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Random circle diffeomorphisms

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1. Introduction

Since the early work of M. Herman [1], we have a very good understanding of the typical features, in the Baire sense, of C^r-diffeomorphisms of the circle $\mathbf{S}^1$.

Here, we propose to depict the landscape in $\mathrm{Diff}_+^1(\mathbf{S}^1)$ from a *measurable* point of view. In fact it is possible to consider a very natural probability measure on the group of C^1-diffeomorphisms of the circle, first introduced by P. Malliavin and E.T. Shavgulidze [3, 5].

Definition 1.1. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and let $(B_s)_{s\in[0,1]}$ be a Brownian bridge on Ω, and $\lambda \in [0,1[$ a uniform random variable independent of B. Then for any $\sigma > 0$, it is possible to define the following random variable f taking values in the space $\mathrm{Diff}_+^1(\mathbf{S}^1)$:

$$f(t) = f_\sigma(t) = \frac{\int_0^t e^{\sigma B_s} ds}{\int_0^1 e^{\sigma B_s} ds} + \lambda\,.$$

The law of f_σ defines a probability Radon measure μ_σ on $\mathrm{Diff}_+^1(\mathbf{S}^1)$ that is called the *Malliavin-Shavgulidze measure.*

The very important property of Malliavin-Shavgulidze measures μ_σ is that they are *Haar-like*: every μ_σ is quasi-invariant under left translations by elements belonging to a dense subgroup (of zero measure!). This implies in particular that every open set has positive measure. We recall that a measure is *quasi-invariant* under a group action if the image of any positive measure set has positive measure.

Theorem 1.2 (Shavgulidze [5])**.** *For any $\sigma > 0$ the measure μ_σ is quasi-invariant under the regular left action of the group of C^1-diffeomorphisms φ with bounded second derivative. When φ is a C^3 diffeomorphism, the Radon-Nykodym cocycle takes the following form:*

$$\frac{d(L_\varphi)_*(\mu_\sigma)}{d\mu_\sigma}(f) = \exp\left\{\frac{1}{\sigma}\int_0^1 \mathcal{S}_\varphi(f(t))\, f'(t)^2\, dt\right\}, \tag{1.3}$$

where $\mathcal{S}_\varphi = D^2 \log D\varphi - \frac{1}{2}(D \log D\varphi)^2$ *denotes the* Schwarzian derivative *of* φ.

QUESTION 1.4. What is the geometrical meaning for the Schwarzian derivative appearing in the expression (1.3)?

By the definition of the measure μ_σ, it follows that the diffeomorphism f is a.s. $C^{1+\alpha}$-regular for any $\alpha < 1/2$. However, Df is a 1/2-Hölder function with probability 0; in particular Df is a.s. not a function of *bounded variation*. Such remarks are very interesting in a dynamical context: there are indeed many results for which the regularities $C^{1+1/2}$ or C^{1+bv} are sharp. Perhaps the first, very well-known example is A. Denjoy's theorem: any C^{1+bv}-circle diffeomorphism without periodic orbits is minimal. Another interesting example is the Godbillon-Vey cocycle gv: S. Hurder and A. Katok showed how to extend the classical definition of gv to $C^{1+1/2+\varepsilon}$-diffeomorphisms [2], but T. Tsuboi explained later that this definition cannot be pursued to lower regularity [6]. Using the Malliavin-Shavgulidze measures and the theory of stochastic integration, it is possible to define gv as an *essential cocycle* on $\mathrm{Diff}^1_+(\mathbf{S}^1)$. However, we do not know whether such cocycle, defined a.e., is not a coboundary (in this measurable setting).

From a dynamical point of view, Malliavin-Shavgulidze measures may allow to *quantify* already known results. For example:

QUESTION 1.5.

1. What is the μ_σ-probability that a diffeomorphism has no periodic orbit?
2. Suppose that for σ sufficiently small, the probability to have no periodic orbit is positive, what is the probability that a diffeomorphism without periodic orbits is not minimal?

Even though such questions arise quite naturally, it is interesting to remark that no other mathematician has thought before about such problems: the Malliavin-Shavgulidze measures were just considered as a very good tool to study the representation theory of the group of smooth circle diffeomorphisms or its Lie algebra. We are very deeply indebted to É. Ghys to have proposed these very beautiful questions.

2. Main results

Our main motivation is to find answers for questions 1.5. Although we have some intuition (and numerical simulations) of what the result should be, we are still rather far away. The following statements are intended as a little step to a global comprehension: we analyse the set of diffeomorphisms *with periodic orbits*. It is worth remarking that in the topological settings, J. Mather and J.-C. Yoccoz gave a very exhaustive description [7].

We define $F_{p/q}$ to be the set of C^1-diffeomorphisms f with rotation number p/q (this implies in particular that f possesses an orbit of period q). It is easy to see that $F_{p/q}$ has non-empty interior and hence positive μ_σ-measure: the subset of *hyperbolic* diffeomorphisms is open and dense. To this purpose, we recall that a diffeomorphism $f \in F_{p/q}$ is hyperbolic if there are finitely many periodic orbits and if $\mathcal{O} = \{x_0, \dots, x_{q-1}\}$ is any q-periodic orbit, its *multiplier*

$$M(\mathcal{O}) = Df(x_0) \cdots Df(x_{q-1})$$

is not equal to 1.

Theorem 2.1. *Let $M > 0$. Then the μ_σ-probability that f has a periodic orbit with multiplier equal to M is zero.*

Corollary 2.2. *The set of hyperbolic diffeomorphisms is of full μ_σ-measure in $F_{p/q}$. In particular a diffeomorphism f has a.s. finitely many periodic orbits and their number is even.*

Corollary 2.3. *The set of diffeomorphisms with trivial C^1-centralizer is of full μ_σ-measure in $F_{p/q}$.*

Actually, Mather and Yoccoz proved that the set of diffeomorphisms with trivial C^1-centralizer in $F_{p/q}$ contains an open dense set.

Proof of corollary 2.3. The proof highly relies on the fact that the random variable f is defined by means of a Brownian bridge.

Take $f \in F_{p/q}$ and suppose f is $C^{1+\alpha}$, for some $\alpha > 0$ and suppose additionally that f has finitely many periodic orbits, each of them hyperbolic. Then by a result of Mather, the C^1-centralizer of f is trivial if and only if its *Mather invariant* has trivial stabilizer. The crucial fact is that the Mather invariant is highly sensible to local modifications of f. This implies that the triviality of the stabilizer of the Mather invariant is an event that depends only on a germ (or tail) filtration of the Brownian bridge B defining f. Then, using *Blumenthal's* 0–1 *law*, we can deduce that such event has zero μ_σ-measure. □

Outline of the proof of theorem 2.1. In order to simplify the details, let us sketch the proof of the theorem for *interval diffeomorphisms*: given a Brownian motion $(B_t)_{t\in[0,1]}$, we can define the random variable f:

$$f(t) = \frac{\int_0^t e^{B_s} ds}{\int_0^1 e^{B_s} ds}.$$

We want to show that for a given $M > 0$, the probability that f has a fixed point t such that $f'(t) = M$ is zero.

It turns out that it is easier to prove that the planar process

$$\left(B_t, B_{f(t)}\right)_{t\in[0,1]}$$

hits the diagonal with zero probability. Indeed, when $f(t) > t$, the variables B_t and $B_{f(t)}$ are almost independent, since for defining $f(t)$, we only have to know how B behaves up to time t (and its geometric average $\int_0^1 e^{B_s} ds$): then, using the Markov property of the Brownian motion, we see that the value of B at time $f(t)$ can be arbitrary. It is actually well known that the planar Brownian motion almost never hits a given point: the tricky proof of this fact (borrowed from [4]) can be adapted to our case. When $f(t) < t$, we can use set $g(t) := 1 - f(1-t)$, which defines an interval diffeomorphism with the same law as f, so that we reduce to the first case. $\square$

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