foliation $\mathcal{F}$ does not admit a compact leaf, then, being $\mathbb{R}$-covered, it admits a unique minimal set, and Theorem 1.5 holds true by Proposition 1.3. Therefore we only consider the case where $\mathcal{F}$ admits a compact leaf. In this case the union X of compact leaves is closed. Let U be a connected component of $M \setminus X$, and let N be the metric completion of U. Then N is a foliated interval bundle, since the one dimensional transverse foliation ϕ is Riemannian.

Therefore we are led to consider the following situation. Let K be a closed C^2 manifold of dimension ≥ 2, equipped with a C^2 Riemannian metric g_K. Let $N = K \times I$, where I is the interval $[0, 1]$. Denote by $\pi : N \to K$ the canonical projection. Consider a continuous foliation $\mathcal{L}$ which is transverse to the fibers $\pi^{-1}(y)$, $\forall y \in K$. Although $\mathcal{L}$ is only continuous, its leaf has a C^2 differentiable structure as a covering space of

K by the restriction of π. Also $\mathcal{L}$ admits a leafwise Riemannian metric g obtained as the lift of g_K to each leaf by π. Such a triplet $(N, \mathcal{L}, g)$ is called a *leafwise C^2 foliated I-bundle* in this talk. Now Theorem 1.5 reduces to the following theorem.

Theorem 1.6. *Assume a leafwise C^2 foliated I-bundle $(N, \mathcal{L}, g)$ does not admit a compact leaf in the interior* $\mathrm{I}nt(N)$. *Then any continuous leafwise harmonic function is constant on N.*

An analogous result for random discrete group actions on the interval was obtained in [FR].

2. Outline of the proof of Theorem 1.6

The proof is by absurdity. Let $(N, \mathcal{L}, g)$ be a leafwise C^2 foliated I-bundle without interior compact leaves, and we assume that there is a continuous leafwise harmonic function f such that $f(K \times \{i\}) = i$, $i = 0, 1$.

A probability measure μ on N is called *stationary* if $\langle \mu, \Delta h \rangle = 0$ for any continuous leafwise C^2 function h.

Proposition 2.1. *There does not exist a stationary measure μ such that*

$$\mu(\mathrm{I}nt(N)) > 0.$$

This can be shown as follows. Denote by X the union of leaves on which f is constant. The subset X is closed in N. L. Garnett [G] has shown that $\mu(X) = 1$ for any stationary measure μ. Therefore if $\mu(\mathrm{I}nt(N)) > 0$, there is a leaf L in $\mathrm{I}nt(N)$ on which f is constant. But since we are assuming that there is no interior compact leaves, the closure of L must contain both boundary components of N. A contradiction to the continuity of f.

The proof of Theorem 1.6 is obtained by studying leafwise Brownian motions. Let us denote by Ω the space of continuous leafwise paths $\omega : [0, \infty) \to N$. For any $t \geq 0$, a random variable $X_t : \Omega \to N$ is defined by $X_t(\omega) = \omega(t)$. For any point $x \in N$, the Wiener probability measure P^x is defined using the leafwise Riemannian metric g. Notice that $P^x\{X_0 = x\} = 1$.

Given $0 < \alpha < 1$, let $V = K \times (\alpha, 1]$, and define a subset Ω_V of Ω by

$$\Omega_V = \{X_{t_i} \in V, \ \exists t_i \to \infty\}.$$

Clearly Ω_V is invariant by the shift map. Then, as is well known, the function $p : M \to [0, 1]$ defined by $p(x) = P^x(\Omega_V)$ is leafwise harmonic. Another important feature of the function p is that p is nondecreasing along the fiber $\pi^{-1}(y)$, $\forall y \in K$, since our leafwise Brownian motion is synchronized, i. e, it is the lift of the Brownian motion on K. The key fact for the proof is the following:

The function p is constant on $\mathrm{I}nt(N)$.

This follows from Proposition 2.1. That is, if we assume p nonconstant, then we can construct a stationary measure μ such that $\mu(\mathrm{I}nt(N)) > 0$. Next an easy observation shows the following:

The function p is 1 on $\mathrm{I}nt(N)$.

This implies that $\limsup_{t\to\infty} f(X_t) = 1$, P^x-almost surely, since the neighbourhood V can be arbitrary. Likewise considering neighbourhoods of $K \times \{0\}$, we have $\liminf_{t\to\infty} f(X_t) = 0$.

But since f is leafwise harmonic, the family $\{f(X_t)\}$ is a P^x-martingale, and the martingale convergence theorem asserts that there exist $\lim_{t\to\infty} f(X_t)$, P^x-almost surely. The contradiction shows Theorem 1.6.

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