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Application of a new integral formula for two distributions with singularities on Riemannian manifolds

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1. The Volume of a Tube

1.1. The second fundamental forms of the tubular hypersurfaces

Definition 1.1. Let P be a topologically embedded sub-manifold (possibly with boundary) in a Riemannian manifold M, then a **tube** $T(P, r)$ of radius $r \geq 0$ about P is the set

$$T(P,r) = \{m \in M : \text{there exists a geodesic } \xi \text{ of length } L(\xi) \leq r \text{ from } m \text{ meeting } P \text{ orthogonally}\}. \tag{1.2}$$

We shall also need a notation closely related to that of tube.

Definition 1.3. We call a hypersurface of the form

$$P_t = \{m \in T(P,r) : \text{distance}(m, P) = t\}$$

the tubular hypersurface at distance t from P.

For $0 < t \leq r$ the tubular hypersurfaces P_t form a natural foliation of the tubular region $T(P, r) - P$.

1.2. The volume of a tube in terms of the infinitesimal change of volume function

We assume that P is topologically embedded submanifold with compact closure of a complete Riemannian manifold M. For all $r \geq 0$ both $T(P, r)$ and P_r are measurable sets. Let

$$\mathrm{V}_P^M(r) = \text{the } n-\text{dimensional volume of } T(P,r),$$

$$\mathrm{A}_P^M(r) = \text{the } (n-1)-\text{dimensional volume of } P_r.$$

It is easy to show that $A_P^M(r)$ is the derivative of $V_P^M(r)$. We use the lemma:

Lemma 1.4. *Suppose that* $\exp_\nu : \{(p,v) \in \nu : \| v \| \leq r\} \longmapsto T(P,r)$ *is a diffeomorphism. Then*

$$A_P^M(r) = r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \mathcal{V}_u(r) du dP.$$

Lemma 1.5. *Suppose that* $\exp_\nu : \{(p,v) \in \nu : \| v \| \leq r\} \longmapsto T(P,r)$ *is a diffeomorphism. Then*

$$\begin{aligned} \frac{d}{dr} V_P^M(r) &= A_P^M(r) \\ &= r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \mathcal{V}_u(r) du dP. \end{aligned}$$

Proofs of this lemmas are available at [3].

2. Riemannian manifolds with singularities

In this section we work with Riemannian geometry of manifolds equipped with a pair of orthogonal plane fields. We want to generalize it to the case of plane fields with singularities, that is defined on a compact manifold except of singular set, the union of submanifolds of lower dimension. Till now, author produced a new integral formula (see[4]) obtained from integration of the divergence of a vector field built from Newton transforms of Weingarten operators applied to the mean curvature vectors of the plane fields under consideration. This formula, in a sense, analogous to the one obtained by Walczak in the 1990 [5].

We get reasonable applications of this formulae leading to provide obstructions for the existence of geometric structures - here, pairs of distributions - satisfying some geometric conditions (for example: being totally geodesic, minimal, umbilical and so on) on some special (locally symmetric, of constant curvature, positively/negatively curved and so on) Riemannian manifolds.

Let M be a Riemannian manifold, $\dim M \geq 3$, equipped with two complementary distributions D_1 and D_2. We assume that

$$p + q = n, \quad \text{where } p = \dim D_1, \; q = \dim D_2 \text{ and } n = \dim M.$$

Let us take a local orthonormal frame $e_1, \ldots, e_n$ adapted to D_1 and D_2, i.e., we assume that e_i is tangent to D_1 for $i = 1, \ldots, p$ and e_α is tangent to D_2 for $\alpha = p+1, \ldots, n$.
The second fundamental forms B_m of D_m ($m = 1, 2$) are defined as follows:

$$B_1(X_1, Y_1) = \frac{1}{2}(\nabla_{X_1} Y_1 + \nabla_{Y_1} X_1)^\perp, \quad B_2(X_2, Y_2) = \frac{1}{2}(\nabla_{X_2} Y_2 + \nabla_{Y_2} X_2)^\top$$

for vector fields X_m and Y_m tangent to D_m.

The integrability tensors T_m of D_m $(m = 1, 2)$ are defined as follows:

$$T_1(X_1, Y_1) = \frac{1}{2}[X_1, Y_1]^{\perp}, \quad T_2(X_2, Y_2) = \frac{1}{2}[X_2, Y_2]^{\top}$$

for vector fields X_m and Y_m tangent to D_m.

Then the mean curvature vectors H_m of D_m are given by

$$H_1 = TraceB_1 = \sum_i B_1(e_i, e_i) = \sum_i (\nabla_{e_i} e_i)^{\perp}$$
$$H_2 = TraceB_2 = \sum_\alpha B_2(e_\alpha, e_\alpha) = \sum_i (\nabla_{e_\alpha} e_\alpha)^{\top}.$$

Let us define the Weingarten operators by

$$A_1 : D_1 \times D_2 \to D_1, \qquad \big\langle A_1(X, N), Y \big\rangle = \big\langle B_1(X, Y), N \big\rangle \quad for\ X, Y \in D_1,\ N \in D_2$$
$$A_2 : D_2 \times D_1 \to D_2, \qquad \big\langle A_2(X', N'), Y' \big\rangle = \big\langle B_2(X', Y'), N' \big\rangle \quad for\ X', Y' \in D_2,\ N' \in D_1.$$

Assume now that M has *bounded geometry* (i.e., bounded sectional curvature and injectivity radii r_x, $x \in M$, separated away from zero). Let A be a finite set of singularities(points, closed curve, etc.) on M and codimension$A = n - 2$. Moreover, let $f : M/A \to [0, +\infty)$ be a function defined on M outside a finite set A.

We denote the tube $T(A, r)$ of radius $r \geq 0$ about set A by $N_A(r)$ and $\delta N_A(r)$ as the tubular hypersurface at a distance $r \geq 0$ from A.

We shall also need an one the well-known formula volume$\delta N_\gamma(r) \backsimeq L(\gamma) \cdot$ volume$S^{n-1}(r)$, where $\gamma \subset A$ is closed curve, $L(\gamma)$ is length of the the curve γ and $S^{n-1}(r) \subset R^n$ is sphere of radius r. In particular:

- in $\mathbf{R}^2$ we obtain volume $\delta N_\gamma(r) = 2r \cdot L(\gamma)$
- in $\mathbf{R}^3$ we obtain volume $\delta N_\gamma(r) = \pi r^2 \cdot L(\gamma)$

It leads to the following and useful lemma.

Lemma 2.1.

$$If \ \liminf_{r \to 0^+} \int_{\delta N_\gamma(r)} f > 0, then \ \int_M f^2 = 0.$$

These lemma will be used extensively and will allow us to proof the following theorems.

Theorem 2.2. *Let M being a compact Riemannian manifold of dimension $n \geq 3$ and A a finite subset of M. If $\int_M \parallel H_1 \parallel < \infty$ and $\int_M \parallel H_2 \parallel < \infty$ then*

$$\int_M ||B_1||^2 + ||B_2||^2 - ||H_1||^2 - ||H_2||^2 - ||T_1||^2 - ||T_2||^2 = \int_M K(D_1, D_2), \tag{2.3}$$

where $K(D_1, D_2)$ is a generalization on the Ricci curvature equal to the sum

$$\sum_{i,\alpha} < R(e_i, e_\alpha)e_\alpha, e_i >$$

and called the mixed scalar curvature.

Theorem 2.4. *Let M being a compact Riemannian manifold of dimension $n \geq 3$ and A a finite subset of M. If $\int_M \parallel A_1 \parallel < \infty$ and $\int_M \parallel A_2 \parallel < \infty$ then*

$$\begin{aligned}
&\int_M \langle Ric(H_2), H_1 \rangle = \\
&\int_M \langle H_1, (\nabla_{H_2} H_1)^\perp \rangle + \langle H_2, (\nabla_{H_1} H_2)^\top \rangle + \\
&\langle Tr^\perp (\nabla_\bullet T_1)(\bullet, H_2), H_1 \rangle + \langle Tr^\top (\nabla_\bullet T_2)(\bullet, H_1), H_2 \rangle + \\
&\langle A_1^{H_1}, \nabla_\bullet^\top H_2 \rangle + \langle A_2^{H_2}, \nabla_\bullet^\perp H_2 \rangle + \\
&\sum_i \langle A_1 (H_2, (\nabla_{e_i} H_1)^\perp), e_i \rangle + \sum_\alpha \langle A_2 (H_1, (\nabla_{e_\alpha} H_2)^\top), e_\alpha \rangle + \\
&2 \sum_i \langle (\nabla_{T_1(e_i, H_2)} e_i)^\perp, H_1 \rangle + 2 \sum_\alpha \langle (\nabla_{T_2(e_\alpha, H_1)} e_\alpha)^\top, H_2 \rangle - \\
&\langle A_2(H_1, H_2), H_1 \rangle - \langle A_1(H_2, H_1), H_2 \rangle.
\end{aligned} \tag{2.5}$$

Corollary 2.6. *Equality (2.3) holds if and only if $K(D_1, D_2) > 0$.*

Proposition 2.7. *If distributions D_1 and D_2 are totally geodesic and D_2 is the orthogonal complement of D_1, then $H_1 = 0$ and $H_2 = 0$ and we get*

$$\int_M K(D_1, D_2) = \int_M \left(||T_1||^2 + ||T_2||^2 \right),$$

where H_m and T_m (m=1, 2) denote mean curvature vectors and integrability tensors of distributions D_m, respectively.

References

[1] F.G.B. Brito, P.G. Walczak *On the energy of unit vector fields with isolated singularities*, Annales Polonici Mathematici Vol. **73** (2000), 269–273

[2] W. Fenchel *Über die Krümmung und Windung geschlossen Raumkurven*, Math. Ann. **101** (1929), 238–252

[3] A. Gray *Tubes* , Advanced Book Program, 44–46

[4] M. Łużyńczyk *New integral formulae for two complementary orthogonal distributions on Riemannian manifolds*, Preprint Faculty of Mathematics and Computer Science University of Lodz (2012)

[5] P.G. Walczak, *An integral formula for Riemannian manifold with two orthogonal complementary distributions*, Coll. Math. Vol. **LVIII** (1990), 243–252.

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