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Generalized Newton transformation and its applications to extrinsic geometry

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Analyzing the study of Riemannian geometry we see that its basic concepts are related with some operators, such as shape, Ricci, Schouten operator, etc. and functions constructed of them, such as mean curvature, scalar curvature, Gauss-Kronecker curvature, etc. The most natural and useful functions are the ones derived from algebraic invariants of these operators, e.g., by taking trace, determinant and in general the r-th symmetric functions σ_r. However, the case $r > 1$ is strongly nonlinear and therefore more complicated. The powerful tool to deal with this problem is the Newton transformation T_r of an endomorphism A (strictly related with the Newton's identities) which, in a sense, enables a linearization of σ_r,

$$(r+1)\sigma_{r+1} = \mathrm{tr}\,(AT_r).$$

Although this operator appeared in geometry many years ago (see, e.g., [21, 29]), there is a continues increase of applications of this operator in different areas of geometry in the last years (see, among others, [1, 2, 3, 8, 10, 17, 18, 23, 24, 25, 28]).

All these results cause a natural question, what happen if we have a family of operators i.e. how to define the Newton transformation for a family of endomorphisms. A partial answer to this question can be found in the literature (operator T_r and the scalar S_r for even r [5, 15]), nevertheless, we expect that this case is much more subtle. This is because in the case of family of operators we should obtain more natural functions as in the case of one operator and consequently more information about geometry. In order to do this, for any multi–index u and generalized elementary symmetric polynomial σ_u we introduce transformations depending on a system of linear endomorphisms. Since these transformations have properties analogous to the Newton transformation (and in the case of one endomorphism coincides with it) we call this new object *generalized Newton transformation* (GNT) and denote by T_u. The concepts of GNT is based on the variational formula for the r–th symmetric function

$$\frac{d}{d\tau}\sigma_{r+1}(\tau) = \mathrm{tr}\left(T_r \cdot \frac{d}{d\tau}A(\tau)\right),$$

Extended Abstract (joint paper with K. Andrzejewski and K. Niedzialomski) which is crucial in many applications and, as we will show, characterize Newton's transformations. Surprisingly enough, according to knowledge of the authors, GNT has never been investigated before.

1. Generalized Newton transformation (GNT)

Let A be an endomorphism of a p–dimensional vector space V. The *Newton transformation* of A is a system $T = (T_r)_{r=0,1,\dots}$ of endomorphisms of V given by the recurrence relations:

$$\begin{aligned} T_0 &= 1_V, \\ T_r &= \sigma_r 1_V - AT_{r-1}, \quad r = 1, 2, \dots \end{aligned}$$

Here σ_r's are elementary symmetric functions of A. If $r > p$ we put $\sigma_r = 0$. Equivalently, each T_r may be defined by the formula

$$T_r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j.$$

Observe that T_p is the characteristic polynomial of A. Consequently, by Hamilton–Cayley Theorem $T_p = 0$. It follows that $T_r = 0$ for all $r \geq p$.

The Newton transformation satisfies the following relations [21]:

(N1) Symmetric function σ_r is given by the formula

$$r\sigma_r = \operatorname{tr}(AT_{r-1}).$$

(N2) Trace of T_r is equal

$$\operatorname{tr} T_r = (p - r)\sigma_r.$$

(N3) If $A(\tau)$ is a smooth curve in $\operatorname{End}(V)$ such that $A(0) = A$, then

$$\frac{d}{d\tau}\sigma_{r+1}(\tau)_{\tau=0} = \operatorname{tr}\left(\frac{d}{d\tau}A(\tau)_{\tau=0} \cdot T_r\right), \quad r = 0, 1, \dots, p.$$

Condition (N3) is the starting point to define generalized Newton transformations.

Let V be a p–dimensional vector space (over $\mathbb{R}$) equipped with an inner product $\langle\, ,\rangle$. For an endomorphism $A \in \operatorname{End}(V)$, let $A^\top$ denote the adjoint endomorphism, i.e. $\langle Av, w\rangle = \langle v, A^\top w\rangle$ for every $v, w \in V$. The space $\operatorname{End}(V)$ is equipped with an inner product

$$\langle\!\langle A, B\rangle\!\rangle = \operatorname{tr}(A^\top B), \quad A, B \in \operatorname{End}(V).$$

Let $\mathbb{N}$ denote the set of nonnegative integers. By $\mathbb{N}(q)$ denote the set of all sequences $u = (u_1, \dots, u_q)$, with $u_j \in \mathbb{N}$. The length $|u|$ of $u \in \mathbb{N}(q)$

is given by $|u| = u_1 + \ldots + u_q$. Denote by $\mathrm{End}^q(V)$ the vector space $\mathrm{End}(V) \times \ldots \times \mathrm{End}(V)$ (q–times). For $\mathbf{A} = (A_1, \ldots, A_q) \in \mathrm{End}^q(V)$, $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$ and $u \in \mathbb{N}(q)$ put

$$t^u = t_1^{u_1} \ldots t_q^{u_q},$$
$$t\mathbf{A} = t_1 A_1 + \ldots + t_q A_q$$

By a *Newton polynomial* of $\mathbf{A}$ we mean a polynomial $P_{\mathbf{A}} : \mathbb{R}^q \to \mathbb{R}$ of the form $P_{\mathbf{A}}(t) = \det(1_V + t\mathbf{A})$. Expanding $P_{\mathbf{A}}$ we get

$$P_{\mathbf{A}}(t) = \sum_{|u| \leq p} \sigma_u t^u,$$

where the coefficients $\sigma_u = \sigma_u(\mathbf{A})$ depend only on $\mathbf{A}$. Observe that $\sigma_{(0,\ldots,0)} = 1$. It is convenient to put $\sigma_u = 0$ for $|u| > p$.

Consider the following (music) convention. For α we define functions $\alpha^\sharp : \mathbb{N}(q) \to \mathbb{N}(q)$ and $\alpha_\flat : \mathbb{N}(q) \to \mathbb{N}(q)$ as follows

$$\alpha^\sharp(i_1, \ldots, i_q) = (i_1, \ldots, i_{\alpha-1}, i_\alpha + 1, i_{\alpha+1}, \ldots, i_q),$$
$$\alpha_\flat(i_1, \ldots, i_q) = (i_1, \ldots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \ldots, i_q),$$

i.e. $\alpha^\sharp$ increases the value of the α–th element by 1 and $\alpha_\flat$ decreases the value of α–th element by 1. It is clear that $\alpha^\sharp$ is the inverse map to $\alpha_\flat$.

Now, we may state the main definition. The *generalized Newton transformation* of $\mathbf{A} = (A_1, \ldots, A_q) \in \mathrm{End}^q(V)$ is a system of endomorphisms $T_u = T_u(\mathbf{A})$, $u \in \mathbb{N}(q)$, satisfying the following condition (generalizing (N3)):

For every smooth curve $\tau \mapsto \mathbf{A}(\tau)$ in $\mathrm{End}^q(V)$ such that $\mathbf{A}(0) = \mathbf{A}$

$$\text{(GNT)} \qquad \begin{aligned} \frac{d}{d\tau}\sigma_u(\tau)_{\tau=0} &= \sum_\alpha \langle\!\langle \frac{d}{d\tau} A_\alpha(\tau)_{\tau=0})^\top | T_{\alpha_\flat(u)} \rangle\!\rangle \\ &= \sum_\alpha \mathrm{tr}\left(\frac{d}{d\tau} A_\alpha(\tau)_{\tau=0} \cdot T_{\alpha_\flat(u)} \right). \end{aligned}$$

From the above definition it is not clear that generalized Newton transformation exists. In order to show the existence of Generalized Newton transformation, we introduce the following notation.

For $q, s \geq 1$ let $\mathbb{N}(q, s)$ be the set of all $q \times s$ matrices, whose entries are elements of $\mathbb{N}$. Clearly, the set $\mathbb{N}(1, s)$ is the set of multi–indices $i = (i_1, \ldots, i_s)$ with $i_1, \ldots, i_s \in \mathbb{N}$, hence $\mathbb{N}(s) = \mathbb{N}(1, s)$. Moreover, every matrix $\mathbf{i} = (i_l^\alpha) \in \mathbb{N}(q, s)$ may be identified with an ordered system $\mathbf{i} = (i^1, \ldots, i^q)$ of multi–indices $i^\alpha = (i_1^\alpha, \ldots, i_s^\alpha)$.

If $i = (i_1, \ldots, i_s) \in \mathbb{N}(s)$ then its *length* is simply the number $|i| = i_1 + \ldots + i_s$. For $\mathbf{i} = (i^1, \ldots, i^q) \in \mathbb{N}(q, s)$ we define its *weight* as an multi–index $|\mathbf{i}| = (|i^1|, \ldots, |i^q|) \in \mathbb{N}(q)$. By the *length* $\|\mathbf{i}\|$ of $\mathbf{i}$ we mean the length of $|\mathbf{i}|$, i.e., $\|\mathbf{i}\| = \sum_\alpha |i^\alpha| = \sum_{\alpha,l} i_l^\alpha$.

Denote by $\mathbb{I}(q,s)$ a subset of $\mathbb{N}(q,s)$ consisting of all matrices $\mathbf{i}$ satisfying the following conditions:

(1) every entry of $\mathbf{i}$ is either 0 or 1,
(2) the length of $\mathbf{i}$ is equal to s,
(3) in every column of $\mathbf{i}$ there is exactly one entry equal to 1, or equivalently $|\mathbf{i}^\top| = (1,\dots,1)$.

We identify $\mathbb{I}(q,0)$ with a set consisting of the zero vector $0 = [0,\dots,0]^\top$.

Let $\mathbf{A} \in \operatorname{End}^q(V)$, $\mathbf{A} = (A_1,\dots,A_q)$, and $\mathbf{i} \in \mathbb{N}(q,s)$. By $\mathbf{A}^{\mathbf{i}}$ we mean an endomorphism (composition of endomorphisms) of the form

$$\mathbf{A}^{\mathbf{i}} = A_1^{i_1^1} A_2^{i_1^2} \dots A_q^{i_1^q} A_1^{i_2^1} \dots A_q^{i_2^q} \dots A_1^{i_s^1} \dots A_q^{i_s^q}.$$

In particular, $\mathbf{A}^0 = 1_V$.

Theorem 1.1. *For every system of endomorphisms* $\mathbf{A} = (A_1,\dots,A_q)$, *there exists unique generalized Newton transformation* $T = (T_u : u \in \mathbb{N}(q))$ *of* $\mathbf{A}$. *Moreover, each* T_u *is given by the formula*

$$T_u = \sum_{s=0}^{|u|} \sum_{\mathbf{i}\in\mathbb{I}(q,s)} (-1)^{\|\mathbf{i}\|} \sigma_{u-|\mathbf{i}|} \mathbf{A}^{\mathbf{i}}, \tag{1.2}$$

where $\sigma_{u-|\mathbf{i}|} = \sigma_{u-|\mathbf{i}|}(\mathbf{A})$.

As a consequence of above theorem we obtain:

Theorem 1.3 (Generalized Hamilton–Cayley Theorem)**.** *Let* $T = (T_u : u \in \mathbb{N}(q))$ *be the generalized Newton transformation of* $\mathbf{A}$. *Then for every* $u \in \mathbb{N}(q)$ *of length greater or equal to* p *we have* $T_u = 0$.

Moreover the generalized Newton transformation $T = (T_u : u \in \mathbb{N}(q))$ of $\mathbf{A}$ satisfies the following recurrence relations:

Theorem 1.4.

$$T_0 = 1_V, \qquad \text{where } 0 = (0,\dots,0), \tag{1.5}$$

$$\begin{aligned} T_u &= \sigma_u 1_V - \sum_\alpha A_\alpha T_{\alpha_\flat(u)} \\ &= \sigma_u 1_V - \sum_\alpha T_{\alpha_\flat(u)} A_\alpha, \end{aligned} \qquad \text{where } |u| \geq 1. \tag{1.6}$$

2. Applications to extrinsic geometry

Let (M, g) be an oriented Riemannian manifold, D a p–dimensional (transversally oriented) distribution on M. Let q denotes the codimension of D. For each $X \in T_xM$ there is unique decomposition

$$X = X^\top + X^\perp,$$

where $X^\top \in D_x$ and $X^\perp$ is orthogonal to D_x. Denote by $D^\perp$ the bundle of vectors orthogonal to D. Let ∇ be the Levi–Civita connection of g. ∇ induces connections $\nabla^\top$ and $\nabla^\perp$ in vector bundles D and $D^\perp$ over M, respectively.

Let $\pi : P \to M$ be the principal bundle of orthonormal frames (oriented orthonormal frames, respectively) of $D^\perp$. Clearly, the structure group G of this bundle is $G = O(q)$ ($G = SO(q)$, respectively).

Every element $(x, e) = (e_1, \ldots, e_q) \in P_x$, $x \in M$, induces the system of endomorphisms $\mathbf{A}(x, e) = (A_1(x, e), \ldots, A_q(x, e))$ of D_x, where $A_\alpha(x, e)$ is the shape operator corresponding to (x, e), i.e.

$$A_\alpha(x, e)(X) = -\left(\nabla_X e_\alpha\right)^\top, \quad X \in D_x.$$

Let $T(x, e) = (T_u(x, e))_{u \in \mathbb{N}(q)}$ be the generalized Newton transformation associated with $\mathbf{A}(x, e)$.

The bundle $\pi : P \to M$ and the vector bundles $TM \to M$, $D \to M$, $D^\perp \to M$ induce the pull–back bundles

$$E = \pi^{-1}TM, \quad E' = \pi^{-1}D \quad \text{and} \quad E'' = \pi^{-1}D^\perp \quad \text{over } P,$$

each with a fiber $(\pi^{-1}TM)_{(x,e)} = T_xM$, $(\pi^{-1}D)_{(x,e)} = D_x$ and $(\pi^{-1}D^\perp)_{(x,e)} = D^\perp_x$, respectively. We have

$$E = E' \oplus E''.$$

Moreover, the connections $\nabla, \nabla^\top, \nabla^\perp$ of g induce pull–back connections $\nabla^E, \nabla^{E'}$ and $\nabla^{E''}$ in E, E' and E'', respectively.

Define the section $Y_u \in \Gamma(E)$, $u \in \mathbb{N}(q)$ as follows

$$Y_u(x, e) = \sum_{\alpha,\beta} T_{\beta_\flat \alpha_\flat(u)}(x, e)(\nabla_{e_\alpha} e_\beta)^\top + \sum_\alpha \sigma_{\alpha_\flat(u)}(x, e) e_\alpha. \tag{2.1}$$

Observe that the first component of Y_u is a section of E', whereas the second component is a section of E''. The section Y_u and the vector field $\widehat{Y_u} \in \Gamma(TM)$ obtained from Y_u by integration on the fibers of P play a fundamental role in our considerations.

Lemma 2.2. *The divergence of Y_u is given by the formula*

$$\mathrm{div}_E Y_u = -|u|\sigma_u + \sum_{\alpha,\beta}\Big[\mathrm{tr}\,(R_{\alpha,\beta}T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'}T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha}e_\beta)^\top)$$
$$- g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\beta)^\top) + \sum_\gamma g((\nabla_{e_\alpha}e_\gamma)^\top, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\gamma}e_\beta)^\top)\Big],$$

where $H_{D^\perp}$ denotes the mean curvature vector of the distribution $D^\perp$.

Put

$$\widehat{\sigma_u}(x) = \int_{P_x}\sigma_u(x,e)\,de = \int_G \sigma_u(x,e_0a)\,da, \tag{2.3}$$

where (x,e_0) is a fixed element of P_x. We call $\widehat{\sigma_u}$'s *extrinsic curvatures* of a distribution D. Moreover, we define *total extrinsic curvatures*

$$\sigma_u^M = \int_M \widehat{\sigma_u}(x)\,dx. \tag{2.4}$$

Since the projection π in the bundle P is a Riemannian submersion, then by Fubini theorem

$$\sigma_u^M = \int_P \sigma_u(x,e)\,d(x,e).$$

Theorem 2.5. *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, the total extrinsic curvature σ_u^M satisfies*

(2.6)
$$|u|\sigma_u^M = \sum_{\alpha,\beta}\int_P \Big(\mathrm{tr}\,(R_{\alpha,\beta}T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'}T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha}e_\beta)^\top)$$
$$- g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\beta)^\top) + \sum_\gamma g((T^*_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\gamma)^\top, (\nabla_{e_\gamma}e_\beta)^\top)\Big),$$

where $H_{D^\perp}$ denotes the mean curvature vector of distribution $D^\perp$.

By Theorem 2.5, we have in particular

$$\sigma^M_{\alpha^\sharp(0,\ldots,0)} = 0$$

and

$$2\sigma^M_{\alpha^\sharp\beta^\sharp(0,\ldots,0)} = \int_P \Big((\mathrm{Ric}_D)_{\alpha,\beta} - g(H_{D^\perp}, (\nabla_{e_\alpha}e_\beta)^\top)$$
$$+ \sum_\gamma g((\nabla_{e_\alpha}e_\gamma)^\top, (\nabla_{e_\gamma}e_\beta)^\top)\Big), \tag{2.7}$$

where $(\mathrm{Ric}_D)_{\alpha,\beta} = \mathrm{Ric}_D(e_\alpha, e_\beta)$ and Ric_D is the Ricci curvature operator in the direction of D, i.e.,

$$\mathrm{Ric}_D(X,Y) = \sum_i g(R(f_i, X)Y, f_i),$$

where (f_i) is an orthonormal basis of D.

If D is integrable i.e. D defines a foliation $\mathcal{F}$ then above theorems generalized some well known facts:

Corollary 2.8. *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, total extrinsic curvature σ_u^M of a distribution D with totally geodesic normal bundle is of the form*

$$|u|\sigma_u^M = \sum_{\alpha,\beta} \int_P \mathrm{tr}\,(R_{\alpha,\beta} T_{\beta_\flat \alpha_\flat(u)}).$$

Corollary 2.9. *Assume (M,g) is closed and of constant sectional curvature κ. Let $\mathcal{F}$ be a foliation on M with totally geodesic and integrable normal bundle $\mathcal{F}^\perp$. Then the total extrinsic curvatures of $\mathcal{F}$ depend on κ, the volume of M and the dimension of $\mathcal{F}$ only.*

Moreover we may also obtain formulae of Brito and Naveira [13] for mean extrinsic curvature S_r

$$\int_M S_r = \begin{cases} \binom{\frac{p}{2}}{\frac{r}{2}} \binom{q+r-1}{r} \binom{\frac{q+r-1}{2}}{\frac{r}{2}}^{-1} \kappa^{\frac{r}{2}} \mathrm{vol}(M) & \text{for } p \text{ even and } q \text{ odd} \\ 2^r \left(\left(\frac{r}{2}\right)!\right)^{-1} \binom{\frac{q}{2}+\frac{r}{2}-1}{\frac{r}{2}} \binom{\frac{p}{2}}{\frac{r}{2}} \kappa^{\frac{r}{2}} \mathrm{vol}(M) & \text{for } p \text{ and } q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

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