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Several problems on groups of diffeomorphisms

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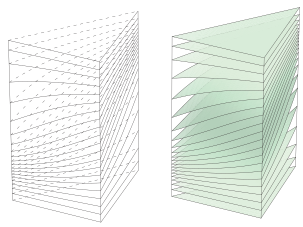
1. Introduction

This is a discussion on several problems related to the study of groups of diffeomorphisms which the author worked on for a while with some hope to find new phenomena.

For a compact manifold M, let $\mathrm{Diff}^r(M)$ ($r = 0$, $1 \leqq r \leqq \infty$, or $r = \omega$) denote the group of C^r diffeomorphisms of M. $\mathrm{Diff}^r(M)$ is equipped with the C^r topology and let $\mathrm{Diff}^r(M)_0$ denote the identity component of it. The family of diffeomorphisms generated by a time dependent vector field is called an isotopy. A diffeomorphism near the identity is contained in an isotopy. $\mathrm{Diff}^r(M)$ has a manifold structure modelled on the space of C^r vector fields. It is worth noticing that the composition $(g_1, g_2) \longrightarrow g_1 \circ g_2$ in $\mathrm{Diff}^r(M)$ ($1 \leqq r < \infty$) is C^∞ with respect to g_1 but not continuous with respect to g_2.

2. Foliated products

A smooth singular simplex $\sigma : \Delta^m \longrightarrow \mathrm{Diff}^r(M)$ corresponds to the multi dimensional isotopy which is the foliation of $\Delta^m \times M$ transverse to the fibers of the projection $\Delta^m \times M \longrightarrow \Delta^m$ whose leaf passing through (t, x) is $\{\sigma(s)\sigma(t)^{-1}(x) \mid s \in \Delta\}$. These multi isotopies naturally match up along the boundary and form the universal foliated M-product over the classifying space $B\overline{\mathrm{Diff}}^r(M)$.



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Let $B\overline{\Gamma}_n^r$ be the classifying space for Haefliger's Γ_n^r structures with trivialized normal bundles. Since $B\overline{\Gamma}_n^r$ classifies C^r foliations with trivialized normal bundles, for an n-dimensional parallelized manifold M^n, we obtain the map $B\overline{\mathrm{Diff}}^r(M^n) \times M^n \longrightarrow B\overline{\Gamma}_n^r$, and hence the map $B\overline{\mathrm{Diff}}^r(M^n) \longrightarrow \mathrm{Map}(M, B\overline{\Gamma}_n^r)$. The deep result by Mather-Thurston says that the last map induces an isomorphism in integral homology.

Theorem 2.1 (Mather-Thurston). *For* $1 \leqq r \leqq \infty$,

$$H_*(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_*(\mathrm{Map}(M^n, B\overline{\Gamma}_n^r); \boldsymbol{Z}).$$

In particular, $H_*(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) \cong H_*(\Omega^n B\overline{\Gamma}_n^r; \boldsymbol{Z})$ *for the group* $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ *of* C^r *diffeomorphisms of* $\boldsymbol{R}^n$ *with compact support.*

On the other hand, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ $(1 \leqq r \leqq \infty,\ r \neq n+1)$ has been shown by Herman-Mather-Thurston. Note that $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_1(B\widetilde{\mathrm{Diff}}^r(M^n)_0^\delta; \boldsymbol{Z})$, where $\widetilde{\mathrm{Diff}}^r(M^n)_0$ is the universal covering group and $^\delta$ means that the group is equipped with the discrete topology when we take its classifying space. In general, the abelianization of a group G is isomorphic to $H_1(BG^\delta; \boldsymbol{Z})$ and a group is said to be perfect if its abelianization is trivial. Moreover, by the fragmentation technique, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ is equivalent to $H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$, and if $\widetilde{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)_0$ is perfect, then $\widetilde{\mathrm{Diff}}^r(M^n)_0$ and $\mathrm{Diff}^r(M^n)_0$ are perfect.

Theorem 2.2 (Herman-Mather-Thurston). $\mathrm{Diff}_c^r(M^n)_0$ $(1 \leqq r \leqq \infty,\ r \neq n+1)$ *is a perfect group. It is a simple group if* M^n *is connected.*

It is known that for $r > 2 - 1/(n+1)$, there is a characteristic cohomology class called the Godbillon-Vey class in $H^{n+1}(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{R})$. $B\overline{\mathrm{Diff}}^r(M^n)$ is conjectured to be n-acyclic. For the higher dimensional homology, it is only known [ASPM (1985), Annals (1989)] that

$$\begin{array}{ll} H_2(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{if } 1 \leqq r < [n/2], \\ H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{if } 1 \leqq r < [(n+1)/m] - 1 \text{ and} \\ H_m(B\overline{\mathrm{Diff}}_c^1(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{for } m \geqq 1. \end{array}$$

The main technical reason of the above regularity conditions can be seen in the infinite iteration construction using $(\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$ action on $\boldsymbol{R}^n$. As is well-known, by the homothety of ratio A, the C^r-norm of a foliated $\boldsymbol{R}^n$-product is multiplied by A^{1-r}. For the easiest case of divisible abelian m-cycle c represented by time 1 maps of commuting vector fields, we divide it into 2^m pieces $[n/m]$ times and we use $\boldsymbol{Z}_+^{2^n}$ action generated by homotheties of ratio $A = 1/(2+\varepsilon)$, then the infinite iteration construction converges in the C^r topology if $2^{-[n/m]}/(2+\varepsilon)^{1-r} < 1$, that is, if $r - [n/m] - 1 < 0$. To treat general cycles we loose a little more regularity.

For the connectivity of $B\overline{\Gamma}^r_n$, it seems that it increases when r tends to 1. It is true that in $\mathrm{Diff}^{1+\alpha}_c(\boldsymbol{R}^n)$, we can construct a $\boldsymbol{Z}^k$ action which permutes open sets, where k tends to infinity as α tends to 0 [JMSJ (1995)], and we think that we can use it to construct infinite iterations of chains. The bound of the rank of such action has been studied by Andrés Navas which gave rise to a new direction of study of group of diffeomorphisms.

For seeking more regular construction, it is necessary to know that abelian cycles are null homologous.

PROBLEM 2.3. For the action $\varphi : \boldsymbol{R}^m \longrightarrow \mathrm{Diff}^r(M^n)$, show that $B(\boldsymbol{R}^m)^\delta \longrightarrow B\overline{\mathrm{Diff}}^r(M^n)$ induces the trivial homomorphism in integral homology.

REMARK 2.4. It is true for $\mathrm{Diff}^\infty_c(\boldsymbol{R})$ [Fourier (1981), Fete of Topology (1988)]. It is probably true for $m = 1$ and $\mathrm{Diff}^\infty_c(\boldsymbol{R}^n)$. The first interesting case is $\boldsymbol{R}^2 \longrightarrow \mathrm{Diff}^\infty_c(\boldsymbol{R}^2)$.

To treat non abelian cycles, we notice that the theorem of Mather-Thurston implies that any class of $H_2(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z})$ $(r \neq n+1)$ can be represented by a foliated M^n product over the surface Σ_2 of genus 2.

For the smooth codimension 1 foliations, there is the interesting problem of determining the kernel of the Godbillon-Vey class.

PROBLEM 2.5. Determine the kernel of $GV : H_2(B\overline{\mathrm{Diff}}^r_c(\boldsymbol{R}); \boldsymbol{Z}) \longrightarrow \boldsymbol{R}$.

REMARK 2.6. There is a group G which contains both $\mathrm{Diff}^r_c(\boldsymbol{R})$ $(r > 1 + 1/2)$ and the group $PL_c(\boldsymbol{R})$ of piecewise linear homeomorphisms of $\boldsymbol{R}$ with compact support, with a metric such that GV cocycle is continuous [Fourier (1992)]. We know that for a G-foliated $\boldsymbol{R}$-product $\mathcal{F}$ over a surface, $GV(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is homologous to a G-foliated $\boldsymbol{R}$-product $\mathcal{H}_0$ over a surface Σ which is the limit of G-foliated $\boldsymbol{R}$-products $\mathcal{H}_k$ over the surface Σ representing 0 in $H_2(G; \boldsymbol{Z})$ [Proc. Japan Acad (1992)]. $\mathcal{H}_k$ are in fact transversely piecewise linear foliations and the topology of $BPL_c(\boldsymbol{R})^\delta$ has been known by the work of Peter Greenberg. It will be nice if we can take $\mathcal{H}_k$ to be C^1 piecewise $PSL(2; \boldsymbol{R})$ foliated S^1-products. The group of C^1 piecewise $PSL(2; \boldsymbol{R})$ diffeomorphisms of S^1 contains the Thompson simple group (consisting of C^1 piecewise $PSL(2; \boldsymbol{Z})$ diffeomorphisms) which gives other interests to study this group.

3. $B\overline{\Gamma}_1^\omega$

Many years ago, Haefliger showed that $B\overline{\Gamma}_1^\omega$ is a $K(\pi,1)$ space. If one understands the definition of the $\overline{\Gamma}_1^\omega$ structures, though π is a huge group, it is easy to show that $H_1(B\overline{\Gamma}_1^\omega;\boldsymbol{Z})=0$.

PROBLEM 3.1. Prove or disprove that $H_2(B\overline{\Gamma}_1^\omega;\boldsymbol{Z})=0$.

REMARK 3.2. The homology group $H_2(B\overline{\Gamma}_1^\omega;\boldsymbol{Z})$ is generated by cycles represented by surfaces Σ_2 of genus 2 with C^ω singular foliations with 2 saddles. Since $H_1(B\overline{\Gamma}_1^\omega;\boldsymbol{Z})=0$ is a $K(\pi,1)$, a homology class represented by the map from S^2 is trivial. A homology class represented by the map from T^2 is homologous to a union of suspensions of C^ω diffeomorphisms of S^1, and these are trivial because $\mathrm{Diff}^\omega(S^1)_0$ is perfect by a result of Arnold.

As for the perfectness of the group $\mathrm{Diff}^\omega(M^n)_0$ of real-analytic diffeomorphisms of M^n, Herman showed that $\mathrm{Diff}^\omega(T^n)_0$ is simple almost 40 years ago. Rather recently, we could show that if M^n admits a nice circle action then $\mathrm{Diff}^\omega(M^n)_0$ is perfect [Ann. ENS (2009)]. These are applications of Arnold's work on the small denominators. With this method, it should be at least generalized to the manifolds with circle actions. There are torus bundles which admits a flow whose orbit closures are fibers. It might be possible to apply the argument of [Ann. ENS (2009)].

4. Uniform perfectness

For a perfect group G, every element g can be written as a product of commutators. The least number of commutators to write g is called the commutator length of g and written as $cl(g)$. A group G is uniformly perfect if cl is a bounded function. The least bound $cw(G)$ is called the commutator width. After the result by Burago-Ivanov-Polterovich [ASPM (2008)], we showed that for a compact n-dimensional manifold M^n which admits a handle decomposition without handles of the middle index $n/2$, $cw(\mathrm{Diff}^r(M^n)_0)\leqq 3$ if n is even, $cw(\mathrm{Diff}^r(M^n)_0)\leqq 4$ if n is odd ($r\neq n+1$). For a compact $2m$-dimensional manifold M^{2m} ($2m\geqq 6$), $cw(\mathrm{Diff}^r(M^{2m})_0)<\infty$ ($r\neq 2m+1$) [CMH (2012)].

PROBLEM 4.1. Estimate $cw(\mathrm{Diff}^r(T^2)_0)$, $cw(\mathrm{Diff}^r(\mathbf{C}P^2)_0)$, $cw(\mathrm{Diff}^r(S^2\times S^2)_0)$, ...

For the group of homeomorphisms, we managed to prove that for the spheres S^n and the Menger compact space μ^n, $cw(\mathrm{Homeo}(S^n)_0)=1$ and

$cw(\mathrm{Homeo}(\mu^n)) = 1$ [Proc. AMS (2013)]. It is probably true that for the Menger-type compact space μ^n_k, $cw(\mathrm{Homeo}(\mu^n_k)_+) = 1$, where $_+$ means a certain condition concerning the orientation. The idea of proof comes from the fact that the typical homeomorphism of such a space is the one with one source and one sink and that the conjugacy class of such a homeomorphism should be unique.

Problem 4.2. Find other examples of groups of commutator width 1.

In 1980, Fathi showed that for the group $\mathrm{Homeo}_\mu(M^n)_0$ of homeomorphisms preserving a good measure μ of M^n ($n \geqq 3$), the kernel of the flux homomorphism $\mathrm{Homeo}_\mu(M^n)_0 \longrightarrow H^{n-1}(M^n; \boldsymbol{R})$ is perfect. It seems that he proved that the kernel is uniformly perfect (at least he proved it for the spheres). For the group $\mathrm{Diff}_{\mathrm{vol}}(M^n)_0$ of volume preserving diffeomorphisms, Thurston showed that the kernel of the flux homomorphism is perfect.

Problem 4.3. Prove or disprove that $\mathrm{Diff}_{\mathrm{vol}}(S^n)_0$ ($n \geqq 3$) is uniformly perfect.

Burago-Ivanov-Polterovich gave the notion of norms on the group and studied its properties. $\nu : G \longrightarrow \boldsymbol{R}_{\geq 0}$ is a (conjugate invariant) norm if it satisfies (i) $\nu(1) = 0$; (ii) $\nu(f) = \nu(f^{-1})$; (iii) $\nu(fg) \leqq \nu(f) + \nu(g)$; (iv) $\nu(f) = \nu(gfg^{-1})$ and (v) $\nu(f) > 0$ for $f \neq 1$. For a symmetric subset $K \in G$ normally generating G, any $f \in G$ can be written as a product of conjugates of elements of K and the function giving the minimum number $q_K(f)$ of the conjugates is a norm. Then $cl(f) = q_K(f)$ for K being the set of single commutators.

For the groups of diffeomorphism with the fragmentation property, the perfectness implies the simplicity. For a simple group G, the norm $q_{\{g,g^{-1}\}} : G \longrightarrow \boldsymbol{Z}_{\geq 0}$ is defined for $g \in G$. If $\{q_{\{g,g^{-1}\}}\}_{g \in G\setminus\{1\}}$ is bounded then G is said to be uniformly simple. In other words, for any $f \in G$ and $g \in G \setminus \{1\}$, f is written as a product of a bounded number of conjugates of g or g^{-1}. We have a distance function d on the set $\{C_{\{g,g^{-1}\}}\}_{g\neq 1}$ of symmetrized nontrivial conjugate classes:

$$d(C_{\{f,f^{-1}\}}, C_{\{g,g^{-1}\}}) = \log \max\{q_{\{f,f^{-1}\}}(g), q_{\{g,g^{-1}\}}(f)\}$$

For simple groups which are not uniformly simple, for example, $\mathrm{Diff}_{\mathrm{vol},c}(\boldsymbol{R}^n)_0$ ($n \geqq 3$), A_∞, etc, it is interesting to study the metric d. For the infinite alternative group A_∞, Kodama and Matsuda told me that d is quasi-isometric to the half line.

A real valued function ϕ on a group G is a homogeneous quasimorphism if $(g_1, g_2) \mapsto \phi(g_2) - \phi(g_1g_2) + \phi(g_1)$ is bounded and $\phi(g^n) = n\phi(g)$ for $n \in \boldsymbol{Z}$. Put

$$D(\phi) = \sup\{|\phi(g_2) - \phi(g_1g_2) + \phi(g_1)| \mid (g_1, g_2) \in G \times G\}.$$

Then Bavard's duality says that

$$\mathrm{scl}(g) = \frac{1}{2} \sup_{\phi \in Q(G)/H^1(G;\boldsymbol{R})} \frac{\phi(g)}{D(\phi)},$$

where $\mathrm{scl}(g) = \lim_{n\to\infty} \dfrac{\mathrm{cl}(g^n)}{n}$ (stable commutator length) and $Q(G)$ is the real vector space of homogeneous quasimorphisms on G. Of course, for groups with infinite commutator width, we need to study their stable commutator length function. If the commutator width of a group G is infinite, G is not uniformly simple, hence the distance function d is unbounded. We might have more information on the distance d by looking at relative quasimorphisms. Let $Q(G, K)$ be the real vector space of homogeneous quasimorphisms on G which vanishes on K. If there is a nontrivial element $\phi \in Q(G, K)$ (for example, if $\dim Q(G)$ is larger than the number of K), then $\phi(f) \leqq (q_K(f) - 1)D(\phi)$ and q_K is not bounded. Since Entov-Polterovich, Gambaudo-Ghys, Ishida, and others have shown that $Q(\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2))$ is infinite dimensional and hence the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2) \longrightarrow \boldsymbol{R}$ is not uniformly simple.

Problem 4.4. For the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2) \longrightarrow \boldsymbol{R}$, show that $\{C_{\{g,g^{-1}\}}\}_{g\neq 1}$ with metric d is not quasi-isometric to the half line.

As for the group $\mathrm{Homeo}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2)$, despite attemps by many people, its simplicity is still an open problem. The following problem seems to be the first step to show it.

Problem 4.5. Using area preserving homeomorphisms with the Calabi invariant being infinity, show that an area preserving diffeomorphism with nontrivial Calabi invariant is a product of commutators.

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