Geometry and Foliations 2013
Komaba, Tokyo, Japan

The minimal volume orientable hyperbolic 3-manifold with 4 cusps

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1. Introduction

For hyperbolic 3-manifolds, their volumes are known to be topological invariants. The structure of the set of the volumes of hyperbolic 3-manifolds is known by the Jørgensen-Thurston theorem: The set of the volumes of orientable hyperbolic 3-manifolds is a well-ordered set of the type ω^ω with respect to the order of $\mathbb{R}$. The volume of an orientable hyperbolic 3-manifold with n-cusps corresponds to an n-fold limit ordinal.

This theorem gives rise to the problem of determining the minimal volume orientable hyperbolic 3-manifolds with n cusps. The answers are known in the cases where $0 \leq n \leq 2$. Agol [1] conjectured which manifolds have the minimal volume in the cases where $n \geq 3$. We present the result that we determined it in the case where $n = 4$.

Theorem 1.1. *The minimal volume orientable hyperbolic 3-manifold with 4 cusps is homeomorphic to the* 8^4_2 *link complement. Its volume is* $7.32... = 2V_8$*, where* V_8 *is the volume of the ideal regular octahedron.*

2. Outline of Proof

8^4_2 link complement is obtained from two ideal regular octahedra by gluing along the faces. Hence we need a lower bound on the volume of an orientable hyperbolic 3-manifold with 4 cusps. The proof relies on Agol's argument used to determine the minimal volume hyperbolic 3-manifolds with 2 cusps [1].

Let M be a finite volume hyperbolic 3-manifold, and let X be a (non-necessarily connected) essential surface in M. After we cut M along X, the relative JSJ decomposition can be performed. The obtained components are characteristic or hyperbolic. The union of hyperbolic components is called the *guts* of $M - X$. The guts admit another hyperbolic metric with totally geodesic boundary. Then we can obtain a lower bound of the volume of M.

Partly supported by "Leading Graduate Course for Frontiers of Mathematical Sciences and Physics" from the Ministry of Education, Culture, Sports, Science and Technology (Mext) of Japan.

Theorem 2.1 (Agol-Storm-Thurston [2]). *Let L be the guts of $M - X$. Then* $\mathrm{vol}(M) \geq \mathrm{vol}(L) \geq \frac{V_8}{2}|\chi(\partial L)|$, *where* $\mathrm{vol}(L)$ *is defined with respect to the hyperbolic metric of L with totally geodesic boundary.*

Therefore it is sufficient that we estimate the Euler characteristic of the boundary of guts. Let M be a finite volume hyperbolic 3-manifold with 4 cusps. At first, we construct an essential surface X such that the guts of $M - X$ have 4 torus or annular cusps. We need to estimate the volume of a hyperbolic manifold L with totally geodesic boundary and 4 cusps. Purely homological arguement shows that L has a non-separating essential surface. Beginning from this surface, we construct an essential surface Y in L such that $\chi(\partial(\text{guts of } L - Y)) \leq -4$.

References

[1] I. Agol, The minimal volume orientable hyperbolic 2-cusped 3-manifolds, *Proc. Amer. Math. Soc.* **138** (2010), no. 10, 3723–3732.

[2] I. Agol, P. Storm and W. Thurston, Lower bounds on volumes of hyperbolic Haken 3-manifolds, *J. Amer. Math. Soc.* **20** (2007), no. 4, 1053–1077.

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