Geometry and Foliations 2013
Komaba, Tokyo, Japan

One dimensional dynamics in intermediate regularity: old and new

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1. Introduction

The study of (pseudo)-group actions on 1-dimensional manifolds by diffeomorphisms of class C^2 is a classical subject that goes back to Denjoy, Schwartz and Sacksteder. By many reasons, it is desirable to extend this to intermediate regularity (smaller than C^2 but larger than C^1 in the Hölder scale). T. Tsuboi and S. Hurder were among the first in being interested on this, both interested in the possibility of extending the Godbillon-Vey class to a certain critical regularity [4, 8]. This made natural several questions of a dynamical natural, formulated by Tsuboi as conjectures. Most of them were solved in collaboration with Deroin and Kleptsyn [2], as for example:

Theorem 1.1 (à la Denjoy). *Every free action of $\mathbb{Z}^d$ by circle diffeomorphisms of class $C^{1+\alpha}$, $\alpha > 1/d$, is minimal and topologically conjugated to a group of rotations.*

Theorem 1.2 (à la Kopell). *Let $f_1, \ldots, f_d, f_{d+1}$ be diffeomorphisms of the closed unit interval for which there exist subintervals $I_{i_1,\ldots,i_d,i_{d+1}}$ disposed lexicographically and such that $f_j(I_{i_1,\ldots,i_j,\ldots,i_{d+1}}) = I_{i_1,\ldots,i_j+1,\ldots,i_{d+1}}$ for all j. If the f_j's, with $j = 1, \ldots, d$, are all of class $C^{1+\alpha}$, where $\alpha > 1/d$, and commute (among them and) with f_{d+1}, then f_{d+1} cannot be of clas C^1.*

In both cases, counter-examples in class $C^{1+\alpha}$, $\alpha < 1/d$, were already been constructed by Tsuboi [9].

In this talk, I will concentrate on recent extensions of these kind of results to more genral groups/regularities. I start with a theorem obtained in collaboration with Kleptsyn [5] concerning the case where the regularities of the maps are different.

Theorem 1.3. *In both theorems above, one may suppose that the regularities of the f_j's are different, say f_j is of class $C^{1+\alpha_j}$, $j = 1, \ldots, d$, provided $\frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_d} > 1$. Moreover, for every combination of exponents satisfying the reverse (strict) inequality, one can construct counter-examples.*

The second extension is very recent and concerns the critical regularity $C^{1/d}$. Although this remains open for the case of the circle, for the interval is completely settled in [7].

Theorem 1.4. *Theorem 1.2 still holds in class $C^{1+1/d}$ (assuming f_{d+1} of class $C^{1+\alpha}$ for some positive α). This remains true for the extension to different regularities above.*

Finally, we consider the case of more complicated groups. We start with a result that completely solves the question concerning growth of groups of diffeomorphisms.

Theorem 1.5. *If Γ is a finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$, then either Γ is almost nilpotent or it contains a free semigroup. Moreover, this is no longer true in class C^1.*

This theorem makes natural the study of nilpotent groups of diffeomorphisms of the interval. In this direction, the classical Kopell-Plante-Thurston theorem establishes that no such group can be contained in $\mathrm{Diff}_+^2([0,1])$ unless it is Abelian. However, it is a classical fact (going back to Malcev and Newmann) that every torsion-free, finitely-generated nilpotent group Γ is left-orderable, hence it acts on the interval. Actually, there is a very natural action, which was proven by B. Farb and J. Franks [3] to be smoothable to the class C^1. This action is constructed as follows: According to a theorem of Malcev, Γ embeds into a group N_d of $d \times d$ lower-triangular matrices with integer entries and 1 in the diagonal. This group acts on $\mathbb{Z}^d$, fixing the hyperplane $\{1\} \times \mathbb{Z}^{d-1}$ and respecting the lexicographic order therein. If we make correspond an interval $I_{i_1,\ldots,i_{d-1}}$ to all such points in $\mathbb{Z}^{d-1}$ and we dispose them on $[0,1]$ in an ordered way, this naturally induces an action of N_d (hence of Γ) on $[0.1]$ (just use infinitely many affine maps and paste them). The next result was shown in collaboration with E. Jorquera and G. Castro.

Theorem 1.6. *The Farb-Franks action of N_d above is not (semi-)conjugate to an action by $C^{1+\alpha}$ diffeomorphisms, where $\alpha \geq \frac{2}{(d-1)(d-2)}$. This is no longer true in lower regularity.*

This result concerns a single action, and it remains the question of determining the best regularity of arbitrary actions of nilpotent groups on the interval. This is work in progress (in collaboration with Castro, Jorquera, C. Rivas and R. Tessera).

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