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Generic pseudogroups on $(\mathbb{C}, 0)$ and the topology of leaves

Helena REIS

This is an extended abstract for the material in the papers [MRR] and [RR] jointly with J.-F. Mattei and J. Rebelo.

In the study of some well-known problems about singular holomorphic foliations, we usually experience difficulties concerning to greater or lesser extent the topology of their leaves. Yet, most of these problems are essentially concerned with pseudogroups generated by certain local holomorphic diffeomorphisms defined on a neighborhood of $0 \in \mathbb{C}$. In this sense, results about pseudogroups of $\mathrm{Diff}\,(\mathbb{C}, 0)$ generated by a finite number of local holomorphic diffeomorphisms are crucial for the understanding of certain singular foliations defined about the origin of $\mathbb{C}^2$. Also, as it will be seen below, for most of these problems it is necessary to consider classes of pseudogroups with a distinguished generating set all of whose elements have fixed conjugacy class in $\mathrm{Diff}\,(\mathbb{C}, 0)$.

In the above mentioned works, some well-known questions about singular holomorphic foliations on $(\mathbb{C}^2, 0)$ are answered. These questions have first arisen as an outgrowth of the problem of classifying germs of plane analytic curves (Zariski problem). The key to answer them will be the introduction of a theory of pseudogroups obtained out of "generic" elements in $\mathrm{Diff}\,(\mathbb{C}, 0)$ having fixed conjugacy class. We shall explain these problems before presenting our main results.

Recall that a local singular holomorphic foliation on a neighborhood of $(0, 0) \in \mathbb{C}^2$ is nothing but the foliation induced by the local orbits of a holomorphic vector field having isolated singularities and defined on the mentioned neighborhood. In particular singular points of a foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ are always isolated and, besides, two holomorphic vector fields representing $\mathcal{F}$ differ by an invertible multiplicative holomorphic function. Assume that the origin is a singular point for a given foliation $\mathcal{F}$ and let X be a representative of $\mathcal{F}$. The eigenvalues of $\mathcal{F}$ at the origin correspond to the eigenvalues of the linear part of X at the same point. It is well-known that every foliation on $(\mathbb{C}^2, 0)$ can be transformed by a finite sequence of blow-up maps into a new foliation $\widetilde{\mathcal{F}}$ possessing singularities that are "simple", i.e. $\widetilde{\mathcal{F}}$ has at least one eigenvalue different from zero at each of its singular points. This sequence of blow-up maps leading to $\widetilde{\mathcal{F}}$ is called the *resolution of* $\mathcal{F}$.

The study of singularities of foliations and of their deformations, paralleling Zariski problem, led to the introduction of the *Krull topology* in the space of these foliations. In this topology, a sequence of foliations $\mathcal{F}_i$ is said to converge to $\mathcal{F}$ if there are representatives X_i for $\mathcal{F}_i$ and X for $\mathcal{F}$ such that X_i is tangent to X, at the origin, to arbitrarily high orders (modulo choosing i large enough). It should be noted that, given a foliation $\mathcal{F}$, its resolution depends only on a finite jet of the Taylor series of X at the singular point. Therefore, if $\mathcal{F}'$ is close to $\mathcal{F}$ in the Krull topology, then these foliations admit exactly the same resolution. Furthermore the position of the singularities of the resolved foliations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ coincide and so do their corresponding eigenvalues.

A prototypical problem in this direction that will also help us to clarify the contents of the above discussion is provided by the nilpotent foliations associated to Arnold singularities A^{2n+1}. These are local foliations $\mathcal{F}$ defined by a (germ of) vector field X having nilpotent linear part, i.e. $X = y\partial/\partial x + \cdots$, and a unique separatrix S that happens to be a curve analytically equivalent to $\{y^2 - x^{2n+1} = 0\}$. Let us discuss the simplest case $n = 1$ in detail (the general case is very similar).

Consider a nilpotent foliation $\mathcal{F}$ associated to Arnold singularity A^3, i.e. a nilpotent foliation admitting a unique separatrix that happens to be a curve analytically equivalent to $\{y^2 - x^3 = 0\}$. For this type of foliation, the desingularization of the separatrix coincides with the resolution of the foliation itself. More precisely, the map associated to the desingularization of the separatrix $E_S : M \to \mathbb{C}^2$ reduces also the foliation $\mathcal{F}$ (see Figure 1 for the corresponding resolution).

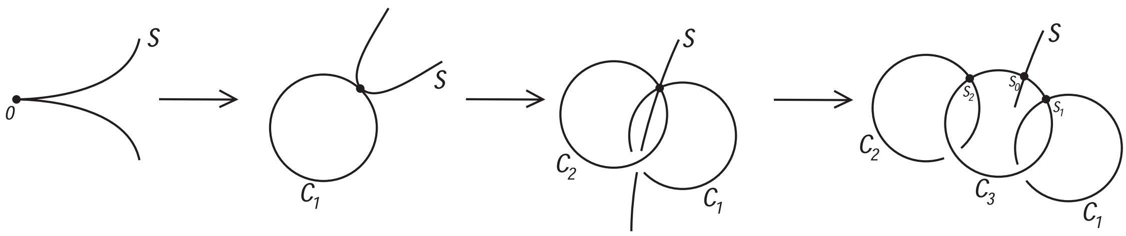

Figure 1

The corresponding exceptional divisor $\mathcal{D} = E_S^{-1}(0)$ consists of the union of 3 rational curves as indicated in Figure 1. The singular points of $\widetilde{\mathcal{F}}$ are the intersection points of consecutive components in the tree along with a point s_0 that corresponds to the intersection of the transformed of the separatrix with $E_S^{-1}(0)$. This intersection takes place in the component C_3 as indicated in Figure 1. All these singular points possess two eigenvalues different from zero. The corresponding eigenvalues can precisely be determined by using the self-intersection of the various components of the exceptional divisor.

For example, the eigenvalues of $\widetilde{\mathcal{F}}$ at s_1 are $1, -3$ whereas the eigenvalues of $\widetilde{\mathcal{F}}$ at s_2 are $1, -2$.

It should be noted that the regular leaf $C_1 \setminus \{s_1\}$ is isomorphic to $\mathbb{C}$ and thus simply connected. This implies that the local holonomy map associated to a path contained in $C_1 \setminus \{s_1\}$ and winding around s_1 coincides with the identity. This assertion combined with the fact that $\widetilde{\mathcal{F}}$ has eigenvalues $1, -3$, guarantees that the germ of $\widetilde{\mathcal{F}}$ at s_1 is linearizable. Thus the local holonomy map f associated to a small loop about s_1 and contained in C_3 must be of finite order equal to 3, i.e. it is conjugate to a rotation of order 3. A similar discussion applies to the component C_2 and leads to the conclusion that the local holonomy map g associated to a small loop around s_2 and contained in C_3 has order equal to 2, i.e. it is conjugate to a rotation of order 2. Since $C_3 \setminus \{s_0, s_1, s_2\}$ is a regular leaf of $\widetilde{\mathcal{F}}$, we conclude that the (image of the) holonomy representation of the fundamental group of $C_3 \setminus \{s_0, s_1, s_2\}$ in $\text{Diff}\,(\mathbb{C}, 0)$ is nothing but the group generated by f, g. The reader will easily convince himself/herself that the dynamics of this holonomy group encodes all the information about the corresponding foliation.

It should be noted that the conclusion above depends only on the configuration of the reduction tree which, in turn, is determined by a finite jet of the Taylor series of X at the singular point. Hence, if the coefficients of Taylor series of the vector field X are perturbed starting from a sufficiently high order, the new resulting vector field X' will still give rise to a foliation whose singularity is reduced by the same blow-up map associated to the divisor of Figure 1. In particular, the holonomy representation of the fundamental group of $C_3 \setminus \{s_0, s_1, s_2\}$ in $\text{Diff}\,(\mathbb{C}, 0)$, obtained from this new foliation, is still generated by two elements of $\text{Diff}\,(\mathbb{C}, 0)$ having finite orders respectively equal to 2 and to 3. Since every local diffeomorphism of finite order as above is conjugate to the corresponding rotation, it follows in particular that their conjugacy classes in $\text{Diff}\,(\mathbb{C}, 0)$ are fixed.

From what precedes, it follows that whenever $\mathcal{F}$ is a foliation as above and $\mathcal{F}'$ is close to $\mathcal{F}$ in the Krull topology, then $\mathcal{F}'$ is also a nilpotent foliation of type A^3. It is then natural to wonder what type of dynamical behavior can be expected from these foliations, or more precisely, from a "typical" foliation in this family. Inasmuch the space of foliations was endowed with the Krull topology, which fails to have the Baire property, questions about "dense sets of foliations" can still be asked. The following is an example of long-standing problem in the area:

Question. Does there exist a nilpotent foliation $\mathcal{F}$ in A^3 whose leaves are simply connected (apart maybe from a countable set)? Is the set of these foliations dense in the Krull topology, i.e. given a nilpotent foliation $\mathcal{F}$ in A^3, does there exist a sequence of foliations $\mathcal{F}_i$ converging to $\mathcal{F}$ in the Krull topology and such that every $\mathcal{F}_i$ has simply connected leaves (with possible exception of a countable set of leaves)?

Our methods are powerful enough to affirmatively settle both questions above. A crucial point is the understanding of groups generated by f, g at level of pseudogroup and not only at germ level. In fact, the local dynamics of the holonomy pseudogroup arising from the leaf $C_3 \setminus \{s_0, s_1, s_2\}$ on a *fixed* neighborhood of $0 \in \mathbb{C}$ must be studied.

In the case of nilpotent foliations in the class A^3, it was seen that pseudogroups given by generating sets with elements possessing fixed conjugacy classes play a central role in the description of the corresponding foliations. This phenomenon is not peculiar to the mentioned family of foliations and, indeed, appears quite often. To have a better insight in the nature of the mentioned phenomenon, suppose we are given a foliation $\mathcal{F}$ and consider $\mathcal{F}'$ very close to $\mathcal{F}$ in the Krull topology. In particular, the resolutions $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ of $\mathcal{F}$, $\mathcal{F}'$ turn out to coincide. The positions of the singular points of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ in the common exceptional divisor coincide as well and so do their corresponding eigenvalues. Suppose now that $\widetilde{\mathcal{F}}$ has only hyperbolic singularities i.e. the singularities of $\widetilde{\mathcal{F}}$ have two eigenvalues different from zero and such that their quotient lies in $\mathbb{C} \setminus \mathbb{R}$. The same holds for $\widetilde{\mathcal{F}}'$ since corresponding singularities of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ have the same eigenvalues. By Poincaré theorem, both singularities are then conjugate to the corresponding linear model and, thus, they are conjugate to each other. Thus the corresponding local holonomy maps arising from a small loop encircling the singularity in question are themselves conjugate by a local diffeomorphism. In other words, the pseudogroups generated by these holonomy maps for $\widetilde{\mathcal{F}}$ and for $\widetilde{\mathcal{F}}'$ naturally have generating sets whose elements have the same conjugacy classes. The latter are, indeed, fixed since it corresponds to the class of a hyperbolic element of $\text{Diff}\,(\mathbb{C}, 0)$ with fixed multiplier.

Having explained the need for considering pseudogroups with generating sets all of whose elements possess a fixed conjugacy class in $\text{Diff}\,(\mathbb{C}, 0)$, we can now proceed to state our main results. Let us begin with the results concerning pseudogroups generated by a finite number of elements in $\text{Diff}\,(\mathbb{C}, 0)$ which will later allow us to answer the above stated questions on nilpotent foliations. For this, let us equip $\text{Diff}\,(\mathbb{C}, 0)$ with the so-called analytic topology, that was first considered by Takens in the context of real diffeomorphisms of an analytic manifold and further discussed in the case of $\text{Diff}\,(\mathbb{C}, 0)$ in [MRR]. Unlike the Krull topology, the analytic topology has the Baire property. Now, consider a k-tuple of local holomorphic diffeomorphisms $f_1, \ldots, f_k$ fixing $0 \in \mathbb{C}$. The first theorem states that the local diffeomorphisms f_i can be perturbed inside their conjugacy classes so as to generate a pseudogroup isomorphic to the free product of the corresponding cyclic groups. Indeed, the perturbation can be made inside a

G_δ-dense subset of $(\mathrm{Diff}\,(\mathbb{C},0))^k$. Also, it can be proved that the mentioned perturbation can be made inside the class of diffeomorphisms tangent to the identity to every a priori fixed order (which for technical reasons is also necessary to solve the corresponding questions on foliations). More precisely, letting $\mathrm{Diff}_\alpha(\mathbb{C},0)$ stand for the normal subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$ consisting of elements tangent to the identity to order α, we have the following:

Theorem A ([MRR]). *Fixed $\alpha \in \mathbb{N}$, let $f_1,\dots,f_k$ be given elements in $\mathrm{Diff}\,(\mathbb{C},0)$ and consider the corresponding cyclic groups $G_1,\dots,G_k$. Then, there exists a G_δ-dense set $\mathcal{V} \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^k$ such that, whenever $(h_1,\dots,h_k) \in \mathcal{V}$, the following holds:*

(1) *The group generated by $h_1^{-1}\circ f_1 \circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k$ induces a group in $\mathrm{Diff}\,(\mathbb{C},0)$ that is isomorphic to the free product $G_1 * \cdots * G_k$.*

(2) *Let $f_1,\dots,f_k$ and $h_1,\dots,h_k$ be identified to local diffeomorphisms defined about $0\in\mathbb{C}$. Suppose that none of the local diffeomorphisms $f_1,\dots,f_k$ has a Cremer point at $0\in\mathbb{C}$. Denote by Γ^h the pseudogroup defined on a neighborhood V of $0\in\mathbb{C}$ by the mappings $h_1^{-1}\circ f_1\circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k$, where $(h_1,\dots,h_k)\in\mathcal{V}$. Then V can be chosen so that, for every non-empty reduced word $W(a_1,\dots,a_k)$, the element of Γ^h associated to $W(h_1^{-1}\circ f_1\circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k)$ does not coincide with the identity on any connected component of its domain of definition.*

Item (1) of the previous result concern groups at the germ level, while item (2) concerns pseudogroups. Note that the assumption that none of the fixed diffeomorphisms $f_1,\dots,f_k$ has a Cremer point at $0\in\mathbb{C}$ is not necessary for the first conclusion of Theorem A. This assumption is, however, indispensable for the second item due to certain examples of dynamics near Cremer points that were constructed by Perez-Marco.

Item (2) ensures the existence of a point p possessing an infinite orbit of hyperbolic fixed points for the pseudogroup Γ^h. In other words, p has an infinite orbit under Γ^h and, for every point q lying in the orbit of p, there is an element $g\in\Gamma^h$ for which q is a hyperbolic fixed point (i.e. $\|g'(q)\|\neq 0$). In fact, the existence of this type of point p associated to a pseudogroup whose germ at $0\in\mathbb{C}$ is not solvable has been known for a while (see [Lo] and their references). However the question on whether or not these pseudogroups exhibit more than one single orbit of hyperbolic "fixed points", at least in the case of "typical" pseudogroups, has remained open. In [RR], we provide "generic" answers for this question and for the question on the nature of the stabilizer of points $p\neq 0$. This is as follows:

Theorem B ([RR]). *Suppose we are given f,g in α and denote by D an open disc about $0\in\mathbb{C}$ where f,g and their inverses are defined. Assume that* *none of the local diffeomorphisms f, g has a Cremer point at $0 \in \mathbb{C}$. Then, there is a G_δ-dense set $\mathcal{U} \subset \mathrm{Diff}_\alpha(\mathbb{C}, 0) \times \mathrm{Diff}_\alpha(\mathbb{C}, 0)$ such that, whenever (h_1, h_2) lies in $\mathcal{U}$, the pseudogroup Γ_{h_1,h_2} generated by $\tilde{f} = h_1^{-1} \circ f \circ h_1$, $\tilde{g} = h_2^{-1} \circ g \circ h_2$ on D satisfies the following:*

(1) *The stabilizer of every point $p \in D$ is either trivial or cyclic.*

(2) *There is a sequence of points $\{Q_i\}$, $Q_i \neq 0$ for every $i \in \mathbb{N}^*$, converging to $0 \in \mathbb{C}$ and such that every Q_n is a hyperbolic fixed point of some element $W_i(\tilde{f}, \tilde{g}) \in \Gamma_{h_1,h_2}$. Furthermore the orbits under Γ_{h_1,h_2} of Q_{n_1}, Q_{n_2} are disjoint provided that $n_1 \neq n_2$.*

Let us now show how the previous theorems can be translated in terms of nilpotent foliations in the class A^{2n+1}. The above conducted discussion can be expanded to show the existence of an injection from the set of nilpotent foliations associated to Arnold singularities A^{2n+1} in the space of subgroups of Diff $(\mathbb{C}, 0)$ generated by two diffeomorphisms such that one of them has order 2 and the other has order $2n + 1$. Denote by Γ the pseudogroup generated by f, g on a neighbourhood V of $0 \in \mathbb{C}$. A necessary condition for a foliation as above to have simply connected leaves (up to a countable set of them), is that every element on Γ cannot coincide with the identity on any connected component of its domain of definition. Owing to Theorem A, the diffeomorphisms f, g can be perturbed into $\tilde{f} = h_1^{-1} \circ f \circ h_1$ and $\tilde{g} = h_2^{-1} \circ g \circ h_2$ so as to satisfy this condition. It remains the problem of realizing these diffeomorphisms as the generators of the holonomy of another nilpotent foliation associated to the Arnold singularity A^{2n+1}. In this direction, we proved that the existence of an actual correspondence between the space of these foliations and the space of subgroups of Diff $(\mathbb{C}, 0)$ generated by two holomorphic diffeomorphims conjugate to the rotations of order 2 and order $2n + 1$ (cf. [MRR]).

To formulate our statement in terms of "Krull denseness", as in the original questions, let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a holomorphic vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}, in particular $\mathcal{F}$ possesses one unique separatrix. Now by putting together the construction in [MRR] with Theorems A and B above, we obtain:

Theorem C ([MRR, RR])**.** *Let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}. Then, for every $N \in \mathbb{N}$, there exists a vector field $X' \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ defining a germ of foliation $\mathcal{F}'$ and satisfying the following conditions:*

(a) $J_0^N X' = J_0^N X$.

(b) *$\mathcal{F}$ and $\mathcal{F}'$ have S as a common separatrix.*

(c) *there exists a fundamental system of open neighborhoods* $\{U_j\}_{j\in\mathbb{N}}$ *of* S*, inside a closed ball* $\bar{B}(0,R)$*, such that the following holds for every* $j \in \mathbb{N}$*:*

(c1) *The leaves of the restriction of* $\mathcal{F}'$ *to* $U_j \setminus S$, $\mathcal{F}'|_{(U_j\setminus S)}$ *are simply connected except for a countable number of them.*

(c2) *The countable set constituted by non-simply connected leaves is, indeed, infinite.*

(c3) *Every leaf of* $\mathcal{F}'|_{(U_j\setminus S)}$ *is either simply connected or homeomorphic to a cylinder.*

The item (c1) in Theorem C appears already in [MRR] whereas items (c2) and (c3) require Theorem B proved in [RR]. The realization of pseudogroups as in the statement of Theorems A and B as holonomy pseudogroups of nilpotent foliations was carried out in [MRR] and relies heavily on the techniques of [MS].

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University of Porto
E-mail: hreis@fep.up.pt