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Uniqueness of the contact structure approximating a foliation

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1. Introduction

We study the relationship between foliations by surfaces and contact structures on oriented 3-manifolds. Let us recall that a positive contact structure ξ is a smooth plane field locally defined by a 1-form α such that $\alpha \wedge d\alpha > 0$. In the following we assume that all plane fields are cooriented (and hence oriented) and all contact structures are positive. The first result indicating that there are connections between foliations and contact structures on 3-manifolds is the following theorem from [3].

Theorem 1.1 (Eliashberg-Thurston). *Let $\mathcal{F}$ be a C^2-foliation on a compact 3-manifold such that $\mathcal{F}$ is not diffeomorphic to a foliation by spheres on $S^2 \times S^1$. Then every C^0-neighbourhood of $\mathcal{F}$ in the space of plane fields contains a positive contact structure.*

Example 1.2. The foliation of $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ given by the 2-tori $\{z = \text{const}\}$ is approximated by the contact structures

$$\xi_{k,\varepsilon} = \ker\left(\alpha_{k,\varepsilon} = dz + \varepsilon\left(\cos(2\pi kz)dx - \sin(2\pi kz)dy\right)\right)$$

as $0 \neq \varepsilon \to 0$ provided that k is a positive integer. According to Gray's theorem, contact structures which are homotopic through contact structures are isotopic. This ensures that $\xi_{k,\varepsilon}$ is independent from ε, so we omit the ε from the notation. However, it is well known that the contact structures ξ_k and ξ_l are isotopic if and only if $k = l$. Therefore one cannot expect that there is a neighbourhood of $\mathcal{F}$ such that all positive contact structures in that neighbourhood are pairwise isotopic.

In this talk we present a complete characterization of those foliations which have a C^0-neighbourhood in the space of plane fields such that all positive contact structures in that neighbourhood are pairwise isotopic. Our result can be applied to show that the space of taut foliations on certain 3-manifolds is not connected. This is of interest in view of the work of H. Eynard [4] and this question was investigated further by J. Bowden [1].

2. Main results

It turns out that the presence of torus leaves as in Example 1.2 is the main source of non-isotopic contact structures in arbitrarily small neighbourhoods of a foliation.

Theorem 2.1 (Vogel). *Let $\mathcal{F}$ be a C^2-foliation on a closed 3-manifold such that*

(i) there is no torus leaf,

(ii) not every leaf is a plane, and

(iii) not every leaf is a cylinder.

Then there is a C^0-neighbourhood of $\mathcal{F}$ in the space of plane fields such that all positive contact structures in that neighbourhhood are pairwise isotopic.

This theorem remains true for confoliations (i.e. smooth plane fields defined by a 1-form α such that $\alpha \wedge d\alpha \geq 0$) instead of foliations. Let us also note that the main use of the C^2-assumption is through Sacksteder's theorem which guarantees the existence of curves with attractive holonomy in exceptional minimal sets. Both the existence result of Eliashberg-Thurston and our uniqueness result remain valid for stable/unstable foliations of Anosov flows on 3-manifolds although these foliations are not C^2-smooth in general.

Let recall that according to theorems of H. Rosenberg and G. Hector, C^2-foliations of the type described in (ii) respectively (iii) occur only on T^3 respectively on parabolic T^2-bundles over S^1. Thus if M is not a torus fibration over S^1, then (i) is the only restriction on the foliation in order to ensure that the contact structures approximating the foliation are unique up to isotopy.

Remark 2.2. It can be shown (by explicit construction) that every neighbourhood of a foliation as in (i),(ii),(iii) of the above theorem contains infinitely many pairwise non-isotopic contact structures.

The uniqueness theorem can be extended to the case when torus leaves are present provided that the torus leaves have attractive holonomy (this condition can be weakened a little bit, however it cannot be omitted completely). Then every two contact structures in a sufficiently small C^0-neighbourhood of $\mathcal{F}$ become isotopic after a stabilization operation is applied to both them.

The proof of Theorem 2.1 is rather intricate. The overall structure is similar to the structure of the proof of Theorem 1.1 but the order of the steps is reversed. For the purposes of this exposition we assume that $\mathcal{F}$ has only one minimal set, namely a closed leaf Σ of genus $g \geq 2$. The two main steps of the proof are then as follows:

1. Fix a pair of tubular neighbourhoods $V_{out}(\Sigma) \supset V_{in}(\Sigma)$ of Σ. Given two contact structures ξ_0, ξ_2 sufficiently close to $\mathcal{F}$ show that there is a contact structure ξ_1 on M such that ξ_0 is isotopic to ξ_1 and $\xi_1 = \xi_2$ on the complement of $V_{in}(\Sigma)$. This step uses an adaptation of the methods used in [2] by V. Colin.
2. Show that the restriction of ξ_1, ξ_2 to $V_{out}(\Sigma) \setminus V_{in}(\Sigma)$ completely determines ξ_1 and ξ_2 on $V_{out}(\Sigma)$ up to isotopy relative to the boundary provided that ξ_2 is sufficiently close to $\mathcal{F}$. For this we appeal to classification results of K. Honda, W. Kazez and G. Matić [6] and we use the technique developed in [5] by E. Giroux.

The above strategy works if a finite list of assumptions on the distance of the contact planes from $\mathcal{F}$ is satisfied. We thus obtain the required neighbourhood of $\mathcal{F}$ in the space of plane fields. Above we have constructed a homotopy through contact structures which is turned into an isotopy by Gray's theorem.

3. Applications and a question

Theorem 1.1 has the following applications: Every construction of an interesting foliation on a 2-manifold can be viewed as construction of a potentially interesting contact structure. Conversely, Theorem 2.1 allows us to associate every invariant of a contact structure to a foliation which satisfies the hypothesis of Theorem 2.1. It is rather easy to show that this invariant does not change when the foliation $\mathcal{F}$ is deformed through a continuous path of foliations satisfying the hypotheses. Therefore, Theorem 2.1 can be used to show that the space of taut foliations is not connected on some manifolds.

For this recall that on the one hand foliations without torus leaves are always taut. On the other hand if a foliation has no Reeb components, then all torus leaves are incompressible. Hence contact invariants can be applied effectively to the study of connectivity properties of spaces of taut foliations on atoroidal manifolds. This should be compared with theorems of H. Eynard which imply that two taut foliations are homotopic through foliations (which may have Reeb components) provided that the two foliations are homotopic through plane fields.

Question 3.1. Theorem 2.1 can be viewed as a statement about the relationship between the topology space of contact structures and the topology of the C^0-closure of the space of contact structures. What else can be said?

References

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