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Leafwise symplectic structures on codimension 1 foliations on S^5

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1. Introduction

In this talk, we present a framework for the existence of leafwise symplectic structures on a codimension 1 foliation associated with a Milnor fibration or with an open book decomposition supporting a contact structure.

Definition 1.1. A *leafwise symplectic structure* on a foliatited manifold $(M, \mathcal{F})$ is a smooth 2-form ω which restricts to a symplectic form on each leaf.

Our previous result was the following.

Theorem 1.2 ([Mi]). *The natural codimension 1 spinnable foliation $\mathcal{F}_k$ on S^5 associated with the simple elliptic hypersurface singularity $\widetilde{E}_k$, admits a leafwise symplectic structure for $k = 6$, 7, 8.*

Corollary 1.3. *There exist regular Poisson stuctures on S^5 whose symplectic dimension is 4.*

In particular for $k = 6$, the associated foliation is so called *Lawson's foliation* which is the first codimension one foliation found on S^5 ([L]).

The three deformation classes $\widetilde{E}_l$ $(l = 6, 7, 8)$ of simple elliptic hypersurface singularities are given by the following polyomials.

$$f_{\widetilde{E}_6} = Z_0^3 + Z_1^3 + Z_2^3 \;\; (+\lambda Z_0 Z_1 Z_2)$$
$$f_{\widetilde{E}_7} = Z_0^4 + Z_1^4 + Z_2^2 \;\; (+\lambda Z_0 Z_1 Z_2)$$
$$f_{\widetilde{E}_8} = Z_0^6 + Z_1^3 + Z_2^2 \;\; (+\lambda Z_0 Z_1 Z_2)$$

As the smooth topology of these objects does not depend on the choice of the constant λ, while λ should avoid finitely many exceptional values, we can ignore it and take it to be 0. In the case of $\widetilde{E}_6$ the objects are homogeneous of degree 3 and the hypersurfaces $f^{-1}(w)$ are all easily dealt with in a geometric sense. For $\widetilde{E}_7$ and $\widetilde{E}_8$, they are quasi homogeneous

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and still easy to handle. In particular we easily see that their links are ismorphic to 3-dimensional nil manifolds $Nil^3(-3)$, $Nil^3(-2)$, and $Nil^3(-1)$ respectively.

The next main target for us is the case of cusp singularities which is defined by the following polynomials.

$$f_{p,q,r} = Z_0^p + Z_1^q + Z_2^r + Z_0 Z_1 Z_2\,, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

whose links are known to be solv manifolds, *i*.e., T^2-bundles over S^1 with monodoromies hyperblic toral auto's. If we try to generalize our consruction to a wider class of Milnor fibrations or open books supporting contact structures including the cusp singularities case, we need to formalize and brush up the construction in [Mi].

For basic facts about hypersurface singularities, the readers may refer to Milnor's seminal text book [M]. For the treatment of cusp singularities in this context, see [H] and [K].

Let (M, K, Φ) be a symplectic open book decomposition of M^{2n+1} supporting a contact structure ξ on M or a Milnor fibration on $M = S^{2n+1}$ associated with a singularity of $n+1$ variables with the link K. In the first case ξ restricts to $\xi_K = \xi|_K$ and in the latter case the ambient complex strucrue of $\mathbb{C}^{n+1}$ determines the standard contact structure $\xi = \xi_0$ on S^{2n+1} which restricts to a natural contact structure $\xi_K = \xi_0|_K$. Let F_θ denote the page over $\theta \in S^1$. The end of F_θ is diffeomorphic to $K \times \mathbb{R}_+$.

Now our main result is stated as follows.

Theorem 1.4. *In the above situation, supppose the following conditions are satisfied. Then M admits a natural 'spinnable' codimension 1 foliation $\mathcal{F}$ with a leafwise symplectic structure and an isotopic family of contact structures starting from the given one ξ on M which converges to $\mathcal{F}$ as a family of almost contact structures.*

(1) *The link K^{2n-1} fibres over S^1 with a symplectic sructure ω_Σ on the fibre Σ^{2n-2} which is invariant under the monodromy.*[1]

(2) *The restriction $H^2_{dR}(F_\theta) \to H^2_{dR}(K \times \mathbb{R}_+) \cong H^2_{dR}(K)$ hits $[\omega_\Sigma]$.*

(3) *ξ_K admits a contact form whose Reeb vector field X_K is tangent to the fibration in (1). For $n > 2$ we need extra quantitative conditions.*[2]

Corollary 1.5. *Associated with any of the cusp singularities of the form $f_{p,q,r} = 0$ at the origin ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$), the natural spinnable foliation on S^5 admits a leafwise symplectic structure.*[3]

[1] In the case $n = 2$, this simply meens K fibres over S^1.

[2] For many cases this extra condtion seems to be easily checked.

[3] This foliation has a unique compact leaf diffeomorphic to a solv manifold.

2. Mori's convergence criterion

Atsuhide Mori's criterions on a symplectic open book supporting a contact structure for the existence of a natural codimension 1 spinnable foliation or for the isotopic convergence of the contact structures to the foliation ([Mo1], [Mo2], and [Mo3]). These criterions are the prototypes of our conditions in the main theorem. Let (M, K, Φ), ξ, ξ_K,and X_K be as in the previous setion.

Theorem 2.1 (Mori)**.** *Let us consider the following two conditions.*

(I) *K admits a Riemannian foliation $\mathcal{G}$ of codimension* 1*.*

(II) *X_K is tangent to $\mathcal{G}$.*

1) *If* (I) *is satisfied, there exists a natural 'spinnable' foliation $\mathcal{F}$ on M which restricts to $\mathcal{G} = \mathcal{F}|_K$ on K.*
2) *Moreover if* (II) *is also satisfied, there exists an isotopic family of ξ which converges to $\mathcal{F}$.*

1) is applicable for any open book decomposition, because the condition (I) comes from Kopell's lemma and Thurston-Rosenberg ([RT]) showed that then $K \times D^2$ can be smoothly foliated. Topologically, (I) implies that K fibres over S^1 (Tischler's theorem). The coincidence of (II) with our condition (3) is rather surprising.

Mori's criterions implies the covergence of contact structures to a spinnable foliation is quite different in dimension 5 or higher than in dimension 3.

3. Further discussions

Naturally next main target for us is a construction of a codimension 1 leaf-wise symplectic foliation on S^7, while the Milnor fibre is simply connected for usual isolated singularities of 4 or more variables. Therefore already the condition (1) in our theorem fails.

Problem 3.1. 1) Does there exist mixed function with an isolated singularity to which we can apply the very basic theory of Milnor fibrations and contact structures which exhibit different features on the topologies of link and Milnor fibres?

2) In particular for 4 or more complex variables whose link K admits a symplectic fibre in the sense of (2) in our main Theorem ?

On the other hand, what is really missing is the search for *impossibility*.

Problem 3.2. 1) Find non-trivial obstructions for codimension 1 foliations to admit a leafwise symplectic structures.

2) Find further obstuctions for Stein manifolds to admit an end-periodic symplectic structures. The 2nd foliated cohomology does not seem to be a good candidate.

3) Does there exist a 2-calibrated codimension 1 foliation on S^5? Here, '2-calibrated' means 'equipped with a *global closed* 2-form which restricts to a symplectic form on each leaf'.

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