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Quasi-invariant measures for non-discrete groups on S^1

Julio C. REBELO

Whereas the discussion is primarily conducted for finitely generated subgroups of $\mathrm{Diff}^{\omega}(S^1)$, almost all results can be adapted to similar subgroups of smooth diffeomorphism of the circle. In terms of higher dimensional manifolds, however, our results will only admit convenient generalizations if the corresponding group of diffeomorphisms are supposed to contain a Morse-Smale element (see [Re-2]).

Consider then a finitely generated subgroup G of $\mathrm{Diff}^{\omega}(S^1)$. The group G is said to be *non-discrete* if it contains a sequence of elements $\{h_i\}$ converging to the identity in the (say) C^{∞}-topology (and such that $h_i \neq \mathrm{id}$ for every $i \in \mathbb{N}$). Concerning the existence of non-discrete groups as before, the following result due to Ghys may be quoted.

Theorem (Ghys [Gh]). *Consider the group* $\mathrm{Diff}^{\omega}(S^1)$ *equipped with the analytic topology. Then there is a neighborhood* $\mathcal{U}$ *of the identity such that every non-solvable group* $G \subset \mathrm{Diff}^{\omega}(S^1)$ *generated by a finite set* $S \subset \mathcal{U}$ *is non-discrete.*

Our purpose will be to study some subtle aspects of the ergodic theory of non-discrete groups as above. In particular, we would like to investigate the structure of quasi-invariant measures (always supposed to be non-atomic) with special interest in the case of stationary measures. Stationary measures are defined as follows. Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a finitely generated group equipped with a probability measure ν which is non-degenerate in the sense that its support generates G as semi-group. A probability measure μ on S^1 is said to be *stationary* for G with respect to ν if the equation

$$\mu(\mathcal{B}) = \sum_{g \in G} \nu(g)\mu(g^{-1}(\mathcal{B})) \; . \tag{1}$$

holds for every Borel set $\mathcal{B} \subset S^1$. A simple adaptation of Krylov-Bogoloubov theorem suffices to ensure that stationary measures always exist. Also, assuming that G has no invariant measure, it easily follows that every stationary measure is quasi-invariant and gives no mass to points. Moreover, this measure is often unique as proved by Deroin, Kleptsyn and Navas:

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Theorem (Deroin-Kleptsyn-Navas [DKN])**.** *Suppose that G is a group of diffeomorphisms of S^1 leaving no probability measure on S^1 invariant. If G is equipped with a non-degenerate probability measure ν, then the resulting stationary measure μ is unique.*

Thus, whereas every theorem valid for quasi-invariant measures will automatically hold for the stationary measure, many easy constructions of singular quasi-invariant measures do not apply to stationary measures. Similarly it is easy to produce several examples of singular measures that are quasi-invariant by non-discrete groups as above but most of the corresponding constructions yield measures that are certainly not stationary. Yet, singular stationary measures for groups as above do exist and the difficulty of providing a criterion to ensure that stationary measures must be regular is illustrated by the following theorem due to Kaimanovich and Le Prince [K-LP]: every Zariski-dense finitely generated subgroup of $\mathrm{PSL}\,(2,\mathbb{R})$ can be equipped with a non-degenerate measure ν giving rise to a singular stationary measure μ on S^1.

Besides stationary measures, Patterson-Sullivan measures are among the best known examples of quasi-invariant measures (in the case for Fuchsian or Kleinian groups). A very distinguished feature of Patterson-Sullivan measure is its *d-quasiconformal character*. Given $d \in \mathbb{R}_+^*$, recall that a probability measure μ on S^1 is said to be d-quasiconformal for G if there exists a constant C such that, for every Borel set $\mathcal{B} \subset S^1$ and every $g \in G$, we have

$$\frac{1}{C}|g'(x)|^d \leq \frac{d\mu}{dg_*\mu}(x) \leq C|g'(x)|^d \,.$$

In particular, d-quasiconformal measures are closely related to the d-dimensional Hausdorff measure. Also we can wonder whether d-quasiconformal measures on the circle exist beyond the class of Fuchsian groups, whether or not we are dealing with "discrete groups". In fact, this problem can be viewed as a far reaching extension of Patterson-Sullivan theory. Concerning the case of non-discrete subgroups of $\mathrm{Diff}^\omega(S^1)$, we have the following theorem.

Theorem (Uniqueness of Lebesgue, [Re-1])**.** *Let $G \subset \mathrm{Diff}^\omega(S^1)$ be a finitely generated non-solvable and non-discrete group. Assume also that G has no finite orbit and that μ is a d-quasiconformal measure for G. Then μ is absolutely continuous and $d = 1$.*

Since we mentioned that some statements for the circle generalize to higher dimensions in the presence of a Morse-Smale dynamics, here it is a good point to give an example of these generalizations. To keep statements as simple as possible, consider non-discrete groups of analytic diffeomorphism of the sphere S^2. Since these diffeomorphisms need not be conformal, the notion of d-quasiconformal measures can be replaced by the following definition:

Definition. Let μ be a probability measure on S^2 and consider a group $G \subset \mathrm{Diff}^{\omega}(S^2)$. Given $d \in \mathbb{R}_+^*$, the measure μ will be called a d-quasi-volume for G if there is a constant C such that for every point $x \in S^1$ and every element $g \in G$, the Radon-Nikodym derivative $d\mu/dg_*\mu$ satisfies the estimate

$$\frac{1}{C}\|\mathrm{Jac}\,[Dg](x)\|^d \leq \frac{d\mu}{dg_*\mu}(x) \leq C\|\mathrm{Jac}\,[Dg](x)\|^d\,,$$

where $\mathrm{Jac}\,[Dg](x)$ stands for the Jacobian determinant of Dg at the point x.

The more general statements in [Re-2] imply the following:

Theorem ([Re-2]). *Suppose that $G \subset \mathrm{Diff}^{\omega}(S^2)$ is non-discrete and contains a Morse-Smale element. Suppose also that G leaves no proper analytic subset of S^2 invariant. Then every d-quasi-volume μ for G is absolutely continuous (in particular $d = 2$).*

Going back to the circle, it was observed that d-quasiconformal measures behave similarly to the d-dimensional Hausdorff measure. Since the examples in Kaimanovich-Le Prince [K-LP] possess Hausdorff dimension comprised between 0 and 1, it is natural to wonder whether these stationary measures are comparable to Hausdorff measures of same dimension. To help to make sense of these possible comparisons, the notion of *Lusin sequences* can be used. A *Lusin sequence* for a probability measure μ consists of a sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ such that $\mu(K_n) \to 1$. The advantage of using Lusin sequences is to work with compact sets as opposed to general Borel sets while recovering the standard definitions of Hausdorff measures/dimensions and so on. Denoting by μ_d the d-dimensional Hausdorff measure, we have:

Theorem ([Re-1]). *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a finitely generated non-solvable and non-discrete group. Suppose that μ is an ergodic (non-atomic) singular quasi-invariant measure for G whose Hausdorff dimension d belongs to $(0, 1]$. Denoting by μ_d the d-dimensional Hausdorff measure, the following alternative holds:*

- *Either there is a Lusin sequence $\{K_n\}$ for μ such that $\mu_d(K_n) = 0$ for every n or*
- *Every Lusin sequence $\{K_n\}$ verifies $\mu_d(K_n) \to \infty$ when $n \to \infty$.*

In particular, the Borel set $K = \bigcup_{n=1}^{\infty} K_n$ is such that $\mu(K) = 1$ and $\mu_d(K)$ is either zero or infinite.

If time permits, we shall conclude with a more detailed discussion of stationary measures along with some regularity criteria for them.

References

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Institut de Mathématiques de Toulouse
E-mail: rebelo@math.univ-toulouse.fr