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Birkhoff sections for geodesic flows of hyperbolic surfaces

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1. Birkhoff section

Definition 1.1. A Birkhoff section for a flow φ_t defined on a closed 3-manifold is an embedded surface satisfying that its interior is transverse to φ_t and that its boundaries are consist of closed orbits of φ_t.

Example 1.2. 1. Let T^2 be a flat torus. Now we construct a Birkhoff section for the geodesic flow g_t of T^2 in the unit tangent vector bundle T_1T^2. We take closed geodesics C_1, C_2, C_3, C_4 of T^2 (see Figure 1). The complement of these closed geodesics is 4 rectangles. We choose two rectangles R_1 and R_2 which are not adjacent. Next we consider a family $\mathcal{C}_i$ $(i = 1, 2)$ of convex simple closed curves which fills the interior of R_i with one singularity deleted. Let S be the closure of the union of unit tangent vectors of all curves of $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, S is a torus with 8 discs deleted and the boundaries of S are close oriented geodesics corresponding to C_1, C_2, C_3, C_4.

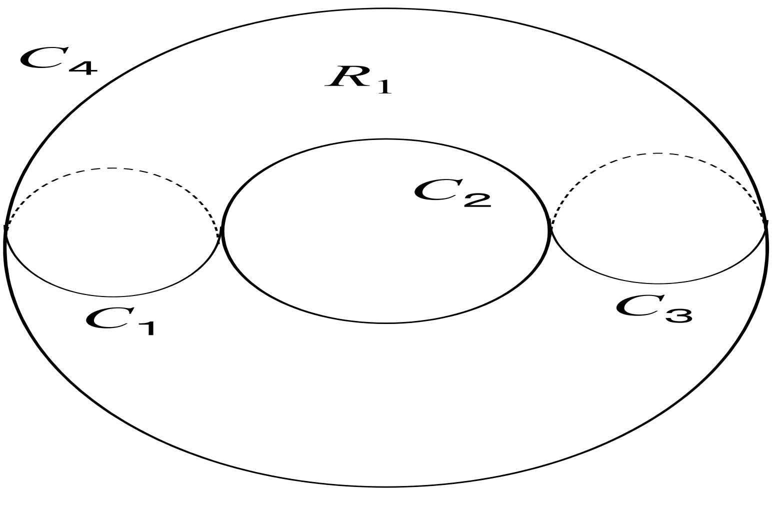

Figure 1: Geodesics and rectangles of T^2

S is a Birkhoff section for g_t. The first return map of g_t associated with S is topologically semiconjugate to the toral automorphism induced by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

2. In the hyperbolic case, we construct a genus one Birkhoff section of the geodesic flow by the same method of the above case.
   Let Σ_g $(g \geq 2)$ be a genus g orientable closed surface with a hyperbolic metric. The geodesic flow of Σ_g has genus one Birkhoff sections [1, 2, 3]. The first return maps associated with these sections are topologically semiconjugate to hyperbolic toral automorphisms. These toral automorphisms are induced by

$$A_g = \begin{pmatrix} 2g^2 - 1 & 2g(g-1) \\ 2g(g+1) & 2g^2 - 1 \end{pmatrix}$$

   and

$$B_g = \begin{pmatrix} 4g^2 - 2g - 1 & 2g^2 - 2g \\ 8g^2 - 2 & 4g^2 - 2g - 1 \end{pmatrix}$$

   [1, 4].

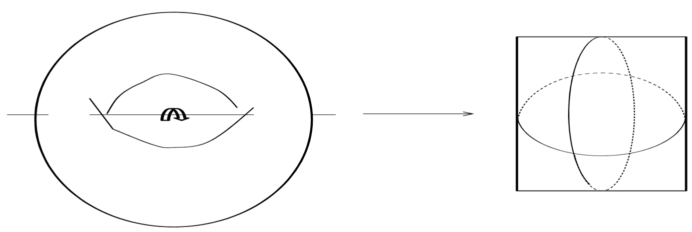

Figure 2: Branched covering $\gamma : T^2 \to P$

3. Let P be a flat pillowcase i.e. a 2-dimensional sphere with 4 singular points. P is also considered as a quotient space $\mathbb{R}^2/\Gamma$ where Γ is the group of isometries of $\mathbb{R}^2$ generated by π-rotations centered at $(0, \pm\frac{1}{2})$ and $(\pm\frac{1}{2}, 0)$. We consider a branched covering $\gamma : T^2 \to P$

(see Figure 2). The differential $T_1\gamma$ of γ preserves geodesic flows and $S' = T_1\gamma(S)$ is also a genus one Birkhoff section for the geodesic flow f_t of P. The double covering $T_1\gamma|_S : S \to S'$ is induced by the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Hence, the first return map of f_t associated with S' is topologically semiconjugate to a toral automorphism induced by $DA_1D^{-1} = \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}$.

2. Main Results

In [1], Brunella showed the method to construct genus one Birkhoff sections. We apply this method to geodesic flows of 2-spheres with singularities.

For any three positive integers p, q, r satisfying that the hyperbolic condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, let $S(p,q,r)$ be a 2-sphere with three singular points whose cone angles are $\frac{2\pi}{p}, \frac{2\pi}{q}, \frac{2\pi}{r}$. If we consider the hyperbolic metric on $S(p,q,r)$, then the geodesic flow F_t of $S(p,q,r)$ is an Anosov flow on a triangular Seifert fibred space.

Using Scott's result about closed geodesics of F_t [5], we have the next theorem.

Theorem 2.1. *If (p,q,r) is not $(2,3,u)$ $(u \geq 7)$ nor $(2,4,u)$ $(u \geq 5)$ up to permutation of p,q,r, then the geodesic flow F_t of $S(p,q,r)$ has a genus one Birkhoff section and F_t is topologically constructed by doing Dehn surgeries along two closed orbits of the suspension of the hyperbolic toral automorphism induced by a matrix $A_{p,q,r} \in SL(2;\mathbf{Z})$.*

In some special cases, we can calculate $A_{p,q,r}$ by the same way of the above flat pillowcase case. There exist branched coverings $\Sigma_g \to S(2g+2, 2g+2, g+1)$ and $\Sigma_g \to S(2g+1, 2g+1, 2g+1)$. Since these branched covering preserve the geodesic flows, they are used to calculate $A_{2g+2,2g+2,g+1}$ and $A_{2g+1,2g+1,2g+1}$.

Theorem 2.2.

$$A_{2g+2,2g+2,g+1} = \begin{pmatrix} 2g^2 - 1 & g(g^2-1) \\ 4g & 2g^2 - 1 \end{pmatrix}$$

$$A_{2g+1,2g+1,2g+1} = \begin{pmatrix} 4g^2 - 2g - 1 & 2g(g-1)(2g+1) \\ 2(2g-1) & 4g^2 - 2g - 1 \end{pmatrix}$$

References

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