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# Genus one Birkhoff sections for suspension Anosov flows

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## 1. Introduction

There are two fundamental examples of Anosov flows of clesed connected 3-manifolds. One is the suspension flow $\phi_{\overline{A}}$ of a hyperbolic toral automorphism $\overline{A}$ induced by an element $A \in SL(2,\mathbb{Z})$ with $trace(A) > 2$ and the other is the geodesic flow $\phi_{\Sigma_g}$ of a closed hyperbolic surface $\Sigma_g$ of genus $g$.

Two flows $\phi$ of $M$ and $\phi'$ of $M'$ are said to be *topologically equivalent* if there exists a homeomorphism $h : M \to M'$ which maps $\phi$-orbits to $\phi'$-orbits preserving their orientation. No suspension Anosov flow is topologically equivalent to a geodesic flow. Two flows $\phi$, $\phi'$ are said to be *topologically almost equivalent* if there exists a finite union $\Gamma$ (resp. $\Gamma'$) of periodic orbits of $\phi$ (resp. $\phi'$) such that $\phi|_{M\setminus\Gamma}$ is topologically equivalent to $\phi'|_{M'\setminus\Gamma'}$. Then, each geodesic flow is topologically almost equivalent to some suspension Anosov flow. This is proved by constructing genus one Birkhoff sections for geodesic flows. A *Birkhoff section* for a flow $\psi$ of a closed connected 3-manifold $M$ is defined to be the pair $(S,\iota)$ of a compact connected surface $S$ with the boundary $\partial S$ and an immersion $\iota : S \to M$ such that (1) $\iota|_{Int(S)}$ is an embedding transverse to the flow, (2) each component of the boundary $\partial S$ covers a periodic orbit of $\psi$ by $\iota$, and (3) every orbit starting from any point of $M$ meets $S$ in a uniformly bounded time. The image $\iota(S)$ is also called a Birkhoff section. These observations may lead one to ask the following question.

Question 1.1. Is any geodesic flow topologically almost equivalent to any suspension Anosov flow?

The aim of this article is to give a positive answer to this question.

## 2. First return maps of Birkhoff sections

Let $(S,\iota)$ be a Birkhoff section of an Anosov flow $\phi$ of a closed 3-manifold $M$. Let $\Gamma$ be a finite union of periodic orbits of $\phi$. For any orbit $\gamma \subset \Gamma$, take a tubular neighborhood $N(\gamma) = D^2 \times S^1$. Let $M^\Gamma$ be the 3-manifold with boundary obtained from $M$ by blowing up each point $p \in \Gamma$ using a polar coordinate of $D^2$. There uniquely exists a flow $\phi^\Gamma$ of $M^\Gamma$ such that $\phi|_{M\setminus\Gamma} = \phi^\Gamma|_{Int(M^\Gamma)}$. The surface $\iota(S)$ gives rise to an embedded surface $S^\Gamma \subset M^\Gamma$ transverse to $\phi^\Gamma$, which is really a cross section of $\phi^\Gamma$. Let $r : S^\Gamma \to S^\Gamma$ be the first return map of the flow $\phi^\Gamma$. Fried has shown that the return map $r$ is topologically conjugate to a pseudo-Anosov diffeomorphism $h$ of $S^\Gamma$ ([2, Theorem 3]). Let $r_S : \hat{S}^\Gamma \to \hat{S}^\Gamma$ be the homeomorphism obtained from $r : S^\Gamma \to S^\Gamma$ by collapsing each component of $\partial S^\Gamma$ to a point. If $S$ is of genus one, $r_S$ is topologically conjugate to a toral automorphism $\overline{B}$ for some hyperbolic element $B \in SL(2,\mathbb{Z})$(see [4, Lemma 1 in Section 3]).

## 3. Main results

Let $A$ be an element of $SL(2,\mathbb{Z})$ with $trace(A) > 2$. Then, $A$ is conjugate, in $GL(2,\mathbb{Z})$, to a matrix of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_2} \cdots\cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_{2n-1}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_{2n}} \qquad (\ a_i \geq 1)$$

(see [3, Section 1]).

Suppose $B_1, B_2 \in SL(2,\mathbb{Z})$ are given. If $B_1$ is conjugate to $B_2$ in $GL(2,\mathbb{Z})$, the induced flow $\phi_{\overline{B}_1}$ is topologically equivalent to $\phi_{\overline{B}_2}$. So we may assume that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \geq b \geq d$, and $a \geq c \geq d$.

**Theorem 3.1.** *If $trace(A) > 3$ , there exists a genus one Birkhoff section $(S, \iota)$ of $\phi_{\overline{A}}$ such that $\sharp Fix(r_S) < \sharp Fix(\overline{A})$.*

Here, $Fix(f)$ denotes the fixed point set of a map $f : X \to X$.

By arguments in Section 2, the map $r_S$ obtained in Theorem 3.1 is topologically conjugate to a hyperbolic automorphism $\overline{B}$ determined by a matrix $B \in SL(2,\mathbb{Z})$. Then we have

$$\begin{aligned} trace(B) - 2 &= \#Fix(\overline{B}) \\ &= \#Fix(r_S) \\ &< \#Fix(\overline{A}) \\ &= trace(A) - 2. \end{aligned}$$

These observations lead one to the following theorem which gives one a positive answer to Question 1.1.

**Theorem 3.2.** *Let $A$ be an element of $SL(2,\mathbb{Z})$ with $trace(A) > 3$. Then, there exists $B \in SL(2,\mathbb{Z})$ such that (1) $\phi_{\overline{A}}$ is topologically almost equivalent to $\phi_{\overline{B}}$, and (2) $2 < trace(B) < trace(A)$.*

## 4. Outline of the proof Theorem 3.1

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $SL(2,\mathbb{Z})$ with $a \geq b \geq d$ and $a \geq c \geq d$. Let $M_A = T^2 \times [0,1]/(p,1) \sim (\overline{A}(p),0)$. recall that the Anosov flow $\phi_{\overline{A}}$ is a suspension flow of $M_A$. A key point of the construction of a genus one Birkhoff section of $\phi_{\overline{A}}$ is to find a suitable rectangle $\square P_1P_2P_3P_4$ in $T^2 = T^2 \times \{0\} \subset M_A$ such that

1. $\phi_{\overline{A}}(\overline{P_1P_2}) = \overline{P_3P_2}$ and $\phi_{\overline{A}}(\overline{P_3P_4}) = \overline{P_1P_4}$, or
2. $\phi_{\overline{A}}(\overline{P_3P_2}) = \overline{P_1P_2}$ and $\phi_{\overline{A}}(\overline{P_1P_4}) = \overline{P_3P_4}$,

where $\overline{PQ}$ denote the edge of a rectangle connecting a vertex $P$ with a vertex $Q$. Such a rectangle, together with two flow bands connecting $\overline{P_1P_2}$ with $\overline{P_3P_2}$ and $\overline{P_3P_4}$ with $\overline{P_1P_4}$ respectively, gives rise to an immersed surface $S_1$ in $M_{\overline{A}}$ such that 1) $S_1$ is homeomorphic to a 2-sphere minus three disks, 2) $int(S_1)$ is embedded in $M_{\overline{A}}$ and is transverse to $\phi_{\overline{A}}$, 3) each component of $\partial S_1$ covers a periodic orbit of $\phi_{\overline{A}}$( see [2, Section 2] ). For any $0 < \epsilon < 1$, $S_1 \cup (T^2 \times \{\epsilon\})$ is a singular surface with double point curves. If you cut and paste the surface $S_1 \cup (T^2 \times \{\epsilon\})$ along the double curves to obtain a required genus one Birkhoff section $(S,\iota)$ of $\phi_{\overline{A}}$. In order to complete the proof of Theorem 3.1, it suffeces to find a rectangle mentioned above. To this end, take $P_1 = (0,0)$, $P_2 = (1/c,0)$, $P_3 = (0,1)$, and $P_4 = (-1/c,1)$ if $b = 1$, and take $P_1 = (1,0)$, $P_2 = \dfrac{1}{a+d-2}(a-1,c)$, $P_3 = (0,0)$, and $P_4 = \dfrac{1}{a+d-2}(d-1,-c)$ if $b \neq 1$. Then the rectangle $\square P_1P_2P_3P_4$ in $\mathbb{R}^2$ gives rise to a required rectangle in $T^2$.

## 5. Question

Given $A \in SL(2,\mathbb{Z})$ with $trace(A) < -3$, you can also make a Birkhoff section $(S,\iota)$ of $\phi_{\overline{A}}$ as in Section 4. In this case, the induced homeomorphism $r_S$ is not an Anosov homeomorphism. Then you have the following question.

QUESTION 5.1. Does there exist a pair of matrices $A$, $B$ in $SL(2,\mathbb{Z})$ such that (1) $\phi_{\overline{A}}$ is topologically almost equivalent to $\phi_{\overline{B}}$, and (2) $trace(A) < trace(B) < -2$.

## References


[1] G. Birkhoff, Dynamical systems with two degrees of freedom, *Trans. Amer. Math. Soc.* **18** (1917), 199–300.

[2] D. Fried, Transitive Anosov flows and pseudo-Anosov maps, *Topology* **22**(1983), 299–303.

[3] W. Froyd and A. Hatcher, Incompressible surfaces in punctured-torus bundles, *Topology and its Applications* **13**(1982), 263–282.

[4] E. Ghys, Sur l'invariance topologique de la classe de Godbillon-Vey, *Ann. Inst. Fourier* **37-4**(1987), 59–76.

[5] H. Minakawa, Examples of pseudo-Anosov homeomorphisms with small dilatations, *J. Math. Sci. Univ. Tokyo* **13**(2006), 95–111.



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