Geometry and Foliations 2013
Komaba, Tokyo, Japan

Hopf Conjecture holds for k-basic, analytic Finsler metrics on two tori

Rafael RUGGIERO, José Barbosa GOMES and
Mário J. DIAS CARNEIRO

1. Introduction

The theory of metric structures in the torus all of whose geodesics are global minimizers was totally understood in the Riemannian case after the solution of the so-called Hopf conjecture: every Riemannian metric in the torus without conjugate points is flat. This statement was proved by Hopf [12] in the 1940's and by Burago-Ivanov [2] in the early 1990's. However, if we widen our scope to the family of Finsler metrics the theory still poses many interesting, unsolved problems.

Definition 1. Let M be a n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In canonical coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM; y \neq 0\}$ be the complement of the zero section. A C^k $(k \geq 2)$ *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:
(i) F is C^k on TM_0;
(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y) \;\forall\; \lambda > 0$$

(iii) The Hessian matrix of $F^2 = F \cdot F$

$$g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Given a Tonelli Hamiltonian in a compact manifold (i.e., a Hamiltonian that is convex and superlinear in the vertical fibers of the cotangent bundle) the Hamiltonian flow in a sufficiently high enery level can be reparametrized

The first author was partly supported CNPq, FAPERJ (Cientistas do Nosso Estado), PRONEX de Geometria, ANR.

to become the geodesic flow of a Finsler metric, so Finsler theory is as general as Hamiltonian theory. Finsler manifolds have curvature tensors which generalize Riemannian curvature tensors, in particular the so-called flag curvature $K(p, v)$ that extends the notion of Riemannian sectional curvature (see [1] for instance for the basic theory of Finsler manifolds). Since the Finsler metric is defined in the tangent bundle of the manifold, the flag curvature depends in general on the vertical variable v. Finsler surfaces where $K(p, v) = K(p)$ are called **k-basic**. Well known examples of non-Riemannian, k-basic Finsler surfaces are given by Randers metrics: those obtained by adding a Riemannian norm and a one form.

Finsler manifolds have geodesics, solutions of the Euler-Lagrange equation defined by the Finsler function F. We say that a complete Finsler manifold has no conjugate points if every geodesic is a global minimizer of the Lagrangian action associated to the Finsler metric (i.e., the Finsler length). Since Busemann examples [3] of non-flat Finsler metrics in the two torus without conjugate points it is known that the Hopf conjecture is false in the Finsler realm. Nevertheless, Finsler metrics in the torus without conjugate points enjoy many properties in common with flat metrics. One of them is their connection with weakly integrable systems in the sense of [11]: there exists a continuous, Lagrangian, invariant foliation by tori of the unit tangent bundle ([6]). The existence of a Lagrangian, C^k invariant foliation of the unit tangent bundle is called in [11] C^k integrability of the geodesic flow of the Finsler metric. Moreover, in all known examples of smooth Finsler metrics without conjugate points ([3], [15] for instance) such foliation is smooth. In the Riemannian case the smoothness of the foliation follows from the rigidity of the metric: since the metric is flat the Riemannian metric is Euclidean. The smoothness of the foliation is not part of the proof of the Hopf conjecture but one of its consequences.

So two questions arise naturally from the above discussion. Do C^0 integrable Finsler geodesic flows on tori are C^k for some $k \geq 1$? Does the C^k integrability of such geodesic flows for $k \geq 1$ play any role in the proof of rigidity results? The first question has been already considered in [6], where it is proved that Lipschitz integrability of the geodesic flow of a Finsler metric on the torus without conjugate points implies C^1 integrability. However, a full answer to the question is still open.

2. Main results

Our main results provide substantial information concerning the above problems. The main contribution of our work is the solution of the Hopf conjecture in the analytic, two dimensional case.

Theorem 1. *An analytic k-basic Finsler metric without conjugate points in the two torus is flat.*

This result is the combination of two results. First of all, the main theorem in [11],

Theorem 2. *$C^{1,L}$ integrable geodesic flows of k-basic Finsler metrics on two tori are flat.*

$C^{1,L}$ means C^1 with Lipschitz first derivatives. The second result [8] deals with the smoothness of the so-called Busemann foliation of Finsler tori without conjugate points.

Theorem 3. *Let (T^2, F) be an analytic, k-basic Finsler metric without conjugate points in the two torus T^2. Then the geodesic flow is analytically integrable, namely, there exists an analytic foliation by invariant tori of the unit tangent bundle of the metric which are graphs of the canonical projection.*

The last Theorem is the first result, as far as we know, to show that a Finsler, non-Riemannian metric in the two torus without conjugate points is smoothly integrable without using geometric rigidity. It is remarkable that in the literature about the link between smoothness of invariant foliations of Hamiltonian flows and geometric rigidity, the most common assumption is hyperbolicity (see for instance [14], [5], [7], [9] with results for surfaces of higher genus. So most of the ideas applied to such manifolds do not hold on tori.

Outline of Proof. Let us give a sketch of the proof of the above results. The proof of Theorem 2 involves the calculation of the Godbillon-Vey number of the Busemann foliation of a Finsler geodesic flow in the two torus provided that the foliation is $C^{1,L}$. This result in itself is very interesting and gives a sort of generalized Gauss-Bonnet formula for Finsler geodesic flows on tori without conjugate points which are smoothly integrable:

Proposition 1. *Let (T^2, F) be a C^∞ Finsler metric without conjugate points whose geodesic flow preserves a codimension 1, $C^{1,L}$ foliation $\mathcal{F}$ of the unit tangent bundle. Then (T^2, F) has no conjugate points and there exists a Riccati operator u associated to the foliation. Moreover, the Godbillon-Vey number of $\mathcal{F}$ is*

$$gv(\mathcal{F}) = \int \eta \wedge d\eta = \int [3(Vu)^2 + u^2]\omega_1 \wedge \omega_2 \wedge \omega_3$$
$$+ \int [4uVJ - 2uXVI - IVK]\omega_1 \wedge \omega_2 \wedge \omega_3,$$

where the integration is taken over T_1M, and

(1) *X is the geodesic vector field, V is a unit vertical field and the triple X, Y, V is a Cartan frame for the unit tangent bundle with $\omega_1, \omega_2, \omega_3$ as their dual Cartan 1-forms.*
(2) *K is the flag curvature, I is the Cartan tensor, and J is the Landsberg tensor.*

This is essentially Proposition 2.1 in [11]. We would like to remark that when the Finsler metric is Riemannian, we get $J = I = 0$ and the Godbillon-Vey formula reduces to Mitsumatsu's formula in [14]. Then we show (Theorem 4.2 in [11]),

Proposition 2. *Let (T^2, F) be a C^∞ Finsler metric without conjugate points whose geodesic flow preserves a codimension 1 C^1 foliation $\mathcal{F}$ in the unit tangent bundle. Then $\mathcal{F}$ is the Busemann foliation and moreover, if $\mathcal{F}$ is $C^{1,L}$ its Godbillon-Vey number is zero.*

Finally, we use Riemann-Finsler geometry to show that when the Finsler metric is k-basic the Landsberg tensor J and the Cartan tensor I are related by the formula

$$J = uI$$

at every point in the unit tangent bundle. Replacing this identity in the Godbillon-Vey formula we get that the Riccati operator must vanish everywhere, and so the flag curvature as well.

The proof of Theorem 3 [8] relies on the application of Riemann-Finsler geometry to link the singularities of the Busemann foliation (as a foliation) with the zeroes of the Cartan tensor in the case of k-basic Finsler metrics. So first of all we show (Proposition 2.2 and Lemma 3.2 in [8])

Proposition 3. *Let (T^2, F) be an analytic k-basic Finsler metric without conjugate points. Then*

(1) *Each leaf of the Busemann foliation is analytic.*
(2) *The Riccati operator associated to Busemann leaves is given by $u = J/I = X(I)/I$ whenever $I \neq 0$.*
(3) *The Busemann foliation is analytic in the set where $I \neq 0$.*

So the study of the analyticity of the Busemann foliation is reduced to show that the function $u = X(I)/I$ has removable singularities in the unit tangent bundle. This is proved in Lemma 3.4 in [8] where we show that the function u has a real analytic extension. □

References

[1] Bao, D., Chern, S.-S., Shen, Z.: *An Introduction to Riemann-Finsler Geometry*, Springer, New York, 2000.

[2] Burago, D., Ivanov, S.: *Riemannian tori without conjugate points are flat*, Geom. Funct. Anal. **4** (1994), 259–269.

[3] Busemann, H.: *The Geometry of Finsler spaces*, Bulletin of the AMS **56** (1950), 5–16.

[4] Busemann, H: *The Geometry of Geodesics*, Pure and Applied Mathematics Vol. 6, Academic Press, New York, NY, 1955.

[5] Ghys, E.: *Rigidité différentiable des groupes Fuchsiens*, Publications Mathématiques I. H. E. S. **78** (1993), 163–185.

[6] Croke, C., Kleiner, B.: *On tori without conjugate points.* Invent. Math. 120 (1995) 241-257.

[7] Gomes, J., Ruggiero, R.: *Rigidity of surfaces whose geodesic flows preserve smooth foliations of codimension 1*, Proceedings of the American Mathematical Society **135** (2007), 507–515.

[8] Dias Carneiro, M. J., Gomes, J. B., Ruggiero R. O: *Hopf conjecture holds for analytic, k-basic Finsler 2-tori without conjugate points.* Preprint PUC-Rio 2013.

[9] Gomes, J., Ruggiero, R.: *Rigidity of magnetic flows for compact surfaces*, Comptes Rendus Acad. Sci. Paris, Ser. I **346** (2008), 313–316.

[10] Gomes, J., Ruggiero, R.: *Smooth k-basic Finsler surfaces with expansive geodesic flows are Riemannian*, Houston J. Math. **37** (3) (2011) 793-806.

[11] Gomes, J., Ruggiero, R.: *Weak integrability of Hamiltonians in the two torus and rigidity.* Nonlinearity 26, 2109 (2013) published online doi:10.1088/0951-7715/26/7/2109.

[12] Hopf, E.: *Closed surfaces without conjugate points*, Proceedings of the National Academy of Sciences **34** (1948), 47–51.

[13] Krantz, S., Parks, H.: *A Primer of real analytic functions.* Second Edition. Birkhäusser, Boston (2002).

[14] Mitsumatsu, Y.: *A relation between the topological invariance of the Godbillon-Vey invariant and the differentiability of Anosov foliations*, Foliations (Tokyo, 1983) 159-167 Advanced Studies in Pure Mathematics **5**, North Holland, Amsterdam (1985).

[15] Zinov'ev, N.: *Examples of Finsler metrics without conjugate points: metrics of revolution*, St. Petersburg Math. J. **20** (2009), 361–379.

Rafael Ruggiero
Pontifícia Universidade Católica do Rio de Janeiro, PUC-Rio,
Rua Marqués de São Vicente 225, Gávea, 22453-900, Rio de Janeiro,
Brasil
E-mail: rorr@mat.puc-rio.br

José Barbosa Gomes
Universidade Federal de Juiz de Fora, Departamento de Matemática,
Universidade Federal de Juiz de Fora, Juiz de Fora, MG,
Brazil, 36036-900
E-mail: barbosa.gomes@ufjf.edu.br

Mário J. Dias Carneiro
Universidade Federal de Minas Gerais,
Ave. Antonio Carlos 6627 CP 702-30 161-970, Belo Horizonte, MG,
Brazil.
E-mail: carneiro@mat.ufmg.br