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On Lagrangian submanifolds in the Euclidean spaces

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1. Introduction

In this paper, we study the problem of realizing an n-manifold M^n as a Lagrangian submanifold in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with a not fixed symplectic structure.

For the standard symplectic structure, there are several conditions on Lagrangian submanifolds.

Theorem 1.1 (Gromov [5]). *Let L^n be a closed Lagrangian submanifold of the $2n$-dimensional Euclidean space with the standard symplectic structure, $(\mathbb{R}^{2n}, \omega_0)$. Then*

$$[\omega_0] \neq 0 \in H^2(\mathbb{R}^{2n}, L; \mathbb{R}),$$

and therefore $H^1(L; \mathbb{R}) \neq 0$.

Theorem 1.2 (Fukaya [3]). *Let $(\mathbb{R}^6, \omega_0)$ be the 6-dimensional Euclidean space with the standard symplectic structure and L be an oriented connected closed prime 3-manifold. Then L can be embedded in $(\mathbb{R}^6, \omega_0)$ as a Lagrangian submanifold if and only if L is diffeomorphic to $S^1 \times \Sigma_g$, where Σ_g is an oriented closed 2-dimensional manifold of genus $g \geq 0$.*

By Theorem 1.1 and Theorem 1.2, the topology of a Lagrangian submanifold of $\mathbb{R}^{2n}$ with the standard symplectic structure is strongly restricted. On the other hand, we will see that almost of all the closed parallelizable manifolds can be Lagrangian submanifolds of the Euclidean spaces with not fixed symplectic structures.

2. Main Result

The main result is the following.

Theorem 2.1. *Let M^n be a closed parallelizable n-manifold. If $n \neq 7$, or if $n = 7$ and the Kervaire semi-characteristic $\chi_{\frac{1}{2}}(M^7)$ is zero, then for any embedding of M^n in $\mathbb{R}^{2n}$, there exists a symplectic structure on $\mathbb{R}^{2n}$ such that the embedding is Lagrangian.*

REMARK 2.2. For $n = 2$, the only closed parallelizable 2-manifold is the 2-torus and for $n = 3$, any closed orientable 3-manifold is parallelizable. There are infinitely many isotopy classes of embeddings of the 2-torus in the 4-dimensional Euclidean space. For $n \geq 3$, there is a surjection from the set of isotopy classes of embeddings of M^n in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ to the homology group $H_1(M^n; \mathbb{Z})$ if n is odd, and to $H_1(M^n; \mathbb{Z}_2)$ if n is even [9], [10].

3. Preliminary

To obtain a Lagrangian embedding of an n-manifold in $\mathbb{R}^{2n}$, we embed its cotangent bundle in $\mathbb{R}^{2n}$ and extend its canonical symplectic structure to $\mathbb{R}^{2n}$.

Proposition 3.1. *Let M^n be a closed parallelizable n-manifold embedded in $\mathbb{R}^{2n}$. Then its normal bundle is trivial.*

Proof. It is an immediate consequence of Kervaire's theorem that for any stably parallelizable manifold K^d embedded in $\mathbb{R}^{2d}$, the normal bundle is trivial [7]. $\square$

Therefore, for a closed parallelizable n-manifold M^n, any embedding of M^n in $\mathbb{R}^{2n}$ extends to an embedding of T^*M^n in $\mathbb{R}^{2n}$. To extend the canonical symplectic structure on T^*M^n, we review Gromov's h-principle for symplectic structures on an open manifold and the space of non-degenerate 2-forms on $\mathbb{R}^{2n}$.

Theorem 3.2 (Gromov [4])**.** *Let N^{2n} be a triangulated open $2n$-manifold and ω be a non-degenerate 2-form on N^{2n}. Then there is a symplectic form $\tilde{\omega}$ on N^{2n}. Moreover, if ω is closed on a neighborhood of a subset M of a core C of N^{2n}, then we can choose $\tilde{\omega}$ which coincides with ω on a neighborhood of M.*

By Theorem 3.2, to extend the canonical symplectic structure, it is sufficient to extend the canonical symplectic structure as a non-degenerate 2-form. We prepare some propositions to apply Theorem 3.2.

Proposition 3.3 (See the section 4.3 of [2])**.** *Let N be a triangulated open manifold. Then there exists a subpolyhedron $C \subset N$ such that* $\dim C < \dim N$ *and N can be compressed by an isotopy $\varphi_t \colon N \to N$, $t \in [0,1]$, into any neighborhood of C.*

We call C a *core* of the open manifold N.

Proposition 3.4. *There is a diffeomorphism from the space of linear symplectic structures on $\mathbb{R}^{2n}$ to the quotient space $\mathrm{GL}(2n;\mathbb{R})/\mathrm{Sp}(2n)$. Moreover, the connected component $\mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n)$ corresponds to the space of linear symplectic structures on $\mathbb{R}^{2n}$ which give the positive orientation on $\mathbb{R}^{2n}$ where*

$$\mathrm{GL}_+(2n;\mathbb{R}) = \{A \in \mathrm{GL}(2n;\mathbb{R}) \mid \det A > 0\}.$$

Proof. For a linear symplectic structure Ω on $\mathbb{R}^{2n}$, we can take a symplectic basis $\langle u_1, v_1, \ldots, u_n, v_n\rangle$ which is determined up to linear transformations by the symplectic group $\mathrm{Sp}(2n)$. That is, the map

$$\Omega \mapsto [A] \in \mathrm{GL}(2n;\mathbb{R})/\mathrm{Sp}(2n) \ \big(A = (u_1 \ \cdots u_n \ v_1 \ \cdots \ v_n)\big)$$

is well defined. Its inverse is given by

$$[A] \mapsto {}^tA^{-1}\Omega_0 A^{-1} \ \left(\Omega_0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \in \mathrm{GL}(2n;\mathbb{R})\right).$$

□

Then we can identify a positive non-degenerate 2-form on $\mathbb{R}^{2n}$ with a smooth map

$$\mathbb{R}^{2n} \to \mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n).$$

We note that the map represents a symplectic basis of the non-degenerate 2-form at each point of $\mathbb{R}^{2n}$.

Proposition 3.5 (See the section 2.2 of [8])**.** *The map*

$$\mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n) \to \mathrm{SO}(2n)/\mathrm{U}(n),\ [A] \mapsto [B],$$

where $({}^tA^{-1}\Omega_0 A^{-1}\,{}^tA^{-1}\,{}^t\Omega_0 A^{-1})^{-\frac{1}{2}}({}^tA^{-1}\Omega_0 A^{-1}) = B\Omega_0 B^{-1}$*, is a homotopy equivalence.*

By Proposition 3.4 and 3.5, we can identify the canonical symplectic structure ω on T^*M^n with the continuous map

$$\omega\colon T^*M^n \to \mathrm{SO}(2n)/\mathrm{U}(n).$$

Actually, the possibility of extending the canonical symplectic structure ω as a non-degenerate 2-form depends only on the homotopy type of ω.

Remark 3.6. For an n-manifold M^n, the existence of a Lagrangian embedding of M^n in $\mathbb{R}^{2n}$ with a not fixed symplectic structure is equivalent to the existence of a totally real embedding of M^n in $\mathbb{C}^n$. Indeed, we can check it by applying Theorem 3.2 and Gromov's h-principle for totally real embeddings [6]. Audin gave a necessary and sufficient condition for the existence of a totally real embedding of M^n in $\mathbb{C}^n$ in [1]. In particular, the existence part of Theorem 2.1 is a part of Audin's theorem if $n \neq 7$.

4. Outline of the Proof of Theorem 2.1

Proof. We prove only the case where $n = 3$. It suffices to show that the map $\omega\colon T^*M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$ is null-homotopic. Let us take a triangulation of M^3, $M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3$ be the skeletons, $f\colon M^3 \to \mathbb{R}^6$ be the embedding. First, we denote the Gauss map of f by g_0. Since M^3 is parallelizable, the Gauss map of f takes the value in the Stiefel manifold $V_{6,3}$,

$$g_0\colon M^3 \to V_{6,3}.$$

The map g_0 is null-homotopic on $M^{(2)}$ because $V_{6,3}$ is 2-connected. Thus there exists a homotopy $g_t^{(2)}\colon M^{(2)} \to V_{6,3}$, $t \in [0,1]$, with $g_0^{(2)} = g_0 \mid_{M^{(2)}}$ and $g_1^{(2)}$ is a constant map. By the covering homotopy property of the fibration $\mathrm{SO}(6) \to V_{6,3}$, we can take the lift $G_t^{(2)}$ of $g_t^{(2)}$,

$$G_t^{(2)}\colon M^{(2)} \to \mathrm{SO}(6).$$

Since the fiber of the fibration $\mathrm{SO}(6) \to V_{6,3}$ is $\mathrm{SO}(3)$ and the homotopy group $\pi_2\big(\mathrm{SO}(3)\big) = 0$, $G_0^{(2)}$ extends to the map $G_0\colon M^3 \to \mathrm{SO}(6)$ which formed by an orthonormal tangent 3-frame field and an orthonormal normal 3-frame field of M^3. On the other hand, $G_1^{(2)}$ extends to a constant map $G_1\colon M^3 \to \mathrm{SO}(6)$. Next, we composes these map with the projection $\pi\colon \mathrm{SO}(6) \to \mathrm{SO}(6)/\mathrm{U}(3)$ which we denote $\bar{G}_0 = \pi \circ G_0$, $\bar{G}_t^{(2)} = \pi \circ G_t^{(2)}$, and $\bar{G}_1 = \pi \circ G_1$. We note that the map $\bar{G}_0 = \omega\colon T^*M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$ and the map $\bar{G}_1$ is a constant map. Lastly, since the homotopy group $\pi_3\big(\mathrm{SO}(6)/\mathrm{U}(3)\big) = 0$, $\bar{G}_t^{(2)}$ extends to the map $\bar{G}_t\colon M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$. Therefore, ω is null-homotopic.

The remaining cases are similar by using the Kervaire semi-characteristic. □

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