Geometry and Foliations 2013
Komaba, Tokyo, Japan

Dehn surgeries along $(-2, 3, 2s + 1)$-type Pretzel knot with no $\mathbb{R}$-covered foliation and left-orderable groups

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1. Main theorem and its background

A codimension one, transversely oriented foliation $\mathcal{F}$ on a closed 3-manifold M is called a Reebless foliation if $\mathcal{F}$ does not contain a Reeb component. By the theorems of Novikov, Rosenberg, and Palmeira, if M is not homeomorphic to $S^2 \times S^1$ and contains a Reebless foliation, then M has properties that the fundamental group of M is infinite, the universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$ and all leaves of its lifted foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ are homeomorphic to a plane. In this case we can consider a quotient space $\mathcal{T} = \widetilde{M}/\widetilde{\mathcal{F}}$, and $\mathcal{T}$ is called a leaf space of $\mathcal{F}$. The leaf space $\mathcal{T}$ becomes a simply connected 1-manifold, but it might be a non-Hausdorff space. If the leaf space is homeomorphic to $\mathbb{R}$, $\mathcal{F}$ is called an $\mathbb{R}$-covered foliation. The fundamental group $\pi_1(M)$ of M acts on the universal cover $\widetilde{M}$ as deck transformations. Since this action maps a leaf of $\widetilde{\mathcal{F}}$ to a leaf, it induces an action of $\pi_1(M)$ on the leaf space $\mathcal{T}$. In fact, it is known that the action has no global fixed point and it acts on $\mathcal{T}$ as a homeomorphism.

In 2004, J.Jun proved the following theorem.

Theorem 1.1. (J. Jun [10, Theorem 2]) *Let K be a $(-2, 3, 7)$-Pretzel knot in S^3 and $E_K(p/q)$ be a closed manifold obtained by Dehn surgery along K with slope p/q. If $p/q \geqq 10$ and p is odd, then $E_K(p/q)$ does not contain an $\mathbb{R}$-covered foliation.*

Dehn surgery along a knot K in S^3 is a procedure which yields a new closed 3-manifold by digging a solid torus along the knot K and successively attaching a solid torus non-trivially along its boundary. The resultant manifold is determined by the knot K and a rational number ρ which represents a slope of meridian of the attaching solid torus on the boundary torus of the digged sphere. For basic definition and properties of Dehn surgery, see Boyer [1].

We proved the following theorem in [14] which is an extension of Theorem 1.1 to the case of $(-2, 3, 2s + 1)$-type Pretzel knot ($s \geqq 3$).

Theorem 1.2. (Main Theorem) *Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot in S^3 ($s \geqq 3$). If $q > 0$, $p/q \geqq 4s+7$ and p is odd, then $E_{K_s}(p/q)$ does not contain an $\mathbb{R}$-covered foliation.*

In [16], Roberts, Shareshian and Stein proved that there exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain a Reebless foliation. J.Jun also proved in [10] conditions for Dehn surgery slopes that $(-2, 3, 7)$-Pretzel knot K yields closed 3-manifolds which do not contain a Reebless foliation. In [6], Fenley showed that there exist infinitely many closed hyperbolic 3-manifolds which do not admit essential laminations.

These theorems are proved by a similar strategy as follows. Let M be a closed 3-manifold and $\mathcal{F}$ be a Reebless foliation in M. Then, the fundamental group $\pi_1(M)$ acts on the leaf space $\mathcal{T}$ of $\mathcal{F}$ as an orientation preserving homeomorphism which has no global fixed point. By the theorem of Palmeira, $\mathcal{F}$ is determined by its leaf space $\mathcal{T}$. Therefore, for any simply connected 1-manifold $\mathcal{T}$, if there exists a point of $\mathcal{T}$ which is fixed by any action of $\pi_1(M)$ then M cannot contain a Reebless foliation.

In order to use above method to prove our main theorem, we will need an explicit presentation of the fundamental group. Moreover, it is better for proving our theorem that its presentation has simpler form because our investigation of existence of a global fixed point becomes easy if its presentation has fewer generators.

In the next section, we will explain how to get a good presentation of the fundamental group of closed 3-manifold obtained by Dehn surgery along our Pretzel knots K_s. We do not explain the proof of Main theorem here, please see [14].

2. A good presentation of fundamental groups

In the proof of [10], Jun uses the presentation of a knot group of $(-2, 3, 7)$-pretzel knot which obtained by the computer program, SnapPea [18]. Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot in S^3. In order to obtain a good presentation of the knot group of K_s and its meridian-longitude pair, we take the following procedure.

We first notice that K_s is a tunnel number one knot for all $s \geqq 3$ by the theorem of Morimoto, Sakuma and Yokota [13]. A knot K is called a tunnel number one knot if there is an arc τ in S^3 which intersects K only on its endpoints and the closure of $S^3 \setminus (K \cup \tau)$ is homeomorphic to a genus two handlebody. Therefore the knot group of K_s can have a presentation which has two generators and one relator.

It is well known that two groups G and G' are isomorphic if there is a sequence of Tietze transformations such that a presentation of G is transformed into its of G' along this sequence. Although it is generally difficult to find such a sequence, we can find the required sequence by applying the procedure which appeared in the paper of Hilden, Tejada and Toro [7] as follows. At the first step, we obtain the Wirtinger presentation G_1 of the knot K. Then we collapse one crossing of the knot diagram and get a graph Γ which is thought as a resulting object $K \cup \tau$ because the exteriors of Γ and $K \cup \tau$ in S^3 are homeomorphic. We modify Γ with local moves in sequence forward to the shape $S^1 \vee S^1$, and in the same time we modify the presentations by a Tietze transformation which corresponds to each local move. In the sequel we finally obtain the graph which is homeomorphic to $S^1 \vee S^1$ and the corresponding presentation which has two generators and one relator.

In order to apply this procedure to the case of K_s, we add some new local moves which are not treated in [7], and we refer the sequence of modifications which appeared in the paper of Kobayashi [11] to obtain our sequence of modifications. Then we obtain the following presentation.

$$G_{K_s} = \pi_1(S^3 \setminus K_s) = \langle c, l \mid cl c\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1}\rangle$$

In order to obtain a presentation of $G_{K_s}(p,q) = \pi_1(E_{K_s}(p/q))$, we have to get a presentation of a meridian-longitude pair. The way of the calculation is as follows. We first fix a meridian c and get a presentation of a longitude L_1 which are compatible with the Wirtinger presentation by using the method which appeared in the book of Burde and Zieschang [3]. Then we continue to modify L_i from $i = 1$ along the steps of sequence of Tietze transformations.

By modifying the last presentation of the longitude slightly, we finally obtain the following presentation.

$$L = \bar{c}^{2s-2}lcl^s cl^s cl\bar{c}^{2s+9}$$

In summary we obtain the following.

Proposition 2.1. *Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot ($s \geqq 3$). Then the knot group of K_s has a presentation*

$$G_{K_s} = \langle c, l \mid clc\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1}\rangle,$$

and an element which represents the meridian M is c and an element of the longitude L is $\bar{c}^{2s-2}lcl^s cl^s cl\bar{c}^{2s+9}$.

By Proposition 2.1, we obtain a presentation of $G_{K_s}(p,q)$ as follows:

$$G_{K_s}(p,q) = \langle c, l \mid clc\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1}, M^pL^q\rangle.$$

3. Problems and related topics

We will discuss some related topics and problems in this section.

We first mention about future problems. In Theorem 1.2 we explore the case when the leaf space $\mathcal{T}$ is homeomorphic to $\mathbb{R}$. It is the first problem that we extend Theorem 1.2 to the general case that a leaf space $\mathcal{T}$ is homeomorphic to a simply connected 1-manifold which might not be a Hausdorff space similar to the result of Jun [10]. In this case, the situation of a leaf space $\mathcal{T}$ is very complicated. But we already obtained the explicit presentation of the fundamental group of $E_{K_s}(p/q)$, we are going to investigate the action of the fundamental group to a leaf space referring the discussions used in [16] and [10]. One of other directions of investigations is that we extend Theorem 1.2 to the case for $(-2, 2r+1, 2s+1)$-type Pretzel knot $K_{r,s}$ ($r \geqq 1$, $s \geqq 3$). Although we need the explicit presentation of the fundamental group $\pi_1(S^3 \setminus K_{r,s})$, A.Tran already presented the presentation of $\pi_1(S^3 \setminus K_{r,s})$ with two generators and one relator in [17]. Using this presentation we are going to calculate the meridian-longitude pair which compatible with this presentation, and also extend Theorem 1.2 to the case of $K_{r,s}$ cooperating with him.

Next we discuss about some related topics. We first discuss our result in the viewpoint of Dehn surgery on knots. A knot K in S^3 has a finite or cyclic surgery if the resultant manifold $E_K(p/q)$ obtained by a non-trivial Dehn surgery along K with a slope p/q has a property that its fundamental group is finite or cyclic respectively. Determining and classifying which knots and slopes have a finite or cyclic surgery are an interesting problem. If $E_K(p/q)$ contains a Reebless foliation, we can conclude that $E_K(p/q)$ does not have a finite and cyclic surgery. For example, Delman and Roberts showed that no alternating hyperbolic knot admits a non-trivial finite and cyclic surgery by proving the existence of essential laminations [5]. Our Pretzel knots K_s are in the class of a Montesinos knot. In [9], Ichihara and Jong showed that for a hyperbolic Montesinos knot K if K admits a non-trivial cyclic surgery it must be $(-2, 3, 7)$-pretzel knot and the surgery slope is 18 or 19, and if K admits a non-trivial acyclic finite surgery it must be $(-2, 3, 7)$-pretzel knot and the slope is 17, or $(-2, 3, 9)$-pretzel knot and the slope is 22 or 23. In contrast, by this theorem, infinitely many knots in the family of pretzel knot $\{K_s\}$ which appeared in Theorem 1.2 do not admit cyclic or finite surgery. Then we have following corollary directly.

Corollary 3.1. *There are infinitely many pretzel knots which does not admit finite or cyclic surgery, but they admit Dehn surgery which produces a closed manifold which cannot contain an $\mathbb{R}$-covered foliation.*

We had expected that proving the existence of Reebless foliations, especially $\mathbb{R}$-covered foliations, or essential laminations is of use for determining and classifying a non-trivial finite or cyclic surgery on other hyperbolic knots in the same way as [5], but Corollary 3.1 means that in the case of pretzel knots, an $\mathbb{R}$-covered foliation is not of use for it. However, we think there are another applications of non-existence of an $\mathbb{R}$-covered foliation, an approach of Cosmetic surgery conjecture [12] (or see [8]) as an example.

Next we discuss our result in the viewpoint of a left-orderable group. A group G is left-orderable if there exists a total ordering $<$ of the elements of G which is left invariant, meaning that for any elements f, g, h of G, if $f < g$ then $hf < hg$. It is known that a countable group G is left-orderable if and only if there exists a faithful action of G on $\mathbb{R}$, that is, there is no point of $\mathbb{R}$ which fixed by any element of G. By this fact, if a closed 3-manifold M contains an $\mathbb{R}$-covered foliation, the fundamental group of M is left-orderable. The fundamental groups $G_{K_s}(p, q)$ which satisfy the assumptions of Theorem 1.2 do not have a faithful action on $\mathbb{R}$ by the proof of Theorem 1.2. Therefore we conclude the following corollary:

Corollary 3.2. *Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot in S^3 $(s \geqq 3)$, $G = G_{K_s}(p, q)$ denotes the fundamental group of the closed manifold which obtained by Dehn surgery along K_s with slope p/q. If $q > 0$, $p/q \geqq 4s + 7$ and p is odd, G is not left-orderable.*

Roberts and Shareshian generalize the properties of the fundamental groups treated in [16]. They present conditions when the fundamental groups of a closed manifold obtained by Dehn filling of a once punctured torus bundle is not right-orderable [15, Corollary 1.5]. These are examples of hyperbolic 3-manifolds which has non right-orderable fundamental groups.

Clay and Watson showed the following theorem.

Theorem 3.3. (A. Clay, L. Watson, 2012, [4, Theorem 28]) *Let K_m be a $(-2, 3, 2m + 5)$-type Pretzel knot. If $p/q > 2m + 15$ and $m \geqq 0$, the fundamental group $\pi_1(E_{K_m}(p/q))$ is not left-orderable.*

By the fact mentioned before, these fundamental groups do not have a faithful action on $\mathbb{R}$, then these $E_{K_m}(p/q)$ do not admit an $\mathbb{R}$-covered foliation. Although the method of the proof of Theorem 3.3 is different from our strategy, it concludes a stronger result than ours in the sense of an estimation of surgery slopes. By the aspects getting from these results, there are many interaction between a study of $\mathbb{R}$-covered foliations and a study of left-orderability of the fundamental group of a closed 3-manifold, so we think that these objects will be more interesting.

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