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Groups of uniform homeomorphisms of covering spaces

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1. Introduction

The uniform topology is one of basic topologies on function spaces. In this note we report some results on local and global deformation properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric manifolds endowed with the uniform topology.

Our main goal is to understand local or global topological properties of groups of uniform homeomorphisms of metric manifolds endowed with the uniform topology (for example, local contractibility, homotopy type, local or global topological type as infinite-dimensional manifolds, etc). Since the notions of uniform continuity and uniform topology depend on the choice of metrics, we are also interested in dependence of those topological properties on the behavior of metrics in neighborhoods of ends of manifolds.

In [6] we studied the formal behaviour of local deformation property in the space of uniform embeddings and showed that this property is preserved by the restriction and union of domains of uniform embeddings. This observation reduces our problem to the study of simpler pieces. In [2] A.V. Černavskiĭ considered the case where the manifold M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. Recently, in [5] we treated the class of metric covering spaces over compact manifolds. In this case we can deduce a local deformation theorem for uniform embeddings from the Edwards-Kirby local deformation theorem for embeddings of compact spaces and the classical Arzela-Ascoli theorem for equi-continuous families of maps ([5, Theorem 1.1]). The additivity of local deformation property implies that any metric manifold with a locally geometric group action also has the same local deformation property ([6, Theorem 4.1]).

The local deformation property for uniform embeddings implies the local contractibility of the group of uniform homeomorphisms. Our next aim is to study its global deformation property. The most standard example is the Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric. Its relevant feature is the existence of similarity transformations. This enables us to deduce a global deformation for uniform embeddings in the Euclidean end from the local one. Since this property is preserved by bi-Lipschitz homeomorphisms, we obtain a global deformation theorem for the group of uniform homeomorphisms of any metric manifold with finitely many bi-Lipschitz Euclidean ends ([5, Theorem 1.2]). This implies, for instance, the contractibility of the identity components of the groups of uniform homeomorphisms of $\mathbb{R}^n$ and any non-compact 2-manifold with finitely many bi-Lipschitz Euclidean ends ([5, Example 1.1]).

In the succeeding sections we explain some details of the statements described in this introduction. Section 2 contains local deformation results for uniform embeddings. Section 3 includes global deformation results for uniform homeomorphisms.

2. Local deformation property for uniform embeddings

2.1. Suppose (X, d) is a metric space. For subsets A, B of X we write $A \subset_u B$ and call B a uniform neighborhood of A in X if B contains the ε-neighborhood $O_\varepsilon(A)$ of A in X for some $\varepsilon > 0$.

A map $f : (X, d) \to (Y, \rho)$ between metric spaces is said to be uniformly continuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, x' \in X$ and $d(x, x') < \delta$ then $\rho(f(x), f(x')) < \varepsilon$. The map f is called a uniform homeomorphism if f is bijective and both f and f^{-1} are uniformly continuous. A uniform embedding is a uniform homeomorphism onto its image.

A metric manifold means a separable topological manifold possibly with boundary assigned a fixed metric. Suppose (M, d) is a metric n-manifold. For subsets X and C of M, let $\mathcal{E}^u_*(X, M; C)$ denote the space of proper uniform embeddings $f : (X, d|_X) \to (M, d)$ such that $f = \mathrm{id}$ on $X \cap C$. This space is endowed with the uniform topology induced from the sup-metric

$$d(f, g) = \sup\big\{d(f(x), g(x)) \mid x \in X\big\} \in [0, \infty] \quad (f, g \in \mathcal{E}^u_*(X, M; C)).$$

Definition 2.1. For a subset A of M we say that A has the local deformation property for uniform embeddings in (M, d) and write $A : (\mathrm{LD})_M$ if the following holds:

$(*)$ for any subset X of A, any uniform neighborhoods $W' \subset W$ of X in (M, d) and any subsets $Z \subset_u Y$ of M there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\phi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ which satisfy the following conditions:

(1) For each $h \in \mathcal{W}$

(i) $\phi_0(h) = h$, (ii) $\phi_1(h) = \mathrm{id}$ on X,

(iii) $\phi_t(h) = h$ on $W - W'$ and $\phi_t(h)(W) = h(W)$ $(t \in [0, 1])$,

(iv) if $h = \mathrm{id}$ on $W \cap \partial M$, then $\phi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0,1])$.

(2) $\phi_t(i_W) = i_W$ $(t \in [0,1])$.

In the case where $A = M$ we omit the subscript M in the symbol $(\mathrm{LD})_M$.

The celebrated Edwards-Kirby local deformation theorem [3] can be restated in the next form.

Theorem 2.2. *(Edwards-Kirby [3]) Any relatively compact subset K of M satisfies the condition $(LD)_M$.*

The condition $(\mathrm{LD})_M$ has the following formal properties:

Proposition 2.3. *([6, Proposition 3.1, Corollary 3.1, Remark 3.2])*

(1) The property (LD) is preserved by any uniform homeomorphism (i.e., if (M,d) is uniformly homeomorphic to (N,ρ), then (M,d) : (LD) $\Longleftrightarrow$ (N,ρ) : (LD).)

(2) (Restriction) (i) Suppose $A \subset B \subset M$. Then, $B : (LD)_M \Longrightarrow A : (LD)_M$.

(ii) Suppose $A \subset_u N \subset M$ and N is an n-manifold. Then, $A : (LD)_N \Longleftrightarrow A : (LD)_M$.

(3) (Additivity) (i) Suppose $A \subset_u U \subset M$ and $B \subset M$. Then, U, $B : (LD)_M \Longrightarrow A \cup B : (LD)_M$.

(ii) Suppose $M = A \cup B$, A, B are n-manifolds and $A - B \subset_u A$. Then, A, $B : (LD) \Longrightarrow M : (LD)$.

(4) Suppose K is a relatively compact subset of M and $A \subset M$. Then, $A : (LD)_M \Longleftrightarrow A \cup K : (LD)_M$.

(5) (Neighborhoods of ends) Suppose $M = K \cup \cup_{i=1}^m L_i$, K is relatively compact, each L_i is an n-manifold and closed in M, and $d(L_i, L_j) > 0$ for any $i \neq j$. Then, $M : (LD) \Longleftrightarrow L_i : (LD)$ $(i = 1, \cdots, m)$.

2.2. Metric covering projections and Geometric group actions

Our next aim is to seek concrete examples which have the local deformation property for uniform embeddings. The following notion is a natural metric version of Riemannian covering projections.

DEFINITION 2.4. A map $\pi : (X,d) \to (Y,\rho)$ between metric spaces is called a metric covering projection if it satisfies the following conditions:

($\natural$)$_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

($\natural$)$_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

($\natural$)$_3$ $\rho(\pi(x), \pi(x')) \leq d(x, x')$ for any $x, x' \in X$.

Here, a subset A of X is said to be uniformly discrete if there exists an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for any distinct points $x, y \in A$. Note that if Y is an n-manifold, then so is X and $\partial X = \pi^{-1}(\partial Y)$. From the Edwards-Kirby local deformation theorem [3] and the Arzela-Ascoli theorem we can deduce the local deformation theorem for uniform embeddings [5, Theorem 1.1].

Theorem 2.5. *If $\pi : (M, d) \to (N, \rho)$ is a metric covering projection and N is a compact metric manifold, then (M, d) satisfies the condition (LD).*

In term of covering transformations, this theorem corresponds to the case of free group actions. For the non-free case, we have the following generalization. Recall that an action Φ of a discrete group G on a metric space X is called geometric if it is proper, cocompact and isometric. (cf. [1, Chapter I.8]). More generally we say that the action Φ is (i) locally isometric if for every $x \in X$ there exists $\varepsilon > 0$ such that each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$, and (ii) locally geometric if it is proper, cocompact and locally isometric.

Corollary 2.6. *([6, Theorem 4.1]) A metric manifold (M, d) satisfies the condition (LD) if it admits a locally geometric group action.*

Example 2.7. The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric admits the canonical geometric action of $\mathbb{Z}^n$ and the associated Riemannian covering projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ onto the flat torus. Therefore, $\mathbb{R}^n$ has the property (LD). From Proposition 2.3 (4) and (3) it follows that the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$ and the half space $\mathbb{R}^n_{\geq 0} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_n \geq 0\}$ also have the property (LD).

3. Groups of uniform homeomorphisms

Suppose (X, d) is a metric space and A is a subset of X. Let $\mathcal{H}^u_A(X, d)$ denote the group of uniform homeomorphisms of (X, d) onto itself which fix A pointwise, endowed with the uniform topology. Let $\mathcal{H}^u_A(X, d)_0$ denote the connected component of the identity map id_X of X in $\mathcal{H}^u_A(X, d)$. We also consider the subgroup

$$\mathcal{H}^u_A(X,d)_b = \{h \in \mathcal{H}^u_A(X,d) \mid d(h,\mathrm{id}_X) < \infty\}.$$

It is easily seen that $\mathcal{H}^u_A(X,d)_0 \subset \mathcal{H}^u_A(X,d)_b$ since the latter is both closed and open in $\mathcal{H}^u_A(X,d)$.

The group $\mathcal{H}^u(M,d)$ is locally contractible if a metric manifold (M,d) satisfies the condition (LD) in Section 2. Hence, our main concern in this section is in the study of its global deformation property.

The most standard model space $\mathbb{R}^n$ has the similarity transformations

$$k_\gamma : \mathbb{R}^n \approx \mathbb{R}^n : \ \ k_\gamma(x) = \gamma x \qquad (\gamma > 0).$$

Conjugation with these similarity transformations enables us to deduce a global deformation property for uniform embeddings in the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$ from the local one. Since this global deformation property is preserved by bi-Lipschitz equivalence, we can transfer to a more general setting of metric spaces with finitely many bi-Lipschitz Euclidean ends.

Recall that a map $h : (X,d) \to (Y,\rho)$ between metric spaces is said to be Lipschitz if there exists a constant $C > 0$ such that $\rho(h(x),h(x')) \leq Cd(x,x')$ for any $x,x' \in X$. The map h is called a bi-Lipschitz homeomorphism if h is bijective and both h and h^{-1} are Lipschitz maps. A bi-Lipschitz n-dimensional Euclidean end of a metric space (X,d) means a closed subset L of X which admits a bi-Lipschitz homeomorphism of pairs, $\theta : (\mathbb{R}^n_1, \partial\mathbb{R}^n_1) \approx ((L, \mathrm{Fr}_X L), d|_L)$ and $d(X - L, L_r) \to \infty$ as $r \to \infty$, where $\mathrm{Fr}_X L$ is the topological frontier of L in X and $L_r = \theta(\mathbb{R}^n_r)$ for $r \geq 1$. We set $L' = \theta(\mathbb{R}^n_2)$ and $L'' = \theta(\mathbb{R}^n_3)$.

Theorem 3.1. *([5, Theorem 1.2]) Suppose X is a metric space and $L_1, \cdots, L_m$ are mutually disjoint bi-Lipschitz Euclidean ends of X. Let $L' = L'_1 \cup \cdots \cup L'_m$ and $L'' = L''_1 \cup \cdots \cup L''_m$. Then there exists a strong deformation retraction ϕ of $\mathcal{H}^u(X)_b$ onto $\mathcal{H}^u_{L''}(X)$ such that*

$$\phi_t(h) = h \ \ \text{on} \ \ h^{-1}(X - L') - L' \ \ \text{for any} \ \ (h,t) \in \mathcal{H}^u(X)_b \times [0,1].$$

This theorem leads to the following conclusions.

Example 3.2. (1) $\mathcal{H}^u(\mathbb{R}^n)_b$ is contractible for every $n \geq 1$. In fact, there exists a strong deformation retraction of $\mathcal{H}^u(\mathbb{R}^n)_b$ onto $\mathcal{H}^u_{\mathbb{R}^n_3}(\mathbb{R}^n)$ and the latter is contractible by Alexander's trick.

(2) Suppose N is a compact connected 2-manifold with a nonempty boundary and $C = \cup_{i=1}^m C_i$ is a nonempty union of some boundary circles of N. If the noncompact 2-manifold $M = N - C$ is assigned a metic d such that for each $i = 1, \cdots, m$ the end L_i of M corresponding to the boundary circle

C_i is a bi-Lipschitz Euclidean end of (M, d), then $\mathcal{H}^u(M, d)_0 \simeq \mathcal{H}^u_{L''}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq *$.

We close the section with a question on the topological type of the group $\mathcal{H}^u(\mathbb{R}^n)_b$. In [4] we studied the topological type of $\mathcal{H}^u(\mathbb{R})_b$ as an infinite-dimensional manifold and showed that it is homeomorphic to ℓ_∞. Example 3.2 (1) leads to the following conjecture.

Conjecture 3.3. *$\mathcal{H}^u(\mathbb{R}^n)_b$ is homeomorphic to ℓ_∞ for any $n \geq 1$.*

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