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Foliations via frame bundles

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1. Introduction

Let (M, g) be a Riemannian manifold. Denote by $L(M)$ and $O(M)$ frame and orthonormal frame bundles over M, respectively. We consider on M the Levi–Civita connection. We can equip these bundles with a Riemannian metric such that the projection $\pi : L(M) \to M$ ($\pi : O(M) \to M$, respectively) is a Riemannian submersion. The classical example is the Sasaki–Mok metric [6, 1, 2]. There are many, so called natural, metrics considered by Kowalski and Sekizawa [3, 4, 5] and by the author [7]. Denote such fixed Riemannian metric by $\bar{g}$.

Assume M is equipped with k–dimensional foliation $\mathcal{F}$. Then $\mathcal{F}$ induces two subbundles $L(\mathcal{F})$ of $L(M)$ and $O(\mathcal{F})$ of $O(M)$ as follows

$$L(\mathcal{F}) = \{u = (u_1, \dots, u_n) \in L(M) \mid u_1, \dots, u_k \in T\mathcal{F}\},$$
$$O(\mathcal{F}) = \{u = (u_1, \dots, u_n) \in O(M) \mid u_1, \dots, u_k \in T\mathcal{F}\}.$$

Hence $L(\mathcal{F})$ and $O(\mathcal{F})$ are submanifolds of the Riemannian manifolds $(L(M), \bar{g})$ and $O(M), \bar{g})$, respectively.

2. Results

For simplicity denote by P the bundle $L(M)$ or $O(M)$ and by $P(\mathcal{F})$ the corresponding subbundle $L(\mathcal{F})$ or $O(\mathcal{F})$.

The objective is to state the correspondence between the geometry of a foliation $\mathcal{F}$ and the geometry of a submanifold $P(\mathcal{F})$ in P. The approach to the stated problem is the following.

1. The submanifold $P(\mathcal{F})$ of P is the subbbundle with the structure group H of matrices of the form
$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}.$$
This induces the vertical distribution of $P(\mathcal{F})$. The horizontal distribution is induced from the horizontal distribution of P. The aim is to obtain the correspondence between the horizontal lifts to $P(\mathcal{F})$ and P. It appears that it depends on the second fundamental form of $\mathcal{F}$.

2. The horizontal lift X^h to P and $X^{h,\mathcal{F}}$ to $P(\mathcal{F})$ and the formula for the Levi–Civita connection of $(P, \bar{g})$ imply the formula for the connection and second fundamental form of $P(\mathcal{F})$. The scope is to derive the explicit formula for these operators in terms of the connection on M and the second fundamental form of $\mathcal{F}$.
3. The formula for the second fundamental of $P(\mathcal{F})$ determines the extrinsic geometry of this submanifold. It appears that the conditions such as being totally geodesic, minimality, umbilicity of $P(\mathcal{F})$ are related with corresponding conditions of foliation $\mathcal{F}$. We state these correspondences.

3. Further research

The further research, initiated by the author, includes the following two problems:

1. Generalize the results to the case of a single manifold. More precisely, any submanifold N of M induces a subbundle $P(N)$ in $L(M)$ or $O(M)$. We may consider the geometry of the submanifold $P(N)$ and the relation with the geometry of N.
2. The subbundle $L(\mathcal{F})$ of $L(M)$ induced by the foliation $\mathcal{F}$ does not require the integrability of $\mathcal{F}$. Hence, we may consider $L(\mathcal{F})$ if $\mathcal{F}$ is non–integrable distribution. In particular, we may choose $\mathcal{F}$ to be the horizontal distribution $\mathcal{H}^\varphi$ of any submersion $\varphi : M \to N$. Therefore we may lift φ to a map $L\varphi : L(\mathcal{H}^\varphi) \to L(N)$ and study the geometry of $L\varphi$. Partial results of the author show that horizontal conformality of $L\varphi$ is equivalent to horizontal conformality of φ under some additional conditions (such as the restriction on $\bar{g}$).

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