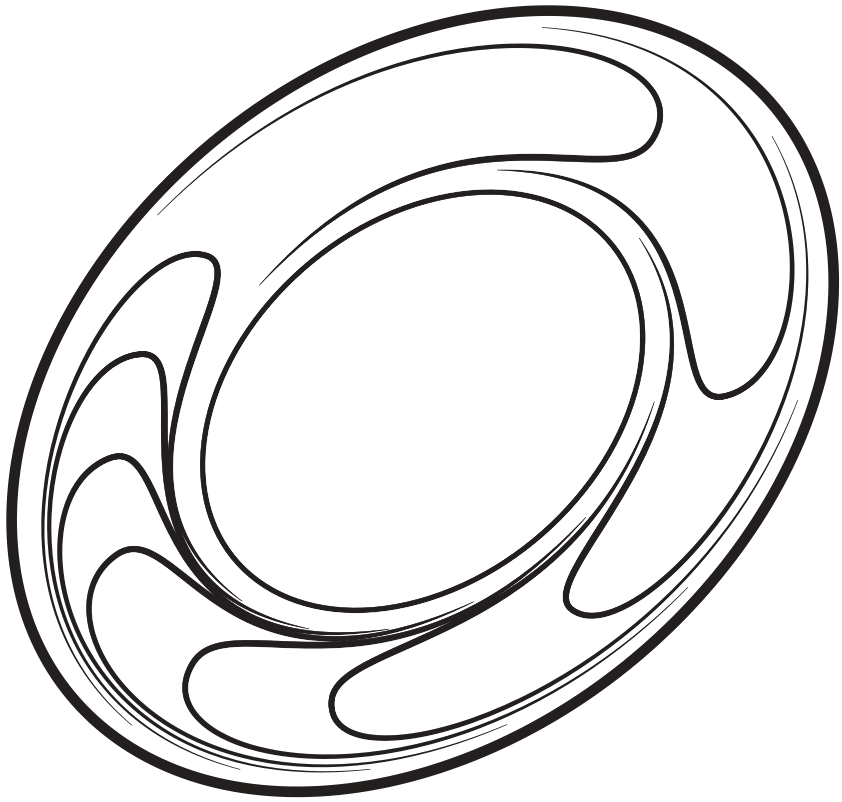
Geometry and Foliations 2013



September 9–14, Tokyo, Japan

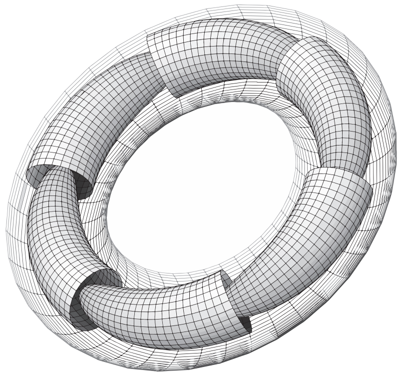
Graduate School of Mathematical Sciences,
University of Tokyo (Komaba Campus)

ABSTRACTS

GF2013

Abstracts for

Geometry and Foliations 2013



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Minicourse

Geometry and Foliations 2013
Komaba, Tokyo, Japan

Brownian motion on foliated complex surfaces, Lyapunov exponents and applications

Bertrand DEROIN

To the birthdays of Steven Hurder and Takashi Tsuboi

Introduction

These lectures are motivated by the dynamical study of differential equations in the complex domain. Most of the topic will concern *holomorphic foliations on complex surfaces*, and their connections with the theory of *complex projective structures on curves*. In foliation theory, the interplay between geometry and dynamics is what makes the beauty of the subject. In these lectures, we will try to develop this relationship even more.

On the geometrical side, we have generalizations of the *foliation cycles* introduced by Sullivan, see [68]: namely the *foliated harmonic currents*, see e.g. [36, 4]. Those currents permit to think of the foliation as if it were a genuine algebraic curve. For instance, one can associate a homology class, compute intersections with divisors on the surface etc... These currents can often be viewed as limits of the (conveniently normalized) currents of integration on large leafwise domains defined via the uniformization of the leaves. This point of view, closely related to Nevanlinna theory, is very fruitful in the applications as we will see. See [5, 28].

On the dynamical side, the *leafwise Brownian motions* (w.r.t. to some hermitian metric on the tangent bundle to the foliation, e.g. coming from uniformization of leaves) generate a Markov process on the complex surface, whose study was begun by Garnett, see [34]. This Markov process seems to play a determinant role in the dynamics of foliated complex surfaces. One reason is that the Brownian motion in two dimensions is conformally invariant. Another reason is that leafwise Brownian trajectories equidistribute w.r.t. the product of a certain foliated harmonic current times the leafwise volume element. This makes the connection with the geometrical side mentioned above.

One of the main theme that will be developed in these lectures is the construction of numerical invariants that embrace these two aspects (dynamical and geometrical) of foliated complex surfaces. The discussion will

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emphasize on the definition and properties of the *foliated Lyapunov exponent* of a harmonic current, which heuristically measures the exponential rate of convergence of leaves toward each other along leafwise Brownian trajectories. A fruitful formula expresses this dynamical invariant in terms of the intersection of some foliated harmonic currents and the normal/canonical bundles of the foliation, see [16]. This formula is a good illustration of the interplay between geometry and dynamics in foliation theory. This will be developed in the first lecture.

In the second and third lectures, we will collect some applications of this formula in different contexts.

The first application concerns *Levi-flats* in complex algebraic surfaces. Those are (real) hypersurfaces that are foliated by holomorphic curves. Most examples occur as three (real) dimensional analytic invariant subsets of singular algebraic foliations. Foliations having Levi-flats are analogous to *Fuchsian groups* (those having an invariant analytic circle in the Riemann sphere) in the context of Kleinian groups or to *Blashke products/Tchebychef polynomials* (having an invariant analytic circle/interval) in the context of iteration of rational functions. Very little is known about Levi-flats in algebraic surfaces. For instance, it is still unknown wether every algebraic surface contains a Levi-flat. A folklore conjecture predicts that the complex projective plane should not have any. Still, there exists a multitude of examples, e.g. in flat ruled bundles over curves, in singular holomorphic fibrations, in ramified covers of these etc. As we will see, some new restrictions concerning the topology of Levi-flats can be deduced from a detailed analysis of the foliated Lyapunov exponent and its relation to the geometry of the ambiant surface. For instance, we will prove that a Levi-flat hypersurface in a surface of general type is not diffeomorphic to the unitary tangent bundle of a two dimensional compact orbifold of negative curvature, nor to a hyperbolic torus bundle, and that its fundamental group has exponential growth. This will be explained in the second lecture, where we'll also construct many examples of Levi-flats, most notably we will realize all the models of Thurston's geometries as Levi-flats in algebraic surfaces appart the elliptic one. All this is based on a work in collaboration with Christophe Dupont, see [20].

The second application concerns *complex projective structures* on curves. These structures are of interest in various problems of uniformization in two or three dimensions. We will define some new invariants associated to complex projective structures: a *Lyapunov exponent*, a *degree*, and a family of *harmonic measures* (analogous to harmonic measure of a compact set in the complex line), and we will see how to relate these invariants. The connexion with foliation theory will be of utmost importance. It comes from the study of the particular class of *transversally holomorphic foliations*: any algebraic curve transverse to such a foliation inherits a complex projective structure by restricting the transverse projective structure of the foliation to the curve. As an illustration of this point of view, an algebraic curve in a Hilbert modular surface of the form $\Gamma\backslash\mathbb{H}\times\mathbb{H}$, where Γ is a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ inherits two (branched) complex projective structures from the two (horizontal and vertical) foliations. We will derive applications of these new invariants, most notably some estimates for the dimension of harmonic measures of complex projective structures. In particular, we will recover the Jones–Wolff and Makarov estimates for classical harmonic measures of limit sets of Kleinian groups. Another application will be to reinforce the analogy between complex projective structures and polynomial dynamics, that was brought to light by McMullen, see [58]. All these developments have been obtained in collaboration with Romain Dujardin, see [19].

Acknowledgments. I warmly thank the organizers of the conference *Geometry and Foliations 2013* who gave me the opportunity to deliver these lectures.

1. Lecture 1 – Lyapunov exponents associated to foliated complex surfaces

1.1. Basic definitions and examples

In this lecture, S will be a complex surface, and $\mathcal{F}$ a non singular holomorphic foliation on S. Recall that $\mathcal{F}$ is a maximal atlas of holomorphic charts $(x,z)\colon U\to\mathbb{D}\times\mathbb{D}$ ($\mathbb{D}\subset\mathbb{C}$ is the unit disc) defined on open subsets U covering S, and overlaping as

$$(x',z')=(x'(x,z),z'(z)).$$

Hence the local fibrations $z=\mathrm{cst}$ are preserved by the change of coordinates. The fibers of these local fibrations, called the *plaques*, are glued together and define Riemann surfaces, called the leaves of the foliation. The sets $\mathbb{D}\ni z$ are called transversal sets, and will be denoted $\mathbb{D}^{\pitchfork}$. We refer to the book [10] for the basics on foliation theory: most notably, the definition of holonomy maps, transverse invariant measures etc...

The data of S and $\mathcal{F}$ will be referred to as *foliated complex surface*. We assume in the sequel that there exists a compact saturated subset $M\subset S$, saturated meaning that it is a union of leaves of $\mathcal{F}$. We have in mind various sources of examples.

Definition 1.1 (Levi-flat). A hypersurface M of class C^2 in a complex surface S inherits a unique distribution by complex lines called the Cauchy–Riemann distribution. It is defined by the formula $TM\cap iTM$ where $i=\sqrt{-1}$. The hypersurface M is called Levi-flat iff the Cauchy–Riemann distribution integrates in a foliation, called the Cauchy–Riemann foliation and denoted by $\mathcal{F}$. If the hypersurface M is Levi-flat and analytic, then $\mathcal{F}$ can be extended in the neighborhood of M as a non singular holomorphic foliation.

In analytic regularity, a more intrinsic view-point is the following

Example 1.2 (Foliated 3-manifolds). A 2-dimensional analytic foliation of a compact 3-manifold equipped with an analytic complex structure on its leaves can be embedded in a germ of foliated complex surface. Such a complex structure can be built using a leafwise orientation plus an analytic metric on $T\mathcal{F}$, since Riemannian surfaces are conformally flat. In analytic regularity, this is a theorem of Gauss, see [15, Théorème I.2.1].

Other examples are

Example 1.3 (Riemann-Hilbert correspondance). Let C be an algebraic curve, and $\pi_1(C) \to \mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ be a representation. We define $S_\rho = C \ltimes_\rho \mathbb{P}^1(\mathbb{C})$ as the flat $\mathbb{P}^1(\mathbb{C})$ bundle over C with monodromy ρ. Recall that S_ρ is defined as the quotient of $\widetilde{C} \times \mathbb{P}^1(\mathbb{C})$ by the action of $\pi_1(C)$ given by

$$\gamma \cdot (x,z) = (\gamma \cdot x, \rho(\gamma) \cdot z),$$

for every $\gamma \in \pi_1(C)$ and $(x,z) \in \widetilde{C} \times \mathbb{P}^1(\mathbb{C})$. Here $\widetilde{C}$ denotes a universal cover of C, and $\pi_1(C)$ the covering group of this covering. The horizontal fibration on $\widetilde{C} \times \mathbb{P}^1(\mathbb{C})$ whose fibers are the subsets $\widetilde{C} \times z$ for $z \in \mathbb{P}^1(\mathbb{C})$, defines on S_ρ a non singular holomorphic foliation $\mathcal{F}_\rho$.

Remark 1.4. In the case the representation ρ takes values in $\mathrm{PSL}(2,\mathbb{R})$, the foliated surface $(S_\rho, \mathcal{F}_\rho)$ contains a Levi-flat, defined as the twisted product $C \ltimes_\rho \mathbb{P}^1(\mathbb{R})$.

1.2. Foliated harmonic currents

As before, let $(S,\mathcal{F})$ be a foliated complex surface and let M be a compact saturated subset of S. We denote by $\mathcal{O}_\mathcal{F}$ the sheaf of continuous functions on M which are holomorphic along the leaves, and by $C^\infty_\mathcal{F}$ the sheaf of functions f which are smooth along the leaves and all whose leafwise derivatives $\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial \overline{x}^\beta}$ in holomorphic foliated coordinates are continuous in (x,z). This definition is independent of the chosen foliated coordinate system. We also denote by $A^p_\mathcal{F}$ (resp. $A^{(p,q)}_\mathcal{F}$) the set of $C^\infty_\mathcal{F}$ forms of degree p (resp. bidegree (p,q)) on $T\mathcal{F}$.

Definition-Proposition 1.5. A foliated harmonic current is a linear form $T\colon A^{1,1}_{\mathcal{F}} \to \mathbb{R}$ which verifies $\partial\overline{\partial}T = 0$ in the weak sense (namely $T(\partial\overline{\partial}f) = 0$ for any smooth function $f\colon S \to \mathbb{R}$), and which is positive on $\mathcal{F}$ (namely $T(\eta) > 0$ if $\eta_{|\mathcal{F}} > 0$). In foliated coordinates, a foliated harmonic current takes the form

$$T(\eta) = \int_{\mathbb{D}^{\pitchfork}} \left[\int_{\mathbb{D}\times z} \varphi(x,z)\eta(dxd\overline{x}) \right] \nu(dzd\overline{z}) \tag{1.6}$$

where ν is a Radon measure on the transversal $\mathbb{D}^{\pitchfork}$ and $\varphi \in L^1(dxd\overline{x} \otimes \nu)$ is harmonic on ν-a.e. plaque $\mathbb{D} \times z$.

Proposition 1.7. *A compact saturated subset supports a foliated harmonic current.*

Proof. The following proof is due to Ghys, see [36], following ideas of Sullivan, see [68]. Let $A^{1,1}_c$ be the set of continuous $(1,1)$-forms along the leaves of M, $\mathcal{P} \subset A^{1,1}_c$ denotes the open convex cone of positive ones, and E be the set of uniform limits of forms of the type $\partial\overline{\partial}f_{|M}$ with $f \in C^{\infty}(S)$. By the maximal principle, $\mathcal{P} \cap E = \emptyset$, hence the Hahn-Banach separation theorem concludes. □

Remark 1.8. The existence of foliated harmonic current has been generalized to singular holomorphic foliations by Berndtsson and Sibony. We refer to [4, Theorem 1.4].

Definition-Proposition 1.9 (Foliation cycles). A foliation cycle is a foliated harmonic current which is d-closed, namely it satisfies $T(d\eta) = 0$ for every $\eta \in A^1_{\mathcal{F}}$. A foliation cycle is expressed locally as

$$T(\eta) = \int_{\mathbb{D}^{\pitchfork}} \left[\int_{\mathbb{D}\times z} \eta \right] \nu(dzd\overline{z}) \tag{1.10}$$

where ν is a Radon measure. The family of measures ν defines a transverse invariant measure for the foliation $(M, \mathcal{F})$.

Example 1.11 (Leaf closed at infinity). The basic example of foliation cycle is the integration current on a leaf. A generalization of this is due to Plante, see [62, Theorem 3.1]. Assume that $A_n \subset L_n$ is a sequence of compact domains contained in leaves L_n of M, and that we have

$$\frac{\mathrm{length}(\partial A_n)}{\mathrm{area}(A_n)} \to_{n\to\infty} 0 \tag{1.12}$$

where the length and area are measured w.r.t. to a hermitian metric along the leaves. Then the family of currents $T_n := \frac{1}{\mathrm{area}(A_n)}[A_n]$ is relatively compact in the weak$*$ topology, and moreover any limit $\lim_{n_k\to\infty} T_{n_k}$ is a foliation cycle. Sullivan generalized this construction, see [68, Theorem II.8].

1.3. Uniformization

Other examples of foliation cycles or foliated harmonic currents come from the uniformization of Riemann surfaces, which is stated as follows.

Theorem 1.13 (Poincaré-Koebe)**.** *Every Riemann surface is covered resp. by $\mathbb{P}^1(\mathbb{C}), \mathbb{C}$ or $\mathbb{D}$. This trichotomy is exclusive. The Riemann surface is resp. called elliptic, parabolic or hyperbolic.*

We refer to the book [15] for the history and the various proofs of this theorem.

Example 1.14 (Ahlfors). If L is a parabolic leaf contained in M, and $f\colon \mathbb{C} \to L$ a uniformization of L, one can extract from the family of currents

$$\forall \eta \in A^{1,1}_{\mathcal{F}}(M), \quad T_r(\eta) := \frac{1}{\mathrm{area}_{f^*g}(\mathbb{D}_r)} \int_{\mathbb{D}_r} f^*\eta \tag{1.15}$$

a subsequence converging in the weak$*$ topology towards a foliation cycle. Here $\mathbb{D}_r := \{x \in \mathbb{C} \mid |x| < r\}$. We refer to [1] and [7, Lemme 0] for a proof of this fact.

Let us now review what happens if the leaves are hyperbolic. We begin by the following theorem of Verjovsky, generalized by Candel in the context of general Riemann surface laminations. Recall that the unit disc has a unique complete conformal metric of curvature -1, given by

$$g_P = \frac{1}{4}\frac{|dx|^2}{(1-|x|^2)^2}. \tag{1.16}$$

This metric is invariant under the group $\mathrm{Aut}(\mathbb{D})$ of automorphisms of the unit disc, hence it defines a conformal metric on any hyperbolic Riemann surface. We have

Theorem 1.17 (Verjovsky-Candel)**.** *Assume that all the leaves of M are hyperbolic. Then the Poincaré metric on each of these leaves defines a continuous metric on $T\mathcal{F}_{|M}$.*

Example 1.18 (Fornaess-Sibony). Assume that all the leaves of M are hyperbolic Riemann surfaces. Let $f\colon \mathbb{D} \to L$ be the uniformization of one leaf of M. Then the family of currents

$$\forall \eta \in A^{1,1}(M), \quad T_r(\eta) = \frac{\int_{\mathbb{D}_r} \log\left(\frac{r}{|x|}\right) f^*\eta(dxd\overline{x})}{\int_{\mathbb{D}_r} \log\left(\frac{r}{|x|}\right) v_P(dxd\overline{x})} \tag{1.19}$$

is relatively compact in the weak$*$ topology and the limit of any convergent subsequence T_{r_n} with $r_n \to 1$ is foliated harmonic. Here v_P refers to the volume element of the Poincaré metric.

1.4. Homology, intersection, and Chern-Candel classes

A foliation cycle being a closed current of dimension 2 on S, it naturally defines a homology class $[T] \in H_2(S, \mathbb{R})$ (by duality) by the formula

$$[T] \cdot [\eta] = T(\eta),$$

for every closed 2-form. In particular, one can consider the intersection product $[T] \cdot c_1(E)$ if $E \to S$ is any complex line bundle over S, and $c_1(E)$ denotes the first Chern class of E. We will denote it succintly by $T \cdot E$. One can compute this intersection by using differential geometry, namely

$$T \cdot E = \frac{1}{2\pi} T(\omega) \tag{1.20}$$

where ω is the curvature form of *any* connexion ∇ on E. In fact, it is sufficient to have a smooth connexion which is only defined along every leaf of $\mathcal{F}$, but we will not verify this here. All this makes sense since the curvature forms of two different connexions on E differ by an exact 2-form.

This does not work this way if T is only assumed to be harmonic, since in this case we only get a homology class in the dual of the Bott-Chern cohomology group

$$H^{1,1}_{\partial\overline{\partial}}(S, \mathbb{C}) = \{\text{closed } (1,1)\text{-forms}\}/\partial\overline{\partial}C^\infty(S). \tag{1.21}$$

Nevertheless, following an observation of Candel, one can define the intersection product of T with E when E is any holomorphic line bundle along the leaves of M (namely every element of $H^1(M, \mathcal{O}^*_{\mathcal{F}})$). This can be achieved by the use of the Chern connexion of a hermitian metric on E, whose expression is on E, whose expression is given locally by

$$\omega_{\|\cdot\|} = \frac{1}{i} \partial\overline{\partial} \log\|s\|^2, \tag{1.22}$$

where s is any local holomorphic section of E. One then defines

$$T \cdot E := \frac{1}{2\pi} T(\omega_{\|\cdot\|}), \tag{1.23}$$

where $\|\cdot\|$ is any hermitian metric on E. Since T is harmonic, the definition does not depend on the chosen hermitian metric on E.

This formula permits to define an important invariant of a harmonic current: its Euler characteristic. This is the intersection of the harmonic current with the tangent bundle of the foliation $\mathcal{F}$. In what follows, we will be more interested in the opposite of this number, namely the intersection of T with the canonical bundle of $\mathcal{F}$ being defined by $K_{\mathcal{F}} := T^*\mathcal{F}$.

An interesting case is where S is a compact Kähler surface, since under this assumption one knows that the group (1.21) is isomorphic to the Dolbeaut cohomology group $H^{1,1}_{\overline{\partial}}(S, \mathbb{C}) \subset H^2(S, \mathbb{C})$, by the $\partial\overline{\partial}$-lemma. Thus we can define a homology class $[T]$ of T belonging to $H_2(S, \mathbb{C})$ (by duality) in that case. Observe that if $E \to S$ is a holomorphic line bundle, the number $T \cdot E$ defined by (1.23) computes the cohomological intersection $[T] \cdot c_1(E)$, where $c_1(E)$ is the Chern class of E.

1.5. Garnett's theory

Here is the basic ingredient that will be needed in this lecture. Let (L, g) be a complete Riemannian manifold with bounded curvature, and $x \in L$ be a point. Then there exists a unique measure W^x, called the *Wiener measure*, on the set Ω_x of continuous paths $\omega \colon [0, \infty) \to L$ starting at $\omega(0) = x$, satisfying the following

(1.24)

$$W^x(\{\omega \mid \omega(t_i) \in B_i\}) = \int_{B_1 \times \cdots \times B_k} \prod_{j=1}^{k} p(x_{j-1}, x_j, t_j - t_{j-1})\, v_g(dx_1) \cdots v_g(dx_k)$$

for every $k \in \mathbb{N}^*$, every non decreasing sequence $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k$, every family $\{B_j\}_j$ of Borel subsets of L, and the convention $x_0 = x$. Here, v_g denotes the volume element, and $p(x, y, t)$ is the heat kernel on L (namely $p(x, \cdot\, , \cdot\,)$ satisfies the heat equation $\frac{\partial u}{\partial t} = \Delta u$ and $p(x, y, t)dy$ weakly tends to the Dirac mass δ_x at x). We refer to [13, Chapter VI].

Let now $(S, \mathcal{F})$ be a foliated complex surface, and M be a $\mathcal{F}$-saturated closed subset of S. Let g be a smooth hermitian metric on $T\mathcal{F}$, defined in a neighborhood of M, and $\Delta_{\mathcal{F}}$ the leafwise Laplacian associated to this metric. A *foliated harmonic measure* on M is a probability measure which satisfies in the weak sense the equation $\Delta_{\mathcal{F}}\mu = 0$. Those are the measures

$$\mu := T \wedge v_g \tag{1.25}$$

where T is a (conveniently normalized) foliated harmonic current and v_g is the leafwise volume element of the Riemannian tensor g. In particular, a foliated harmonic measure always exists, by Proposition 1.7.

Let Ω be the set of continuous paths $\omega\colon [0,\infty) \to M$ which are contained in a leaf of M, and Ω^w those conditioned to begin at $\omega(0) = w$. Shifting the time defines a semi-group $\sigma = \{\sigma_t\}_{t\geq 0}$ of transformations acting on Ω by the formula $\sigma_t(\omega)(\,\cdot\,) := \omega(t+\,\cdot\,)$. Given a probability measure μ on M, let $\overline{\mu}$ be the measure on Ω defined by $\overline{\mu} := \int_M W^w \mu(dw)$. An easy observation shows that if μ is harmonic, then the measure $\overline{\mu}$ is σ-invariant. We can then apply ergodic theory to the system $(\Omega, \sigma, \overline{\mu})$. Garnett proved the following version of the random ergodic theorem in this context:

Theorem 1.26 (Random Ergodic Theorem)**.** *If the foliated harmonic measure μ is extremal in the compact convex set of harmonic measures, then the system $(\Omega, \sigma, \overline{\mu})$ is ergodic.*

We refer to [34] and to the survey paper by Candel [12]. A foliated harmonic measure satisfying the assumptions of the theorem will be called *ergodic*. Observe that in particular, for a.e. point w w.r.t. a foliated ergodic harmonic measure, W^w-a.e. Brownian path starting at x equidistributes w.r.t. μ.

1.6. The foliated Lyapunov exponent

In this section, we endow the tangent bundle $T_{\mathcal{F}}$, resp. the normal bundle $N_{\mathcal{F}}$, with smooth hermitian metrics. Recall that if $\omega\colon [0,t] \to L$ is a continuous path in a leaf of L, there is a holonomy map $h_\omega\colon \tau_{\omega(0)} \to \tau_{\omega(t)}$ from a transversal $\tau_{\omega(0)}$ at $\omega(0)$ to a transversal $\tau_{\omega(t)}$ at $\omega(t)$. See the book [10] for the definition of holonomy map. The derivative of h_ω at $\omega(0) \in \tau_{\omega(0)}$ will be denoted $Dh_\omega(\omega(0))$.

Definition-Proposition 1.27. Let T be an ergodic foliated harmonic current on M, and $\mu = T \wedge v_g$ the associated foliated harmonic measure. There exists a number $\lambda = \lambda(T)$, such that for μ-a.e. point $w \in M$, and W^w-almost every path $\omega\colon [0,\infty) \to L_w$ starting at $\omega(0) = w$, we have

$$\frac{1}{t}\log\|Dh_{\omega|_{[0,t]}}(\omega(0))\| = \lambda. \tag{1.28}$$

The proof of this fact relies on the ergodic theorem applied to the cocyle

$$H_t(\omega) := \log\|Dh_{\omega|_{[0,t]}}(\omega(0))\|, \tag{1.29}$$

which satisfies the relation $H_{t+s}(\omega) = H_t(\omega) + H_s(\sigma_t(\omega))$ for every $\omega \in \Omega$ and every $s,t \geq 0$. To get the result one needs to verify that H_t is $\overline{\mu}$-integrable. This relies on Cheng-Li-Yau estimates for the heat kernel:

$$p(x,y,t) \leq C\exp(-\alpha d(x,y)^2),$$

where $C, \alpha > 0$ are constant depending only on t and the local geometry of the manifold. See [14].

In the case all the leaves of $\mathcal{F}$ are hyperbolic Riemann surfaces, one can parametrize Brownian motions using the Poincaré metric. In this case, the Lyapunov exponent depends on cohomological quantities.

Proposition 1.30 (Cohomological formula for the Lyapunov exponent)**.** *Let $(S, \mathcal{F})$ be a foliated complex surface and M be a minimal set. Assume that the leaves of M are hyperbolic Riemann surfaces. We endow its tangent bundle with the Poincaré metric. Then for every ergodic foliated harmonic current T on M, we have*

$$\lambda(T) = -\frac{T \cdot N_{\mathcal{F}}}{T \cdot K_{\mathcal{F}}}.$$

In this formula, $N_{\mathcal{F}} = TS/T\mathcal{F}$ and $K_{\mathcal{F}} = T^*\mathcal{F}$ stand for the normal bundle and the canonical bundle of $\mathcal{F}$.

Proof. We repoduce here the proof given in [16, Appendice A]. Observe that the formula depends only on T modulo multiplication by a positive constant, so we can assume that the measure $\mu := T \wedge v_g$ has mass one. Introduce some coordinates (x, z) where the foliation is defined by $dz = 0$, and consider the infinitesimal distance between leaves, namely the function $\left\|\frac{\partial}{\partial z}\right\|$. This function depends on the foliated coordinates, but when changing coordinates, it is multiplied by a positive function which is constant on the leaves. In particular, the function $\Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ is well-defined on M. Similarly $d_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ is a well-defined 1-form along the leaves of $\mathcal{F}$.

Lemma 1.31. $\lambda = \int_M \Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\| \, d\mu$.

Proof. The starting point of the proof relies on the fact that $\int H_t d\overline{\mu} = \lambda t$, hence

$$\lambda = \frac{d}{dt}_{|t=0} \int H_t d\overline{\mu}.$$

Now, we have

$$\int H_t d\overline{\mu} = \int_X \left[\int_{\Omega^w} H_t dW^w \right] \mu(dw).$$

So we deduce

$$\lambda = \int_X \left[\frac{d}{dt}_{|t=0} \int_{\Omega^w} H_t dW^w \right] \mu(dw).$$

Fix w and introduce the universal covering $\widetilde{L_w}$ of L_w, viewed as the set of homotopy classes of paths $\omega \colon [0,1] \to L_w$ starting at w with fixed extremities. Let φ be a primitive of the form $d_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ which vanishes at w. The

Laplacian of φ is invariant by the covering group and gives the function $\Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|$ on the quotient. Moreover, we have $H_t(\omega) = \varphi(\omega_{|[0,t]})$. Hence we get

$$\frac{d}{dt}_{|t=0} \int_{\Omega^w} H_t dW^w = \frac{d}{dt}_{|t=0} \mathbb{E}^w(\varphi(\omega(t))) = \Delta_{\mathcal{F}}\varphi(w) = \Delta_{\mathcal{F}} \log\left\|\frac{\partial}{\partial z}\right\|(w).$$

This proves the formula. $\square$

Proposition 1.30 follows from Lemma 1.31 and from the following elementary identity $2i\partial\overline{\partial} = \Delta_g \cdot v_g$. $\square$

Remark 1.32. The existence of an analogous Lyapunov exponent for singular holomorphic foliations (say on algebraic surfaces) is not obvious at all. Assume for instance we are in the following situation. Let $(S, \mathcal{F})$ be a singular holomorphic foliation of a compact complex surface, whose leaves are hyperbolic Riemann surfaces, and whose singularities are linearizable. Then the product $T \wedge v_P$ is finite, see [22], and Garnett's theory can be extended almost line by line, by using the fact that the Poincaré metric is continuous in that case. The only problem to define the Lyapunov exponent in this context is the integrability of the cocyle (1.29). The integrability can be proved when the singularities are in the Siegel domain, namely conjugate to ones of the form $xdy - \alpha ydx$ where $\alpha \in \mathbb{R}$. Then Proposition 1.30 holds with a correction term involving some indices defined at each singularity. However, in the hyperbolic case $\Im\alpha \neq 0$, the integrability remains an open problem.

1.7. Unique ergodicity

A general principle is that foliated harmonic currents associated to minimal sets are unique. This fact was already observed in the work of Garnett (unique ergodicity of the weak stable foliation of the geodesic flow of a compact surface of constant curvature -1, see [34, Proposition 5]). Here is a general result that we obtained in collaboration with Victor Kleptsyn:

Theorem 1.33 (Unique ergodicity). *Let $(S, \mathcal{F})$ be a foliated complex surface, and M be a minimal set. Assume that $\mathcal{F}$ does not support any foliation cycle on M. Then there exists a unique harmonic current on M up to multiplication by a constant. Moreover, given a hermitian metric on $T\mathcal{F}$, there exists a number $\lambda < 0$ such that for every point $w \in M$, and W^w-a.e. leafwise Brownian path ω starting at w, the limit* (1.28) *exists and equal λ.*

We refer to [21] for the proof of this result, the main step being the existence of at least one harmonic current whose associated Lyapunov exponent is negative. This being done, a second step (the similarities between

Brownian motions on different leaves) permits to infer unique ergodicity. A weak version of the contraction statement was used by Thurston for the construction of his universal circle theorem, see [70].

Observe that under the assumption of Theorem 1.33, the leaves of M are hyperbolic Riemann surfaces since otherwise there would exist a foliation cycle. In particular, for every uniformization $f\colon \mathbb{D} \to L$ of a leaf, the family of currents T_r defined by (1.19) converge to a certain harmonic current T. In the context of flat $\mathbb{P}^1$-bundles over a compact curve C, Bonatti and Gomez-Mont have obtained a much more precise equidistribution statement, namely that of large leafwise discs. See [5]. Recall that a representation from an abstract group to $\mathrm{PSL}(2,\mathbb{C})$ is *non elementary* iff it does not preserve any probability measure on $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.34 (Equidistribution of large leafwise discs)**.** *Let C be an algebraic curve and $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation sending the peripheral curves to parabolic transformations. Assume that ρ is non elementary. Then for every sequence of points $w_n \in S_\rho = C \ltimes_\rho \mathbb{P}^1$, and every sequence of positive numbers R_n tending to infinity, we have the following*

$$\frac{1}{V(R_n)}[B_{\mathcal{F}}(w_n, R_n)] \to_{n\to\infty} T, \tag{1.35}$$

where $V(R)$ is the volume of a ball of radius R in hyperbolic plane, and T is the unique harmonic current normalized so that $\int T \wedge v_P = 1$.

Remark 1.36. Theorem 1.34 can be generalized when the base curve C is a quasi-projective curve, but we will not state this version of the result here.

We end this lecture by insisting on the fact that the dynamical method based on the Lyapunov exponent does not work to prove unique ergodicity in the context of singular holomorphic foliations on compact complex surfaces since, as was already mentioned, the definition of the Lyapunov exponent is unclear in this case. Fornaess and Sibony succeeded proving a similar unique ergodicity statement for generic singular holomorphic foliations of the complex projective plane, see [27, 28]. Their proof is based on a completely different approach (a computation of the self-intersection of a foliated harmonic current together with Hodge index theorem), which nevertheless does not extend to all compact complex surfaces: it necesitates a non trivial automorphism group of the ambiant surface.

2. Lecture 2: Topology of Levi-flats in algebraic surfaces

2.1. A rough guide to complex algebraic surfaces

A smooth complex algebraic manifold is a compact complex manifold which embeds holomorphically in a complex projective space $\mathbb{P}^N(\mathbb{C})$ for some $N \geq 1$. By the GAGA principle, such a compact complex submanifold is defined by algebraic homogeneous equations.

An important character in the understanding of an algebraic manifold X is its canonical bundle, namely the bundle $K_X := \bigwedge^d T^*X$, where d is the dimension of X. The plurigenera of X are defined by the dimensions $P_n(X) = h^0(X, nK_X)$ of the spaces of holomorphic sections of the powers nK_X of the canonical bundle (the tensor product of line bundles is denoted additively in the sequel). Their asymptotics when n tends to $+\infty$ is governed by the Kodaira dimension $k(X)$, which is defined by $k(X) := \lim_{n\to\infty} \frac{\log P_n}{\log n}$. The Kodaira dimension can assume any value $k \in \{-\infty, 0, 1, \ldots, d\}$, where by convention $k(X) = -\infty$ means that the plurigenera vanishes for every n.

As we have seen, algebraic curves can be classified into three classes, depending upon the type of their universal covering: $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$. This trichotomy can be detected by the Kodaira dimension, being respectively equal to $-\infty, 0$ or 1.

Algebraic surfaces are more difficult to classify. The surfaces with Kodaira dimension being $-\infty, 0, 1$ are relatively well understood, thanks to the classification of Enriques-Kodaira, and fall into eight classes: rational, ruled, $K3$, Enriques, Kodaira, toric, hyperelliptic, and properly quasi-elliptic. We refer to [3] for a complete treatment of this topic. Concerning the class of surfaces with Kodaira dimension 2, not much is known about their classification, though many examples have been found. These surfaces are called *surfaces of general type*, and in a sense, are the most commun surfaces.

Examples of general type surfaces are smooth hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^3(\mathbb{C})$, quotients of bounded domains in $\mathbb{C}^2$, double covers of $\mathbb{P}^2(\mathbb{C})$ ramified along a non singular curve of even degree ≥ 8 etc. Surfaces with a metric of negative holomorphic curvature are of general type. There is a weak converse to this statement: a theorem of Mumford states that the canonical bundle of a (minimal) surface of general type admits a metric whose curvature is positive on all complex directions appart from a finite union of (-2)-rational curves.

2.2. Thurston's eight geometries as Levi-flats in algebraic surfaces

We will say that a 3-manifold possesses a *geometry* if it admits a complete locally homogeneous metric (homogenous meaning that two different points admit isometric neighborhoods). Thurston classified in eight classes the compact 3-manifolds possessing a geometry, depending on the isometric class of their universal cover among:

$$\mathbb{S}^3,\ \mathbb{R}^3,\ \mathbb{H}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \mathrm{Nil},\ \widetilde{\mathrm{SL}(2,\mathbb{R})},\ \mathrm{Sol}. \tag{2.1}$$

The spaces $\mathbb{S}^p, \mathbb{R}^p$ and $\mathbb{H}^p$ for $p \in \{2,3\}$ stand for the complete simply connected Riemannian manifolds of dimension p of constant sectional curvature, resp. $1, 0, -1$. The last three models are Lie groups equipped with left invariant metrics. We refer to the article of Scott [65] for a more complete treatment. Let $\mathcal{M}$ be one of the eight simply connected manifolds in the list (2.1). We say that a compact 3-manifold M *carries the geometry* of $\mathcal{M}$ if M is the quotient of $\mathcal{M}$ by a discrete group of isometries of $\mathcal{M}$.

All the geometries (2.1) are carried by Levi-flats in algebraic complex surfaces, appart $\mathbb{S}^3$. The fact that $\mathbb{S}^3$ does not appear is an observation by Inaba and Michshenko, see [46, Theorem 1], which relies on the Kähler property for algebraic surfaces, together with the famous theorem of Novikov on existence of Reeb components, see 2.5.

Proposition 2.2 (Inaba-Michshenko)**.** *A Levi-flat in a Kähler surface has an infinite fundamental group. In particular, such a Levi-flat does not carry the geometry* $\mathbb{S}^3$*.*

Let us review the argument. We adopt the following definition:

Definition 2.3 (Reeb component). A Reeb component is a domain contained in M which is saturated by the foliation and diffeomorphic to the solid torus.

Recall that a Kähler form on a surface S is a closed $(1,1)$-form ω which is positive on complex lines of the tangent bundle, namely $\omega(u, iu) > 0$ for every $u \neq 0 \in TS$. A complex surface is called Kähler iff it admits a Kähler form.

Lemma 2.4. *The Cauchy–Riemann foliation of a Levi-flat in a Kähler surface does not have any Reeb component.*

Proof. By contradiction, the integral of ω on the boundary would both be positive (by Kähler property) and zero (by Stokes formula). □

Hence, Proposition 2.2 is a consequence of Lemma 2.4 and of the following result:

Theorem 2.5 (Novikov)**.** *Let M be a compact orientable 3-manifold endowed with a transversally orientable 2-dimensional foliation $\mathcal{F}$ of class C^2. The following assertions are equivalent*

(a) The foliation $\mathcal{F}$ contains a Reeb component.

(b) There exists a leaf $L \in \mathcal{F}$ such that the inclusion map $\pi_1(L) \to \pi_1(M)$ between the fundamental groups has a non-trivial kernel.

Moreover, if there exists a closed and homotopically trivial loop transverse to $\mathcal{F}$, then the foliation $\mathcal{F}$ contains a Reeb component. This occurs in particular when the fundamental group of M is finite.

We now review examples showing that all of the geometries (2.1) except $\mathbb{S}^3$ are carried by Levi-flats in algebraic surfaces. First we recall that the geometries Nil, Sol and $\mathbb{H}^3$ are supported by non trivial surface bundles. A *surface bundle* is the quotient of $[0,1] \times \Sigma$ by the relation $(0,x) \sim (1,\Phi(x))$, where Σ is a compact oriented surface and Φ is a diffeomorphism of Σ preserving the orientation.

We shortly denote a surface bundle $\mathbb{S}^1 \ltimes_\Phi \Sigma$. Its *monodromy* is the projection $[\Phi]$ of Φ in the mapping class group $\mathrm{MCG}(\Sigma)$. An element $[\Phi] \in \mathrm{MCG}(\Sigma)$ is called *elliptic* if its order is finite, *reducible* if there is a finite collection of pairwise disjoint simple closed curves in Σ whose union is invariant by a diffeomorphism in $[\Phi]$, and *pseudo-Anosov* in the other cases, see [69, Section 2].

If Σ has genus 1, the surface bundle is called a *torus bundle*. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\Sigma \simeq \mathbb{R}^2/\mathbb{Z}^2$ by linear transformations and captures all the classes of $\mathrm{MCG}(\Sigma)$. A *unipotent torus bundle* is a torus bundle whose monodromy comes from a unipotent matrix in $\mathrm{SL}(2,\mathbb{Z})$ (reducible monodromy), it carries the Nil geometry. A *hyperbolic torus bundle* is a torus bundle whose monodromy comes from a hyperbolic matrix in $\mathrm{SL}(2,\mathbb{Z})$ (pseudo-Anosov monodromy), it carries the S*ol* geometry.

We shall realize such surface bundles in *singular holomorphic fibrations.* Such a fibration stands for a holomorphic map $f\colon S \to B$ where S is a complex surface and B is a compact Riemann surface, see [3, Chapter V]. Let $p_1, \ldots, p_n$ be the singular values of f (it may be empty). A *fibered Levi-flat hypersurface* is a Levi-flat hypersurface of the form $f^{-1}(\gamma)$, where $f\colon S \to B$ is a singular holomorphic fibration and $\gamma \subset B \setminus \{p_1, \ldots, p_n\}$ is a simple closed path. Such hypersurfaces were already considered by Poincaré in his study of cycles on algebraic surfaces, see [63].

Proposition 2.6. *Every geometry $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil or Sol is carried by a fibered Levi-flat hypersurface. Moreover, $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ are* *carried by fibered Levi-flat hypersurfaces in surfaces of general type.*

We give the sketch of proof of this fact. It is easy to realize $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ by using products of compact Riemann surfaces $S = \Sigma \times B$. To exhibit fibered Levi-flat hypersurfaces with the geometries Nil and Sol, we use the following classical proposition, see [31, Chapter II, Section 2.3]. Here the complex surface S comes from a singular holomorphic fibration by elliptic curves over the Riemann sphere.

Proposition 2.7. *Let $f\colon S \to \mathbb{P}^1(\mathbb{C})$ be a singular elliptic fibration. Let $p_1, \dots, p_n$ be the singular values of f, assume that this set is not empty. Then the monodromy representation from the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \dots, p_n\}$ to $\mathrm{SL}(2, \mathbb{Z})$ is surjective.*

Using this proposition, one easily constructs Levi-flat hypersurfaces of the form $f^{-1}(\gamma)$ (up to finite coverings of f) carrying the geometries Nil or Sol. We refer to [20]. To realize $\mathbb{H}^3$ we use Thurston's theorem, see [69, Theorem 0.1].

Theorem 2.8 (Thurston)**.** *Let Σ be a compact oriented surface of genus $g \geq 2$. A surface bundle $\mathbb{S}^1 \ltimes_\Phi \Sigma$ carries the geometry $\mathbb{H}^3$ if and only if its monodromy $[\Phi]$ is pseudo-Anosov.*

By using the same arguments as before, the following theorem provides fibered Levi-flat hypersurfaces modelled on $\mathbb{H}^3$, see [67, Corollary 1].

Theorem 2.9 (Shiga)**.** *Let B be a compact Riemann surface with genus larger than or equal to 2. Let $f\colon S \to B$ be a singular holomorphic fibration, such that the generic fiber has genus ≥ 2 and its modulus is not locally constant (e.g. a Kodaira fibration). Let $p_1, \dots, p_n$ be the critical values of f. Then there exists an immersed simple closed curve γ in $B \setminus \{p_1, \dots, p_n\}$ whose monodromy is pseudo-Anosov.*

Note that the surface S in this theorem is of general type, since the genus of the base and the fibers of f is larger than 1, see [3, Chapter 3, Theorem 18.4]. This completes the proof of Proposition 2.6.

It remains to treat the geometry of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. This geometry is supported for instance by non-trivial circle bundles over compact oriented surfaces of genus $g \geq 2$, see [65, Theorem 5.3]. There exists Levi-flat hypersurfaces with this topology in flat $\mathbb{P}^1(\mathbb{C})$-bundles over compact Riemann surfaces. Namely, we consider a representation $\rho\colon \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$, and the flat $\mathbb{P}^1(\mathbb{C})$ bundle $S_\rho = C \ltimes_\rho \mathbb{P}^1(C)$, see 1.3; the subset $M_\rho := C \ltimes_\rho \mathbb{P}^1(\mathbb{R}) \subset S_\rho$ is an analytic Levi-flat hypersurface, having the structure of an oriented circle bundle over C. We denote e the Euler class of M_ρ. We recall that this invariant belongs to $H^2(C,\mathbb{Z}) \simeq \mathbb{Z}$ and characterizes the circle bundle up to isomorphism, see e.g. [57, Section 2]. Note that $|e| = 2g-2$ if and only if ρ is an isomorphism between $\pi_1(C)$ and a Fuchsian group. In this case M_ρ is diffeomorphic to the unitary tangent bundle of C, see [72, Proposition 6.2].

Proposition 2.10. *Let C be a compact oriented surface of genus $g \geq 2$ and let $e \in \mathbb{Z}$ satisfying $|e| \leq 2g-2$. There exists a flat $\mathbb{P}^1(\mathbb{C})$-bundle S over C and a Levi-flat hypersurface $M \subset S$ which is diffeomorphic to a circle bundle over C with Euler class e.*

Proof. If $|e| \leq 2g-2$ then there exists a representation $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{R})$ such that M_ρ has Euler class e, see [38, Theorems A and B]. □

2.3. Levi-flat circle bundles in surfaces of general type

We begin with an upper bound on the Euler class of Levi-flat circle bundles.

Proposition 2.11. *Let S be a surface of general type and M be a Levi-flat hypersurface of class C^2 in S. Assume that M is an oriented circle bundle over a compact oriented surface C of genus $g \geq 2$. Then the Euler class of M satisfies $|e| \leq 2g-2$.*

Sketch of proof. We can assume $e \neq 0$. We first prove that the Cauchy–Riemann foliation has no compact leaf. As we will see later, see 2.4, the general type assumption implies that every leaf is hyperbolic. Assuming by contradiction that there exists a compact leaf L, it would have genus $g \geq 2$, and the Euler class being different from 0, it is easy to see that L would be compressible, namely the map $\pi_1(L) \to \pi_1(M)$ would not be injective. Novikov's theorem would then provide a Reeb component, which contradicts the fact that the surface is Kähler. Hence, there are no compact leaves, and the result follows from the combination of the next two results. □

Theorem 2.12 (Thurston)**.** *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. Assume that $\mathcal{F}$ is an oriented 2-dimensional foliation on M of class C^2, and that $\mathcal{F}$ does not have any compact leaf. Then there exists a diffeomorphism Ψ of M of class C^2 isotopic to the identity such that $\Psi_*\mathcal{F}$ is transverse to the circle fibration.*

Theorem 2.13 (Milnor-Wood)**.** *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. If M supports a transversally oriented 2-dimensional foliation which is transverse to the circle fibration,* *then its Euler class satisfies* $|e| \leq 2g-2$.

Remark 2.14. The question of the existence of Levi-flats in algebraic surfaces diffeomorphic to circle bundles over hyperbolic compact surface with arbitrarily large Euler class, obtained by the technique called in french "tourbillonement de Reeb", remains open.

The following result provides a construction of Levi-flat hypersurfaces in surfaces of general type with a non trivial Euler class.

Theorem 2.15. *For every* $\epsilon > 0$ *there exist a surface of general type* S_ϵ *and a Levi-flat hypersurface* $M_\epsilon \subset S_\epsilon$ *which is diffeomorphic to an oriented circle bundle* M_ϵ *over a compact oriented surface* C_ϵ *of genus* ≥ 2*. We have* $|e(M_\epsilon)/\chi(C_\epsilon)| \in [1/5-\epsilon, 1/5]$*, where* $e(M_\epsilon)$ *denotes the Euler class of* M_ϵ *and* $Eu(C_\epsilon)$ *denotes the Euler characteristic of* C_ϵ*.*

Sketch of proof. Here we only prove that there exists a Levi-flat in a surface of general type which is diffeomorphic to a non trivial circle bundle, hence carrying the geometry $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Let C be a compact algebraic curve of genus $g \geq 2$. By the uniformization theorem, see 1.13, there is a biholomorphism $D\colon \widetilde{C} \to \mathbb{H}$ which is equivariant w.r.t. some representation $\rho\colon \pi_1(C) \to \mathrm{Aut}(\mathbb{H}) \subset \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$. Let $(S_\rho, \mathcal{F}_\rho)$ be the flat $\mathbb{P}^1(\mathbb{C})$-bundle over C of monodromy ρ, defined as in 1.3. There is a Levi-flat defined by $M_\rho = C \ltimes \mathbb{P}^1(\mathbb{R})$, which is diffeomorphic to the unitary tangent bundle of the surface C equipped with e.g. its Poincaré metric. The bundle $S_\rho \to C$ has a holomorphic section $s\colon C \to S_\rho$ defined as the level of the universal covers by $s(x) = (x, D(x))$. Of course we are not done since the Kodaira dimension of S_ρ is $-\infty$, hence S_ρ is not of general type.

We construct $(S_\varepsilon, M_\varepsilon)$ as a double ramified covering of (S_ρ, M_ρ). To define such a ramified cover, let $E \to S_\rho$ be a holomorphic line bundle and $h\colon S_\rho \to 2E$ (recall our additive notation for tensor product of line bundles) be a holomorphic section, whose zero divisor $h^{-1}(0)$ is a smooth reduced algebraic curve in S_ρ. The algebraic surface

$$S_\varepsilon = \{(w,\zeta) \in E \mid \zeta^2 = h(w)\}. \tag{2.16}$$

is a $2:1$ ramified cover (defined by $\pi(x,\zeta) = x$), ramifying over $h^{-1}(0)$. We easily verify that the pull-back of $\mathcal{F}_\rho$ is a singular holomorphic foliation $\mathcal{F}_\varepsilon$ whose singularities are the pull-back in S_ε of the points of tangency between $\mathcal{F}_\rho$ and $h^{-1}(0)$. Hence assuming that $h^{-1}(0)$ intersects M_ρ transversally, the set $M_\varepsilon = \pi^{-1}(M_\rho)$ is a Levi-flat hypersurface of S_ε. To understand its topology, one has to understand the topology of the link $h^{-1}(0) \cap M_\rho$ in M_ρ.

It is well-known that if E is sufficiently ample[1] then the surface S_ε constructed above is of general type (e.g. if F is ample, then the sufficiently large powers of F will work). For such a line bundle, choosing at random the section h of its square would probably lead to a hyperbolic manifold M_ε. Hence we will need to make a very particular choice. Define $F = \mathcal{O}\big(ks + \sum_{j\in J} f_j\big)$ where k is an integer, and f_j are distinct fibers of the fibration $S_\rho \to C$. If we assume furthermore that k and the number $|J|$ of fibers f_j are both even, then it is possible to find a line bundle E such that $2E = F$. By definition of F there exists a holomorphic section $h_0\colon S_\rho \to F$ such that $h_0^{-1}(0) = s \cup \bigcup_j f_j$. Observe that the zero set of h_0 is transverse to M_ρ and that its intersection with M_ρ is a union of $|J|$ fibers of the circle fibration $M_\rho \to C$, hence is a quite simple link. The section h_0 is not convenient for our purpose, since its zero set is not smooth (at the intersection points of f_j and s). Nevertheless, we can show that if k and $|J|$ are large enough, the line bundle E is ample, and one can make a small perturbation h of h_0 with a smooth zero set. For such a choice, the couple (E, h) yields the desired Levi-flat $M_\varepsilon \subset S_\varepsilon$ diffeomorphic to a non trivial circle bundle. See details in [20]. $\square$

The sup of the ratios $|e(M)/\mathrm{Eu}(C)|$, where M is a Levi-flat in a surface of general type diffeomorphic to a circle bundle of Euler class $e(M)$ over a hyperbolic compact surface C, is unknown. The following result shows that the value $|e(M)/\mathrm{Eu}(C)| = 1$ (the maximal permitted by Proposition 2.11) is not reached:

Theorem 2.17. *A Levi-flat hypersurface of class C^2 in a surface of general type is not diffeomorphic to the unitary tangent bundle of a hyperbolic compact two dimensional orbifold.*

The proof of this result uses a foliated Lyapunov exponent associated to the Cauchy–Riemann foliation and its sketch is postponed to Corollary 2.25. See [20] for details.

2.4. Hyperbolicity and topological consequences

The following result will be crucial for studying the topology of Levi-flats in surfaces of general type.

Proposition 2.18. *Let M be a Levi-flat of class C^2 in a surface of general type. Then the Cauchy–Riemann foliation of $\mathcal{F}$ has hyperbolic leaves.*

[1] Ample means that it carries a metric of positive curvature.

Sketch of proof. We prove Proposition 2.18 under the assumption that K_S is ample, namely that it has a metric of positive curvature. Assume that $\mathcal{F}$ has a compact leaf L. Adjunction formula then gives

$$\mathrm{Eu}(L) = -L \cdot K_S - L \cdot N_L.$$

The first term of the right hand side is < 0 because K_S has a metric of positive curvature, and the second one is zero because the normal bundle of L has a flat connexion (the Bott connexion induced by the foliation), hence C is hyperbolic.

Assume now that there exists a parabolic leaf L. A theorem of Candel shows that there exists an Ahlfors current T such that $T \cdot K_{\mathcal{F}} = 0$ (see [11]). Using the leafwise adjunction formula we obtain

$$T \cdot K_{\mathcal{F}} = T \cdot K_S + T \cdot N_{\mathcal{F}}$$

The right hand side is > 0 for the same reason as before (take the Bott connexion on $N_{\mathcal{F}}$ in equation (1.20)). This yields a contradiction. □

We deduce the following application:

Theorem 2.19. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface of class C^2 in S. Then the fundamental group of M has exponential growth. In particular M does not carry the geometries $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{R}^3$ nor Nil.*

Sketch of proof. Since there is no Reeb component, Novikov's theory shows that the leaves of the pull-back of the Cauchy–Riemann foliation in the universal cover of M are simply connected, and that moreover they are quasi-isometrically embedded in $\widetilde{M}$. Hence, $\widetilde{M}$ has exponential growth, by Proposition 2.18 and by Verjovsky-Candel result on the continuity of the Poincaré metric, see 1.17. □

Remark 2.20. The hyperbolicity of the Cauchy–Riemann foliation is related to the following open conjecture.

Conjecture 2.21 (Green-Griffiths)**.** *Let S be a surface of general type. There exists a proper subvariety $Y \subset S$ such that every entire curve $f\colon \mathbb{C} \to S$ satisfies $f(\mathbb{C}) \subset Y$.*

This problem was solved by McQuillan [59] for surfaces of general type satisfying $c_1^2(S) > c_2(S)$. He proved that every non-degenerate entire curve $f\colon \mathbb{C} \to S$ is tangent to a singular holomorphic foliation on (a finite cover of) S. A contradiction is deduced from positivity properties of the tangent bundle of the foliation. Brunella provided an alternative proof in [7] by using the normal bundle of the foliation. An important difficulty in these works is that $f(\mathbb{C})$ can contain a singular point of the foliation. In our non-singular context the proof is simpler because we directly use adjunction formula. We refer to the survey [23] for recent results concerning Green-Griffiths conjecture.

2.5. The Anosov property and application to the topology of Levi-flats

A Levi-flat $M \subset S$ in a complex surface is called *Anosov* if its Cauchy–Riemann foliation is topologically conjugate to the weak unstable foliation of a 3-dimensional Anosov flow on some compact 3-manifold N. Classical examples of Anosov flows are the geodesic flow on the unitary tangent bundle of compact orientable surfaces of genus ≥ 2 and the horizontal flow on hyperbolic torus bundles. There are many other examples, for instance on hyperbolic 3-manifolds and graph 3-manifolds, see [30, 39, 41]. One can verify that Anosov Levi-flat hypersurfaces do not have any transverse invariant measure, their foliation $\mathcal{F}$ is therefore hyperbolic. We have the following upper bound for the Lyapunov exponent.

Theorem 2.22. *Let S be a complex surface and M be an immersed Anosov Levi-flat hypersurface in S. We endow the leaves of the Cauchy–Riemann foliation $\mathcal{F}$ with the Poincaré metric g_P. Let T be an ergodic foliated harmonic current of $\mathcal{F}$. Then the Lyapunov exponent of T satisfies $\lambda(T) \leq -1$.*

Sketch of proof. We use that the trajectories of the Anosov flow in the hyperbolic uniformizations of the leaves are quasigeodesics for the Poincaré metric, to produce a new flow by stretching these trajectories to geodesics. We obtain a continuous flow on M whose orbits are leafwise geodesics for the Poincaré metric. Let v_P the leafwise Poincaré volume form. Since the result does not depend the projective class of T, we can assume that the foliated harmonic measure $T \wedge v_P$ has mass one. This latter is shown to be a SRB measure for the stretched flow. Moreover, the Lyapunov exponents of this measure are $1, 0, \lambda$. (The Lyapunov exponents are not a priori defined since the stretched flow is only continuous. However, it is smooth along the leaves, which gives the exponents 1 and 0, and using the C^1 transverse structure of the foliation we can define another exponent, which we identify with λ). The ingredients for this computation involve the shadowing property of geodesics by Brownian paths due to Ancona, see [2, théorème 7.3, p. 103]. The bound $\lambda(T) + 1 \leq 0$ to be proved then relies on volume estimates in the spirit of Margulis-Ruelle's inequality. □

Corollary 2.23. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface in S. Then M is not Anosov.*

Sketch of proof. We indicate the proof when K_S has a metric of positive curvature. The proof then relies on the leafwise adjunction formula, which gives $T \cdot K_{\mathcal{F}} = T \cdot N_{\mathcal{F}} + T \cdot K_S > T \cdot N_{\mathcal{F}}$. We deduce that the Lyapunov exponent verifies the following pinching estimates

$$-1 < \lambda(T) \leq 0 \tag{2.24}$$

which is contradictory with being Anosov by Theorem 2.22. □

Corollary 2.25. *A Levi-flat in a surface of general type is not diffeomorphic to a quotient of the Lie groups Sol or* $\mathrm{PSL}(2, \mathbb{R})$ *by a cocompact lattice.*

Sketch of proof. The proof is by contradiction. Assuming that a Levi-flat is diffeomorphic to one of those manifolds, we use deep results of resp. Ghys/Sergiescu, see [37], and Matsumoto, see [57], which enable to prove that the Levi-flat is Anosov. Hence the contradiction comes from Corollary 2.23. In order to apply the mentioned theorems, one needs to verify that the Cauchy–Riemann foliation has no compact leaf, which is done by using the hyperbolicity of the leaves together with Novikov's theory. □

3. Lecture 3 – Complex projective structures: Lyapunov exponent, degree and harmonic measure

3.1. A rough guide to complex projective structures

Let C be a smooth complex quasi-projective curve of negative Euler characteristic. We denote by g its genus and by n its number of punctures. A complex projective structure on C is a maximal atlas of holomorphic charts $z_j : U_j \subset C \to \mathbb{P}^1(\mathbb{C})$ (called projective charts) which overlap as

$$z_j = \frac{az_k + b}{cz_k + d},$$

on the intersection $U_j \cap U_k$, where a, b, c, d are complex numbers such that $ad - bc \neq 0$. We will denote $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, and will refer to $\mathbb{P}^1$-structures instead of complex projective structures. Two $\mathbb{P}^1$-structures on C are equivalent if they define the same atlas of projective charts.

It is convenient to define a $\mathbb{P}^1$-structure on C in terms of the so-called development-holonomy pair $(\mathsf{dev}, \mathsf{hol})$. Each projective chart can be extended analytically as a locally injective meromorphic map $\mathsf{dev} : \widetilde{C} \to \mathbb{P}^1$, satisfying the equivariance property $\mathsf{dev} \circ \gamma = \mathsf{hol}(\gamma) \circ \mathsf{dev}$, where hol is a representation $\pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$. A development-holonomy pair is not unique for a given projective structure. Namely, if $A \in \mathrm{PSL}(2,\mathbb{C})$, $(A \circ \mathsf{dev}, A \circ \mathsf{hol} \circ A^{-1})$ gives another development-holonomy pair. We refer here to the survey paper by Dumas, see [24] for a comprehensive treatment of this notion.

When the surface C is not compact (hence by assumption it is biholomorphic to a compact Riemann surface punctured at a finite set), we restrict ourselves to the subclass of *parabolic* $\mathbb{P}^1$-structures. Such a structure has the following well-defined local model around the punctures: each puncture has a neighborhood which is *projectively* equivalent to the quotient of the upper half plane by the translation $z \mapsto z+1$.

A $\mathbb{P}^1$-structure on C can be understood by the way of the Schwarzian derivative. Indeed, introduce the following differential operator called the Schwarzian:

$$S(f) := \{f,z\}dz^2 \text{ where } \{f,z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{3.1}$$

for every holomorphic local diffeomorphism $f\colon U \subset \mathbb{C} \to \mathbb{C}$. We have the following two fundamental properties

(1) $S(g \circ f) = S(f) + f^*S(g)$ for every local diffeomorphisms $f\colon U \subset \mathbb{C} \to V \subset \mathbb{C}$ and $g\colon V \subset \mathbb{C} \to W \subset \mathbb{C}$.
(2) $S(f) = 0$ iff $f(z) = \frac{az+b}{cz+d}$ for some complex numbers a, b, c, d such that $ad - bc \neq 0$.

In particular, let σ_1 and σ_2 be two $\mathbb{P}^1$-structures on C. Pick projective charts z_1 and z_2 defined on some commun open set $U \subset C$ of σ_1 and σ_2 respectively, and define the holomorphic quadratic differential $q_{\sigma_1,\sigma_2} = \{z_2, z_1\}dz_1^2$. Properties (1) and (2) show that q_{σ_1,σ_2} does not depend on the chosen projective charts z_1 and z_2, and thus defines a holomorphic quadratic differential on the curve C. Reciprocally, given a $\mathbb{P}^1$-structure σ_1 and a holomorphic quadratic differential q on C, there exists a unique $\mathbb{P}^1$-structure σ_2 on C such that $q = q_{\sigma_1,\sigma_2}$. In particular, at least when C is compact, the set of projective structures on C is an affine space directed by the vector space of holomorphic quadratic differentials on C. This shows that the set $P(C)$ of $\mathbb{P}^1$-structures on a compact algebraic curve of genus $g \geq 2$ is isomorphic to $\mathbb{C}^{3g-3}$. We will not discuss here the analogous computation in the punctured case, which relies on results of Fuchs and Schwarz, but we state the result: the set $P(C)$ of parabolic $\mathbb{P}^1$-structures on C is isomorphic to $\mathbb{C}^{3g-3+n}$.

One of the interests in studying complex projective structures comes from their relations to uniformization problems in two or three dimensions. The main illustration of this is certainly given by the uniformization theorem of Poincaré-Koebe, which in particular defines a canonical projective structure σ_{Fuchs} (by viewing C as a quotient of $\mathbb{H}$ under a Fuchsian group). Other kind of uniformizations have been considered, e.g. Schottky uniformizations, and lead to parabolic $\mathbb{P}^1$-structures as well. More generally, the Ahlfors finiteness theorem provides many examples of parabolic $\mathbb{P}^1$-structures:

Theorem 3.2 (Ahlfors finiteness theorem)**.** *Let Γ be a finitely generated discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$*. Then the quotient of the discontinuity set $\Omega \subset \mathbb{P}^1$ by Γ is a finite type Riemann surface. Moreover, if Γ is torsion free, the natural $\mathbb{P}^1$-structure that it inherits is parabolic.*

The last (less known) part of the theorem is proved in [1, Lemma 1]. The structures produced by Theorem 3.2 have been known as *covering projective structures*, because they are characterized by the fact that the developing map is a covering onto its image [49, 50]. A particular example is given by quasi-Fuchsian deformations of the canonical structure σ_{Fuchs}. These structures play an important role in Teichmüller theory. Recall that the Teichmüller space $T(C)$ is defined as the set of equivalence classes of couples $(D,[\Psi])$ where D is a Riemann surface and $[\Psi]$ is a homotopy class of diffeomorphism between C and D. Two couples $(D_1,[\Psi_1])$ and $(D_2,[\Psi_2])$ are considered as equivalent if $\Psi_2 \circ \Psi_1^{-1}$ is homotopic to a biholomorphism from D_1 to D_2. Recall the following important result.

Theorem 3.3 (Bers simultaneous uniformization theorem)**.** *For every $(D,[\Psi]) \in T(C)$, there exists a unique representation ρ from $\pi_1(C)$ to* $\mathrm{PSL}(2,\mathbb{C})$ *(up to conjugation) preserving a partition $\mathbb{P}^1 = \mathcal{D}_C \cup \Lambda \cup \mathcal{D}_D$, where Λ is a topological circle, and $\mathcal{D}_C$ (resp. $\mathcal{D}_D$) is the image of a ρ-equivariant univalent holomorphic (resp. anti-holomorphic) map from $\widetilde{C}$ (resp. $\widetilde{D}$, observe that we have an identification of $\pi_1(D)$ with $\pi_1(C)$ induced by Ψ) to $\mathbb{P}^1$.*

Let $P(C)$ be the set of (parabolic) $\mathbb{P}^1$-structures on C. Observe that for every $(D,[\Psi]) \in T(C)$, the holomorphic univalent ρ-equivariant mapping given by Theorem 3.3 produces a (parabolic) $\mathbb{P}^1$-structure, and that this later determines the element $(D,[\Psi])$. This defines an embedding $B\colon T(C) \to P(C)$, called the Bers embedding. Bers proved that the map B is holomorphic, and that its image $B(C)$ is relatively compact in $P(C)$. This later is called the *Bers slice*.

There are many other examples of parabolic $\mathbb{P}^1$-structures. For instance surgery operations such as grafting (see Hejhal's original construction in [42]) may produce a parabolic $\mathbb{P}^1$-structure with holonomy a Kleinian group that is not of covering type.

Theorem 3.4 (Hejhal)**.** *There exist $\mathbb{P}^1$-structures on compact curves such* *that the developing map is not a covering onto its image, but whose holonomy has image a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$.

Such projective structures are usually called *exotic*. The prototype of such an exotic projective structure is obtained by inserting a Hopf annulus after cutting a given $\mathbb{P}^1$-structure along a simple closed curve. More precisely start with the quotient C_u of $\mathbb{H}$ by a lattice $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ containing as a primitive element the hyperbolic transformation $\gamma(z) = \alpha z$ for $\alpha > 1$, and consider

$$C = \left(\overline{C_u \setminus \gamma_u} \cup \overline{H \setminus \gamma_H}\right)/\{\gamma_u^{\pm} \simeq \gamma_H^{\mp}\},$$

where $\gamma_u = \alpha\backslash i\mathbb{R}^{+*} \subset C_u$, $H = \alpha\backslash\mathbb{C}^*$ is the Hopf torus, and $\gamma_H = \alpha\backslash i\mathbb{R}^{+*} \subset H$. The set of exotic $\mathbb{P}^1$-structures in $P(C)$ is organized as a countable union of non empty connected open subsets called *exotic components*.

Using the point of view of the Schwarzian derivative, one can construct yet other examples of $\mathbb{P}^1$-structures on C. For instance, one can prove that there exists a non empty open subset of $P(C)$ formed by $\mathbb{P}^1$-structures on C whose holonomy has image a dense subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We refer to [9] for a proof of this fact in the case of the fourth punctured sphere, which readily extends to all algebraic curves.

There are nice pictures of the decomposition of $P(C)$ into the various subsets described above: Bers slice, exotic components, etc. We refer e.g. to [48].

3.2. The degree of a $\mathbb{P}^1$-structure

Let g_P be the unique complete conformal metric of curvature -1 on C. It is well known that when C is of finite type, the hyperbolic metric has finite volume. Recall that a representation $\pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is non elementary if it does not preserve any probability measure on the Riemann sphere. The holonomy of a parabolic projective structure always non elementary: see [33, Theorem 11.6.1, p. 695] for the compact case, and [9, Lemma 10] for the punctured case.

If σ is a parabolic projective structure, we want to define $\delta(\sigma)$ as a nonnegative number counting the average asymptotic covering degree of $\mathsf{dev}_\sigma \colon \widetilde{C} \to \mathbb{P}^1$. For any $x \in \widetilde{C}$ we denote by $B(x,R)$ the ball centered at x of radius R in the Poincaré metric, and by vol the hyperbolic volume.

Definition-Proposition 3.5. Let C be a Riemann surface of finite type and σ be a parabolic $\mathbb{P}^1$-structure on X. Choose a universal convering $c \colon \widetilde{C} \to C$, and a developing map $\mathsf{dev} \colon \widetilde{C} \to \mathbb{P}^1$. Let (x_n) be a sequence of points in $\widetilde{C}$ whose projections $c(x_n)$ stay in a compact subset of C, R_n be a sequence of radii tending to infinity, and (z_n) be an arbitrary sequence in

$\mathbb{P}^1$. Then the limit

$$\delta = \lim_{n\to\infty} \frac{\#B(x_n, R_n) \cap \mathrm{dev}^{-1}(z_n)}{\mathrm{vol}(B(x_n, R_n))} \tag{3.6}$$

exists, and does not depend on the chosen sequences (x_n), (R_n) nor on the developing map dev. The number δ is invariant by taking finite coverings, so does not behave like a degree. We define $\deg(\sigma) = \mathrm{vol}(C)\delta$, and call this number the *degree* of the $\mathbb{P}^1$-structure.

The very reason for the normalization $\deg(\sigma) = \mathrm{vol}(X)\delta$ is clearer when dealing with branched projective structures. Such structures are defined by non constant equivariant meromorphic maps defined on the universal cover w.r.t. a representation of the covering group to $\mathrm{PSL}(2,\mathbb{C})$. The most basic example of a branched projective structure is a non constant meromorphic function $f\colon C \to \mathbb{P}^1$. For such a structure, one verifies that the limit (3.6) exists, and that the average degree in the sense of 3.5 coincides with the topological degree of the map f.

The existence of the limit in (3.6) is not obvious, in particular due to the possibility of boundary effects. The proof ultimately relies on the equidistribution theorem of Bonatti and Gomez-Mont [5] mentioned in the first lecture, Theorem 1.34.

It also makes use of the following dictionary between projective structures on curves and transverse sections of flat $\mathbb{P}^1$-bundles over curves, which was developped in depth in [53].

Suppose that σ is a $\mathbb{P}^1$-structure. Introduce the flat $\mathbb{P}^1$-bundle $(S_{\mathrm{hol}}, \mathcal{F}_{\mathrm{hol}})$, see 1.3, where $(\mathrm{dev}, \mathrm{hol})$ is a development-holonomy pair for the structure σ. Observe that the bundle map $S_{\mathrm{hol}} \to C$ has a section $s\colon C \to S_{\mathrm{hol}}$ defined at the level of the universal covers by $x \mapsto (x, \mathrm{dev}(x))$. This section – we identify the section and its image here – is transverse to the foliation $\mathcal{F}_{\mathrm{hol}}$.

Reciprocally, if $\rho\colon \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is any representation, a section of S_ρ transverse to the foliation $\mathcal{F}_\rho$ gives rise to a projective structure on C, by restricting the transverse projective structure of the foliation $\mathcal{F}_\rho$ to the section. This operation is the inverse of the one described in the last paragraph.

Sketch of proof of 3.5. After these preliminaries, let us sketch the proof of the convergence (3.6). We will give the proof only in the case C is compact. The punctured case necesitates a separate technical analysis. We refer to [19] for the details. Let σ a $\mathbb{P}^1$-structure and s its corresponding section of S_{hol}. We denote by T a foliated harmonic current on $(S_{\mathrm{hol}}, \mathcal{F}_{\mathrm{hol}})$ normalized so that its product with the Poincaré volume form is 1. The number $\#B(x_n, R_n) \cap \mathrm{dev}^{-1}(z_n)$ is easily seen to be the number of intersection of points of the leafwise ball $B_{\mathcal{F}}(w_n, R_n)$ with s, where w_n is the projection in S_ρ of the point (x_n, z_n). Hence since the leafwise balls normalized by their volume (considered as currents) tend to T (Theorem 1.34), one shows (with a little additional technical work) that $\frac{\#B(x_n,R_n)\cap \mathrm{dev}^{-1}(z_n)}{\mathrm{vol}(B(x_n,R_n))}$ tends to the *geometric* intersection product of T with s. This product is defined in the following way: T can be thought of as a family of transverse measures for the foliation $\mathcal{F}_\rho$, and it induces a Radon measure on any curve of S_ρ. The mass of this measure is by definition the intersection product of T with s and is denoted $T \dot\wedge s$. □

A corollary from the proof of 3.5 yields the following.

Corollary 3.7. *The degree vanishes iff σ is a covering projective structure.*

3.3. Lyapunov exponent of $\mathbb{P}^1$-structures

Fix a basepoint $\star \in C$, in particular an identification between the covering group $\pi_1(C)$ and the usual fundamental group $\pi_1(C,\star)$. As C is endowed with its Poincaré metric, Brownian motion on C is well-defined. Let $W_\star$ be the Wiener measure on the set of continuous paths $\omega\colon [0,\infty) \to X$ starting at $\omega(0) = \star$.

Definition-Proposition 3.8. Let C and σ be as above. Define a family of loops as follows: for $t > 0$, consider a Brownian path ω issued from $\star$, and concatenate $\omega|_{[0,t]}$ with a shortest geodesic joining $\omega(t)$ and $\star$, thus obtaining a closed loop $\widetilde{\omega}_t$. Then for $W_\star$ a.e. Brownian path ω the limit

$$\chi(\sigma) = \lim_{t\to\infty} \frac{1}{t} \log \left\|\mathsf{hol}\,(\widetilde{\omega}_t)\right\| \tag{3.9}$$

exists and does not depend on ω. This number is by definition the Lyapunov exponent of σ.

Here $\|\cdot\|$ is any matrix norm on $\mathrm{PSL}(2,\mathbb{C})$. The existence of the limit in (3.9) was established in [18, Definition-Proposition 2.1]. As expected it is a consequence of the subadditive ergodic theorem. In the notation of [18], $\chi(\sigma) = \chi_{\mathrm{Brown}}(\mathsf{hol})$. Another way to define $\chi(\sigma)$ goes as follows (see [18, Remark 3.7]: identify $\pi_1(C)$ with a Fuchsian group Γ and choose independently random elements $\gamma_n \in \Gamma \cap B_{\mathbb{H}}(0, R_n)$, relative to the counting measure. Here (R_n) is a sequence tending to infinity as fast as, say n^α for $\alpha > 0$. Then almost surely

$$\frac{1}{d_{\mathbb{H}}(0,\gamma_n(0))} \log \|\mathsf{hol}(\gamma_n)\| \underset{n\to\infty}{\longrightarrow} \chi(\sigma).$$

The following formula relates the Lyapunov exponent $\chi(\sigma)$ to the degree defined in the last subsection.

Theorem 3.10. *Let σ be a parabolic holomorphic $\mathbb{P}^1$ structure on C. Let as above $\chi(\sigma)$, $\delta(\sigma)$, and $\deg(\sigma)$ respectively denote the Lyapunov exponent, the unnormalized degree and the degree of σ. Then the following formula holds:*

$$\chi(\sigma) = \frac{1}{2} + 2\pi\delta(\sigma) = \frac{1}{2} + \frac{\deg(\sigma)}{|\mathrm{eu}(\mathrm{X})|}. \tag{3.11}$$

Theorem 3.10 could be understood as the analogue of the familiar Manning-Przytycki formula [55, 64] for the Lyapunov exponent of the maximal entropy measure of a polynomial. Recall that this formula states that for a polynomial P of degree d in one variable

$$\chi = \log d + \sum_{P'(c)=0} G(c),$$

where G is the Green function. See [55, 64]. The term $\log d$ is constant on parameter space (equal to the entropy of the polynomial P), as the term $\frac{1}{2}$ in formula (3.11), and the term $\sum_c G(c)$ is non negative, as well as the degree.

This reinforces an analogy between Mandelbrot sets and Bers slices that was brought to light by McMullen [58]. Namely, the Lyapunov exponent is minimal on these sets (equal to $\log d$ for the Mandelbrot set and to $1/2$ for the Bers slice). We will develop more on this analogy later on.

Sketch of proof. Surprisingly enough, the proof is based on the ergodic theory of holomorphic foliations. Again we will indicate the proof only when C is compact, and refer to [19] for the punctured case. Recall that there is a dictionary between $\mathbb{P}^1$-structures and transverse sections of flat $\mathbb{P}^1$-bundles. In this dictionary, there is a simple relation between the Lypaunov exponent χ defined in 3.8 and the foliated Lyapunov exponent defined in 1.6.

Lemma 3.12. *Let σ be a $\mathbb{P}^1$-structure, $(\mathrm{dev}, \mathrm{hol})$ a development-holonomy pair, and $\lambda(\sigma)$ be the Lyapunov exponent of the foliated complex surface $(S_{\mathrm{hol}}, \mathcal{F}_{\mathrm{hol}})$ computed w.r.t. the leafwise Poincaré metric. Then $\chi(\sigma) = -2\lambda(\sigma)$.*

The proof of this lemma essentially follows from the formula of the derivative of a Moebius map in the spherical metric, namely if $h(z) = \frac{az+b}{cz+d}$, then $\|Dh(z)\| = \frac{|ad-bc|}{|az+b|^2+|cz+d|^2}$. We refer to [19] for the detailed proof of Lemma 3.12.

Next, the proof of Theorem 3.10 relies on cohomological computations in $H^{1,1}(S_{\mathrm{hol}}, \mathbb{C})$. Recall that a $\mathbb{P}^1$-bundle is an algebraic surface, by the GAGA principle, and in particular is Kähler. Also recall that by the $\partial\overline{\partial}$-lemma, in a Kähler compact surface, a closed $(1,1)$-form is exact iff it is

$\partial\overline{\partial}$-exact. This means that $T \cdot E = T \cdot F$ if E and F have the same Chern classes, see 1.4.

The cohomology of S_{hol} is easy to compute. Indeed, a $\mathbb{P}^1$-bundle over a curve is diffeomorphic to a product as soon as there exists a section of even self-intersection. In our situation, we have such a section at hand: the section s being (at the level of the universal covers) the graph of dev. We claim: $s^2 = \mathrm{Eu}(C)$. This is due to the fact that there is an isomorphism between the tangent bundle of C and its normal bundle, since C is both transverse to the foliation $\mathcal{F}_{\mathrm{hol}}$ and to the fibration $S_{\mathrm{hol}} \to C$. In particular, we infer $H^{1,1}(S_{\mathrm{hol}}, \mathbb{C}) = \mathbb{C}[s] \oplus \mathbb{C}[f]$, where f is any fiber of the fibration. The intersection product on $H^{1,1}(S_{\mathrm{hol}}, \mathbb{C})$ is given by $s^2 = \mathrm{Eu}(C)$, $f^2 = 0$, and $f \cdot s = 1$.

After these preliminaries, let us use the combination of Lemma 3.12 and Proposition 1.30, to get

$$\chi = \frac{1}{2}\frac{T \cdot N_{\mathcal{F}}}{T \cdot K_{\mathcal{F}}}.$$

We have $N_{\mathcal{F}} \cdot f = 2$ and $N_{\mathcal{F}} \cdot s = \mathrm{Eu}(C)$. So we infer $[N_{\mathcal{F}}] = 2[s] - \mathrm{Eu}(C)[f]$. Let T be the unique harmonic current whose product with the Poincaré volume form is equal to 1. We then have $T \cdot f = \frac{1}{\mathrm{vol}(C)}$ and $T \cdot K_{\mathcal{F}} = \frac{|\mathrm{Eu}(C)|}{\mathrm{vol}(C)}$. This gives

$$\chi = \frac{\mathrm{vol}(C)}{2\mathrm{Eu}(C)}(2\ T \cdot s + |\mathrm{Eu}(C)|\ T \cdot f) = 2\pi\ T \cdot s + \frac{1}{2}.$$

The proof is completed by showing that the cohomological intersection $T \cdot s$ coincides with the geometric intersection $\delta = T \dot{\wedge} s$. This last fact is not immediate since one cannot regularize the current of integration on s (recall $s^2 < 0$) but this is done by hand. We refer to [19] for more details. □

3.4. Harmonic measures of $\mathbb{P}^1$-structures

Let C be a smooth quasi-projective curve of negative Euler characteristic and σ a parabolic type projective structure on C. As before, we endow C and its universal covering with the Poincaré metric. We associate to σ a family of *harmonic measures* $\{\nu_x\}_{x \in \widetilde{X}}$ on the Riemann sphere, indexed by $\widetilde{C}$. It can be defined in several ways. The following appealing presentation was introduced by Hussenot in his PhD thesis [45]:

Definition-Proposition 3.13 (Hussenot). Let C be a Riemann surface of finite type and σ be a parabolic projective structure on C. Choose a representing pair $(\mathsf{dev}, \mathsf{hol})$. Then for every $x \in \widetilde{C}$, and W_x a.e. Brownian path starting at $\omega(0) = x$, there exists a point $\mathrm{e}(\omega)$ on $\mathbb{P}^1$ defined by the property that

$$\frac{1}{t}\int_0^t \mathsf{dev}_*\left(\delta_{\omega(s)}\right) ds \underset{t\to+\infty}{\longrightarrow} \delta_{\mathrm{e}(\omega)}.$$

The distribution of the point $\mathrm{e}(\omega)$ subject to the condition that $\omega(0) = x$ is the measure ν_x. In particular, due to the conformal invariance of Brownian motion, for a covering $\mathbb{P}^1$-structure, we recognize the classical harmonic measures on the limit set of a Kleinian group.

Another definition of the harmonic measures is based on the so-called Furstenberg boundary map, which was designed in [32], based on the discretization of Brownian motion in the hyperbolic plane $\mathbb{H}$ (see also Margulis [56, Theorem 3] for a different approach). Furstenberg showed that if Γ is a cofinite Fuchsian group and $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is a non-elementary representation, there exists a unique measurable equivariant mapping $\theta\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1$ defined a.e. with respect to Lebesgue measure. Choose a biholomorphism $\widetilde{C} \simeq \mathbb{H}$, thereby identifying $\pi_1(C)$ with a cofinite Fuchsian group. For every $x \in \mathbb{H}$, let m_x be the classical harmonic measure (i.e. the exit distribution of Brownian paths issued from x), which is a probability measure with smooth density on $\mathbb{P}^1(\mathbb{R})$. The harmonic measure ν_x is then defined by $\nu_x = \theta_* m_x$. From this perspective it is clear that, the measures ν_x are mutually absolutely continuous and depend harmonically on x.

Theorem 3.14. *Let C be compact algebraic curve and σ be a parabolic projective structure on C. Let as above χ be its Lyapunov exponent and $(\nu_x)_{x\in\widetilde{X}}$ be the associated family of harmonic measures. Then for every x,*

$$\dim_H(\nu_x) \leq \frac{1}{2\chi} \leq 1.$$

Furthermore $\dim_H(\nu_x) = 1$ *if and only if the developing maps are injective.*

So, as in the polynomial case, formula (3.11) provides an alternate approach to the classical bound $\dim_H(\nu) \leq 1$ for the harmonic measure on boundary of discontinuity components of finitely generated Kleinian groups, which follows from the famous results of Makarov [54] and Jones–Wolff [47]. In addition, with this method we are also able to show that $\dim_H(\nu) < 1$ when the component is not simply connected. Indeed we have the more precise bound $\dim_H(\nu) \leq \frac{A}{2\chi}$, where $0 \leq A \leq 1$ is an invariant of the flat foliation, and $A < 1$ when hol is not injective. This A has been defined by Frankel and is called the *action*, see [29].

We also see that the value of the dimension of the harmonic measures detects exotic quasifuchsian structures, that is, projective structures with quasifuchsian holonomy which do not belong to the Bers slice.

Sketch of proof. The curve C will be assumed to be compact, we refer to [19] for the punctured case. The main observation is to see the family of harmonic measures of a $\mathbb{P}^1$-structure as a foliated harmonic current. This is summarized in the following statement.

Proposition 3.15. *Let σ be a $\mathbb{P}^1$-structure on a compact C, and let $(\mathrm{dev}, \mathrm{hol})$ be a development-holonomy pair. Let $(S_{\mathrm{hol}}, \mathcal{F}_{\mathrm{hol}})$ be the flat $\mathbb{P}^1$-bundle constructed in 1.3. Let T' be the unique foliated harmonic current whose intersection with the fibers of S_{hol} is 1. The family of harmonic measures of σ is the family of desintegration of a (lift) of T' to $\widetilde{C} \times \mathbb{P}^1$ on the fibers $x \times \mathbb{P}^1$.*

Observe that the current T' in this proposition is equal to $T' = \mathrm{vol(C)}T$, where T is the current such that the foliated harmonic measure $\mu = T \wedge v_P$ has mass one. The proof of proposition relies on the fact that the map $x \mapsto \nu_x$ is harmonic, which is clear from the Furstenberg/Margulis point of view.

We now review an invariant of the harmonic current T that was introduced by Frankel, under the name of *action*. See [29]. It is defined as the non negative number

$$A = A(T) = \int_{S_{\mathrm{hol}}} \|\nabla_{\mathcal{F}} \log \varphi\|^2 \, d\mu, \tag{3.16}$$

where the functions φ are the densities of the desintegration of T along the leaves. The function φ are positive harmonic functions, so that the integral (3.16) is convergent. More precisely, by observing that the functions φ can be extended analytically on the universal cover of the leaves, and applying the Schwarz Pick lemma, one shows that $A(T) \leq 1$. See [16] for more details.

Using the fact that φ is harmonic, one finds the formula $\|\nabla \log \varphi\|^2 = -\Delta \log \varphi$, so that

$$\int_{S_{\mathrm{hol}}} \Delta(\log \varphi) \, d\mu = -A.$$

Using exactly the same argument as in the proof of Lemma 1.31, we infer the following result:

Lemma 3.17. *For μ-a.e. $w \in S_{\mathrm{hol}}$, and W^w-a.e. leafwise Brownian path ω starting at w, we have*

$$\lim_{t \to \infty} \frac{1}{t} \log D_T(h_{\omega_{|[0,t]}})(w) = -A,$$

where $D_T h := \frac{h^{-1}\nu_{\omega(t)}}{\nu_{\omega(0)}}$ is the Radon-Nikodyn derivative with respect to the measure induced by T on $\mathbb{P}^1$-fibers, namely the family of harmonic measures.

Hence, for every $\varepsilon > 0$, the maps $h_{\omega_{|[0,t]}}$ contract conformally the spherical distances by the factor $\exp((\lambda \pm \varepsilon)t)$, whereas they contract the harmonic measures by the factor $\exp((-A(T) \pm \varepsilon)t)$. We deduce the heuristic

$$\dim(\nu_x) = \frac{A}{|\lambda|} = \frac{A}{2\chi} \le \frac{1}{2\chi}.$$

Using a weak notion of dimension, the so-called *essential dimension* (denoted by $\dim_{\mathrm{ess}}$), one can prove part of this heuristic, namely the inequality

$$\dim_{\mathrm{ess}}(\nu_x) \le \frac{A}{2\chi}. \tag{3.18}$$

This uses an argument of Ledrappier [52, Thm 1] in the context of random product of matrices that we adapt to our setting. The proof of Theorem 3.14 then follows from (3.18) and the fact that the Hausdorff dimension is bounded by the essential dimension. □

3.5. Geometry of Bers slices

As another application of formula (3.11), we recover a result due to Shiga [66].

Theorem 3.19 (Shiga)**.** *Let C be a hyperbolic Riemann surface of finite type (of genus g with n punctures). The closure of the Bers embedding $B(C)$ is a polynomially convex compact subset of the space $P(C) \simeq \mathbb{C}^{3g-3+n}$ of holomorphic projective structures on C. As a consequence, $B(C)$ is a polynomially convex (or Runge) domain.*

Recall that a compact set K in $\mathbb{C}^N$ is polynomially convex if $\widehat{K} = K$, where

$$\widehat{K} = \left\{ z \in \mathbb{C}^N,\ |P(z)| \le \sup_K |P| \text{ for every polynomial } P \right\}.$$

An open set $U \subset \mathbb{C}^N$ is said to be polynomially convex (or Runge) if for every $K \Subset U$, $\widehat{K} \subset U$. The theorem may be reformulated by saying that $\overline{B(C)}$ is defined by countably many polynomial inequalities of the form $|P| \le 1$. This is not an intrinsic property of Teichmüller space, but rather a property of its embedding into the space $P(C)$ of holomorphic projective structures on C (as opposed to the Bers-Ehrenpreis theorem that Teichmüller is holomorphically convex).

Shiga's proof is based on the Grunsky inequality on univalent functions. Only the polynomial convexity of $B(C)$ is asserted in [66], but the proof covers the case of $\overline{B(C)}$ as well. Our approach is based on the elementary fact that the locus of minima of a global psh function on $\mathbb{C}^N$ is polynomially convex.

Sketch of proof. We just prove here the polynomial convexity of the Bers slice $B(C)$. The polynomial convexity of $\overline{B(C)}$ is more involved, we refer to [19]. It was shown in [18] that $\sigma \mapsto \chi(\sigma)$ is a continuous (Hölder) plurisubharmonic (psh for short) function on $P(X)$, hence it follows from formula 3.11 that deg is continuous and psh, too. In addition we see that $\chi(\sigma)$ reaches its minimal value $\frac{1}{2}$ exactly when $\deg(\sigma) = 0$, see 3.7. This already proves that the interior of $\{\delta = 0\}$, namely the set of covering $\mathbb{P}^1$-structures, is polynomially convex. But this set is exactly the Bers slice, so we are done. □

We finish this lecture by reviewing yet another application of formula (3.11) to equidistribution properties in $P(C)$. In [18] we showed that $T_{\text{bif}} := dd^c\chi$ is a *bifurcation current*, in the sense that its support is precisely the set of projective structures whose holonomy representation is not locally structurally stable in $P(X)$. The support of this current is the exterior of the Bers slice $B(C)$.

Analogous bifurcation currents have been defined for families of rational mappings on $\mathbb{P}^1$. It turns out that the exterior powers T^k_{bif} are interesting and rather well understood objects in that context (see [26] for an account). In particular, in the space of polynomials of degree d, the maximal exterior power T^{d-1}_{bif} is a positive measure supported on the boundary of the connectedness locus, which is the right analogue in higher degree of the harmonic measure of the Mandelbrot set [25].

For bifurcation currents associated to spaces of representations, nothing is known in general about the exterior powers T^k_{bif}. In our situation, we are able to obtain some information.

Theorem 3.20. *Let C be a compact Riemann surface of genus $g \geq 2$. Let $T_{\text{bif}} = dd^c\chi$ be the natural bifurcation current on $P(C)$. Then $\partial B(C)$ is contained in* $\mathrm{Supp}(T^{3g-3}_{\text{bif}})$.

Notice that $3g - 3$ is the maximum possible exponent. It is likely that the support of T^{3g-3}_{bif} is much larger than $\partial B(C)$. The reason for the compactness assumption here is that the proof requires some results of Otal [61] and Hejhal [43] that are known to hold only when X is compact.

If γ is a geodesic on C w.r.t. to the Poincaré metric, we let $Z(\gamma)$ be the subvariety of $P(C)$ defined by the property that $\mathrm{tr}^2(\mathsf{hol}(\gamma)) = 4$ (i.e. $\mathsf{hol}(\gamma)$ is parabolic or the identity). As a consequence of Theorem 3.20 and of the equidistribution theorems of [18] we obtain the following result, which contrasts with the description of $\partial B(C)$ "from the inside" in terms of maximal cusps and ending laminations ([60, 6], see [51] for a nice account).

Corollary 3.21. *For every $\varepsilon > 0$ there exist $3g-3$ closed geodesics $\gamma_1, \dots, \gamma_{3g-3}$ on C such that $\partial B(C)$ is contained in the ε-neighborhood of $Z(\gamma_1) \cap \dots \cap Z(\gamma_{3g-3})$.*

We observe that the value 4 for the squared trace is irrelevant here. As the proof will show, the result holds a.s. when $\gamma_1, \dots, \gamma_k$ are independent random closed geodesics of length tending to infinity.

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Talks

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On the uniformly perfectness of diffeomorphism groups preserving a submanifold and its applications

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1. Introduction

In this talk we shall describe the recent results on the uniformly perfectness of diffeomorphism groups of smooth manifolds preserving a submanifold.

Let M be a smooth connected manifold without boundary. Let $D^\infty(M)$ denote the group of C^∞-diffeomorphisms of M with compact support which are isotopic to the identity through C^∞-diffeomorphisms with compact support. It is known that M.Herman [5] and W.Thurston [6] proved $D^\infty(M)$ is perfect.

Let (M, N) be a manifold pair and $D^\infty(M, N)$ be the group of C^∞-diffeomorphisms of M preserving N which are isotopic to the identity through compactly supported C^∞-diffeomorphisms preserving N. In [1], we proved that the group $D^\infty(M, N)$ is perfect if the dimension of N is positive. In this talk we consider the conditions for $D^\infty(M, N)$ to be uniformly perfect. A group G is said to be uniformly perfect if each element of G is represented as a product of a bounded number of commutators of elements in G.

In [7], [8] T.Tsuboi obtained an excellent results on the uniform perfectness of the group $D^r(M)$. He proved that it is uniformly perfect $1 \leq r \leq \infty$ $(r \neq \dim M + 1)$ when M is an odd dimensional manifold or an even dimensional manifold with the appropriate conditions.

In [1], [2] we studied the conditions for $D^\infty(M, N)$ to be uniformly perfect when M is a compact manifold. If the group $D^\infty(M, N)$ is uniformly perfect, then both $D^\infty(N)$ and $D^\infty(M - N)$ are uniformly perfect. We need the another conditions for the converse. Let $p : D^\infty(M, N) \to D^\infty(N)$ be the map given by the restriction. If the connected components of $\ker p$ are finite, then $D^\infty(M, N)$ is a uniformly perfect group for $n \geq 1$. There exist many examples satisfying this condition.

If N is the union of circles in M and the connected components of $\ker p$ are infinite, then we can prove that $D^\infty(M, N)$ is not a uniformly perfect group. We can apply the result for various cases. If M is an oriented surface and N a disjoint union of circles in M, we can determine the uniformly perfectness of the group $D^\infty(M, N)$ ([2]). Finally we consider the case when $M = S^3$ and N is a knot in S^3, Then we prove that $D^\infty(S^3, N)$ is uniformly perfect if and only if N is a torus knot.

2. Statement of the main results

Let (M, N) be a manifold pair. Then $D^\infty(M, N)$ is perfect only if $\dim N \geq 1$ ([1], Theorem 1.1). Thus we assume that $\dim N \geq 1$ and investigate the conditions that $D^\infty(M, N)$ is uniformly perfect.

Theorem 2.1 ([1], [2])**.** *Let M be an m-dimensional compact manifold without boundary and N an n-dimensional C^∞-submanifold such that both groups $D^\infty(M - N)$ and $D^\infty(N)$ are uniformly perfect. If the connected components of* $\ker p$ *are finite, then $D^\infty(M, N)$ is a uniformly perfect group for $n \geq 1$.*

The converse of Theorem 2.1 is valid when N is a disjoint union of circles in M.

Theorem 2.2 ([2])**.** *Let M be an m-dimensional compact manifold without boundary and N be a disjoint union of circles in M. If the connected components of* $\ker p$ *are infinite, then $D^\infty(M, N)$ is not a uniformly perfect group.*

Now we apply Theorem 2.1 and Theorem 2.2 for studying the uniformly perfectness of the group $D^\infty(M, N)$ when M is an orientable surface and N is a disjoint union of circles.

Theorem 2.3 ([3])**.** *$D^\infty(M, N)$ is uniformly perfect if and only if*
(1) *$M = S^2$ and $k = 1$ and,*
(2) *$M = T^2$, $k = 1$ and N represents a non-trivial element of $\pi_1(T^2)$.*

Finally we consider the case where K is a knot in S^3. Using the result by G. Burde and H. Zieschang [4], we have the following.

Theorem 2.4. *$D^\infty(S^3, K)$ is uniformly perfect if and only if K is a torus knot.*

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On the ampleness of positive CR line bundles over Levi-flat manifolds

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1. Background

A closed real hypersurface M in a complex manifold X is said to be *Levi-flat* if M has a foliation $\mathcal{F}$ (called the *Levi foliation*) whose leaves are non-singular complex hypersurfaces of X. By the Frobenius theorem, this definition is equivalent to saying that M is locally pseudoconvex from both sides. Therefore, by its definition, the study of Levi-flat real hypersurfaces is of two natures: intrinsic one of the theory of foliations, and extrinsic one of function theory of several complex variables.

A problem in the study of Levi-flat real hypersurfaces is to understand the interplay between complexity of the Levi foliation $\mathcal{F}$ and pseudoconvexity of the complement $X \setminus M$, which was first pointed out explicitly by Barrett [2]. He studied several explicit families of Levi-flat real hypersurfaces in compact complex surfaces, and showed that, in these examples, the existence of a leaf with non-trivial holonomy corresponds to the 1-convexity of the complement. Our motivation of this study is to refine this suggested connection and to describe it in quantitative way.

On the other hand, complexity of the Levi foliation $\mathcal{F}$ should also be reflected on transverse regularity of leafwise meromorphic functions on M. Here, what we mean by a *leafwise meromorphic function* is a leafwise holomorphic section of a $\mathcal{C}^\infty$ *CR line bundle* L over M (a $\mathcal{C}^\infty$ $\mathbb{C}$-line bundle over M with $\mathcal{C}^\infty$ leafwise holomorphic transition functions) and when we say *leafwise holomorphic*, it is in the distribution sense. If the Levi foliation $\mathcal{F}$ is enough complicated, leafwise meromorphic functions may lose transverse regularity since they are analytically continued along leaves and can behave wildly in the transverse direction. Actually, Inaba [4] showed that if we impose continuity on leafwise holomorphic functions on compact Levi-flat manifolds, they must be constant along leaves.

In this note, we take the latter viewpoint and focus on the following

Question 1.1. How does pseudoconvexity of the complement $X \setminus M$ affect transverse regularity of leafwise meromorphic functions on M?

This short note is an announcement of [1], to which we refer the reader for the details.

2. Results

First we recall a Kodaira type embedding theorem of Ohsawa and Sibony, which holds for not only compact Levi-flat hypersurfaces but also abstract compact Levi-flat manifolds.

Theorem 2.1 ([7] Theorem 3, refined in [5])**.** *Let M be a compact $\mathcal{C}^\infty$ Levi-flat manifold equipped with a $\mathcal{C}^\infty$ CR line bundle L. Suppose L is positive along leaves, i.e., there exists a $\mathcal{C}^\infty$ hermitian metric on L such that the restriction of the curvature form to each leaf is everywhere positive definite. Then, for any $\kappa \in \mathbb{N}$, L is $\mathcal{C}^\kappa$-ample, i.e., there exists $n_0 \in \mathbb{N}$ such that one can find leafwise holomorphic sections $s_0, \cdots, s_N$ of $L^{\otimes n}$, of class $\mathcal{C}^\kappa$, for any $n \geq n_0$, such that the ratio $(s_0 : \cdots : s_N)$ embeds M into $\mathbb{CP}^N$.*

For arbitrarily large $\kappa \in \mathbb{N}$, by taking n_0 sufficiently large depending on κ, we can obtain so many $\mathcal{C}^\kappa$ leafwise holomorphic sections of $L^{\otimes n_0}$; in fact, they form an infinite dimensional vector space. Note that the existence of a positive-along-leaves $\mathcal{C}^\infty$ CR line bundle over M is equivalent to the tautness of the Levi foliation of M; our setting is not too restrictive.

A natural question on the Ohsawa–Sibony embedding theorem is whether or not we can improve the regularity to $\kappa = \infty$. This question asks the dependence of n_0 on κ as $\kappa \to \infty$, and at this point we will face a subtle interplay between transverse regularity of leafwise meromorphic functions, and complexity of the Levi foliation or pseudoconvexity of the complement.

Now we introduce a notion of pseudoconvexity that we are going to focus on.

Definition 2.2 (Takeuchi 1-complete space). Let D be a relatively compact domain in a complex manifold X with $\mathcal{C}^2$ boundary. We say that D is *Takeuchi 1-complete* if there exists a $\mathcal{C}^2$ defining function r of ∂D defined on a neighborhood of D with $D = \{z \mid r(z) < 0\}$ such that, with respect to a hermitian metric on X, all of the eigenvalues of the Levi form of $-\log(-r)$ are bounded from below by a strictly positive constant entire on D.

Takeuchi 1-completeness not only implies that the domain is Stein, but also implies that it behaves as if it is in complex Euclidean space.

Theorem 2.3 (cf. [6] Theorem 1.1)**.** *Let D be a Takeuchi 1-complete domain with defining function r. Then, $-\partial\overline{\partial}\log(-r)$ gives a complete Kähler metric on D, and it follows that $-(-r)^\varepsilon$ with sufficiently small $\varepsilon > 0$ becomes a strictly plurisubharmonic bounded exhaustion function on D, i.e., D is hyperconvex.*

We do not have any compact Levi-flat real hypersurface in $\mathbb{C}^n$ if $n \geq 2$. Nevertheless, there exist compact Levi-flat real hypersurfaces in compact complex surfaces whose complements are Takeuchi 1-complete.

Theorem 2.4 ([1]). *Let Σ be a compact Riemann surface of genus ≥ 2 and $\rho : \pi_1(\Sigma) \to \mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$ a group homomorphism. Denote by $\mathcal{D}_\rho = \Sigma \times_\rho \mathbb{D}$ the holomorphic unit disc bundle obtained by the suspension of ρ over Σ. Suppose there exists a unique non-holomorphic harmonic section $h : \Sigma \to \mathcal{D}_\rho$ whose $\mathrm{rank}_{\mathbb{R}}\, dh = 2$ on a non-empty open set. Then, $\mathcal{D}_\rho$ is Takeuchi 1-complete in its associated $\mathbb{CP}^1$-bundle $X_\rho = \Sigma \times_\rho \mathbb{CP}^1$.*

The boundary of $\mathcal{D}_\rho$ is a flat S^1-bundle $M_\rho = \Sigma \times_\rho S^1$, thus, a Levi-flat real hypersurface. The assumption is fulfilled for any non-trivial quasiconformal deformation ρ of Γ where Γ is a Fuchsian representation of $\Sigma = \mathbb{D}/\Gamma$.

The proof of Theorem 2.4 is by explicitly constructing a suitable defining function, in which the harmonic section h is the essential ingredient. This technique originates in the work of Diederich and Ohsawa [3].

We can observe the following Bochner–Hartogs type phenomenon for Levi-flat real hypersurfaces with Takeuchi 1-complete complements.

Theorem 2.5. *Let X be a compact complex surface, L a holomorphic line bundle over X, and M a $\mathcal{C}^\infty$ compact Levi-flat real hypersurface of X which splits X into two Takeuchi 1-complete domains $D \sqcup D'$. Then, there exists $\kappa \in \mathbb{N}$ such that any $\mathcal{C}^\kappa$ leafwise holomorphic section of $L|M$ extends to a holomorphic section of L.*

This theorem tells us that the space of $\mathcal{C}^\kappa$ leafwise holomorphic sections of $L|M$ is finite dimensional for sufficiently large κ; in particular, the space of $\mathcal{C}^\infty$ leafwise holomorphic sections of $L|M$ is always finite dimensional. This description is a qualitative answer to Question 1.1 for Levi-flat real hypersurfaces with Takeuchi 1-complete complements.

The proof of Theorem 2.5 can be done with established techniques in function theory of several complex variables. A simple proof is given in [1].

As a corollary, we give an example that shows that the Ohsawa–Sibony embedding theorem cannot hold for $\kappa = \infty$ in general.

Corollary 2.6 ([1]). *Let Σ be a compact Riemann surface of genus ≥ 2, and $\rho : \pi_1(\Sigma) \to \mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$ a group homomorphism. Denote the suspended $\mathbb{CP}^1$ bundle by $\pi \colon X_\rho \to \Sigma$. Take a positive line bundle L over Σ. Suppose $\mathcal{D}_\rho$ has a unique non $\pm$holomorphic harmonic section h whose $\mathrm{rank}_{\mathbb{R}}\, dh = 2$, then $\pi^* L|M_\rho$ is positive along leaves, but never $\mathcal{C}^\infty$ ample.*

3. Further Questions

We conclude this short note with further questions. The following notion is a quantitative version of Takeuchi 1-completeness according to Theorem 2.3.

Definition 3.1. Let D be a Takeuchi 1-complete domain with defining function r. We denote by $\varepsilon_{DF}(r)$ the supremum of $\varepsilon \in (0, 1)$ such that $-(-r)^{\varepsilon}$ is a strictly plurisubharmonic bounded exhaustion function on D, and call it the *Diederich–Fornaess exponent* of the defining function r.

Our questions are quantitative or intrinsic versions of Question 1.1.

Question 3.2. Can we estimate the κ in Theorem 2.5 in terms of the Diederich–Fornaess exponent of some defining function?

Question 3.3. What is the counterpart of the Diederich–Fornaess exponent in the theory of foliations? By using it, can we prove Corollary 2.6 without looking the natural Stein filling $\mathcal{D}_{\rho}$?

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A cocycle rigidity lemma for Baumslag-Solitar actions and its applications

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1. A cocycle rigidity lemma

Let $\mathrm{Diff}(\mathbf{R}^n, 0)$ be the group of local diffeomorphisms of $\mathbf{R}^n$ at the origin. In many situations in study of foliations, we encounter with $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycles over a group action. A typical case is the following. Consider an action of simply connected Lie group whose orbits form a smooth codimension-n foliation with trivial normal bundle. Then, the holonomy map of the foliation with respect to a fixed family of transverse coordinates defines a $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycle. In this case, the existence of a transverse geometric structure is equivalent to the condition that the cocycle can be reduced to a subgroup of $\mathrm{Diff}(\mathbf{R}^n, 0)$ which preserves the geometric structure.

In this talk, we show a rigidity lemma for $\mathrm{Diff}(\mathbf{R}^n, 0)$-valued cocycle over actions of the Baumslag-Solitar group $BS(1,k)$. We also apply it to rigidity problem of several group actions.

For integers $k \geq 2$, **the Baumslag-Solitar group** $BS(1,k)$ is the group presented as

$$\langle a, b \mid aba^{-1} = b^k \rangle.$$

There are many copies of $BS(1,k)$ is contained in the group $\mathrm{CAff}(\mathbf{R}^n)$ of conformal affine transformations of $\mathbf{R}^n$. In fact, let f_k and g_v be elements of $\mathrm{CAff}(\mathbf{R}^n)$ given by $f_k(x) = kx$ and $g_v(b) = x+v$. Then, the correspondence $a \mapsto f_k$ and $b \mapsto g_v$ gives an inclusion from $BS(1,k)$ to $\mathrm{CAff}(\mathbf{R}^n)$.

Let Γ and H be topological groups and X a topological space. For a given action $\rho : \Gamma \times X \to X$, a map $\alpha : \Gamma \times X \to H$ is called a **cocycle** over ρ if $\alpha(1_\Gamma, x) = x$ and $\alpha(\gamma\gamma', x) = \alpha(\gamma, \gamma' x) \cdot \alpha(\gamma', x)$ for any $\gamma, \gamma' \in G$ and $x \in X$ (1_Γ is the unit element of Γ). The space of H-valued cocycle over ρ admits a topology as a subspace of $C^0(\Gamma \times X, H)$. Let H' be a subgroup of H. Two H-valued cocycles α and β over ρ are H'-equivalent if there exists $h \in H'$ such that $\beta(\gamma, x) = h \cdot \alpha(\gamma, x) \cdot h^{-1}$ for any $\gamma \in \Gamma$ and $x \in X$.

For an element F of $\mathrm{Diff}(\mathbf{R}^n 0)$, we denote the r-jet of F at the origin by $j_0^r F$. Let $j^r \mathrm{Diff}(\mathbf{R}^n, 0)$ is the group of r-jets of elements of $\mathrm{Diff}(\mathbf{R}^n, 0)$ at the origin. The group $\mathrm{Diff}(\mathbf{R}^n, 0)$ is endowed with the weakest topology such that the projection to $j^r \mathrm{Diff}(\mathbf{R}^n, 0)$ is continuous for any $r \geq 1$ (it is not Hausdorff). We denote the identity map of $\mathbf{R}^n$ by Id. For $r \geq 1$, let $G^{(r)}$ be the subgroup of $\mathrm{Diff}(\mathbf{R}^n, 0)$ consisting of elements with trivial r-jet.

Cocycle Rigidity Lemma *There exists a universal constant $\epsilon_k > 0$ such that the following assertion holds: Let X be a topological space, $\rho : BS(1,k) \times X \to X$ a continuous $BS(1,k)$-action. If continuous cocycles $\alpha, \beta : BS(1,k) \times X \to \mathrm{Diff}(\mathbf{R}^n, 0)$ over ρ satisfies that*

1. $j_0^2(\alpha(\gamma, x)) = j_0^2(\beta(\gamma, x))$ *for any $\gamma \in BS(1,k)$ and $x \in X$, and*
2. $\|j_0^1(\alpha(a,x)) - (1/k)\,\mathrm{Id}\,\| < \epsilon_k$ *and* $\|j_0^1(\alpha(b,x)) - \mathrm{Id}\,\| < \epsilon_k$*, where* Id *is the identity map on $\mathbf{R}^n$,*

then two cocycles α and β are $G^{(2)}$-equivalent. If $\alpha(a,\cdot) = \beta(a,\cdot)$ in addition then α and β coincide as cocycles.

In other words, a cocycle whose linear part is close to the linear representation given by $a \mapsto (1/k)I$ and $b \mapsto I$ is determined by its 2-jet up to $G^{(2)}$-equivalence.

2. Applications

The first application of the above cocycle rigidity lemma is rigidity of certain conformal local action of a Baumslag-Solitar-like group. For $k \geq 2$ and $n \geq 1$, let $\Gamma_{n,k}$ be the discrete group presented as

$$\langle a, b_1, \ldots, b_n \mid a b_i a^{-1} = b_i^k, b_i b_j = b_j b_i \ (i,j = 1, \ldots, n) \rangle.$$

Each subgroup generated by a and b_i is isomorphic to $BS(1,k)$. Let f_k and g_v be conformal affine maps on $\mathbf{R}^n$ given in the previous section. They naturally extends to the sphere $S^n = \mathbf{R}^n \cup \{\infty\}$. For a basis $B = (v_1, \ldots, v_n)$ of $\mathbf{R}^n$, we define a smooth $\Gamma_{n,k}$-action ρ_B on S^n (*i.e.* a homomorphism from $\Gamma_{n,k}$ to $\mathrm{Diff}(S^n)$) by $\rho_B(a) = f_k$ and $\rho_B(b_i) = g_{v_i}$. Let $\phi : S^n \setminus \{0\} \to \mathbf{R}^n$ be a coordinate at ∞ given by $\phi(x) = x/\|x\|^2$. We define a local $\Gamma_{n,k}$-action P_B (*i.e.* a homomorphism from $\Gamma_{n,k}$ to $\mathrm{Diff}(\mathbf{R}^n, 0)$) by $P_B(\gamma) = \phi \cdot \rho_B(\gamma) \cdot \phi^{-1}$. Remark that the local action P_B preserves the standard conformal structure on $\mathbf{R}^n$.

Theorem 2.1 ([1]). *If a local action $P : \Gamma_{n,k} \to \mathrm{Diff}(\mathbf{R}^n, 0)$ is sufficiently close to P_B, then there exists a basis B' of $\mathbf{R}^n$ and a local diffeomorphism $H \in \mathrm{Diff}(\mathbf{R}^n, 0)$ such that $P(\gamma) = H \cdot P_{B'}(\gamma) \cdot H^{-1}$ for any $\gamma \in \Gamma_{n,k}$. In particular, the local $\Gamma_{n,k}$-action P preserves a smooth conformal structure on $\mathbf{R}^n$.*

Outline of Proof. Notice that a local action is a $\mathrm{Diff}(\mathbf{R}^n, 0)$-cocycle over the trivial action on a point. We can check that the sub-action generated by a and b_i satisfies the assumptions of the rigidity lemma. So, it is sufficient to show that P coincides with a conjugate of some $P_{B'}$ up to 2-jet. Finding the basis B' can be done using a variant of Weil's rigidity theorem of homomorphisms between Lie groups [5]. $\square$

Using the persistence of global fixed point ∞, we can derive a global rigidity theorem from the above theorem.

Theorem 2.2 ([1])**.** *If a smooth $\Gamma_{n,k}$-action ρ is sufficiently close to ρ_B, then there exists a basis B' of $\mathbf{R}^n$ and a diffeomorphism h of S^n such that $\rho(\gamma) = h \cdot \rho_{B'}(\gamma) \cdot h^{-1}$.*

A similar local or global rigidity theorem can be shown for a $\Gamma_{n,k}$-action on the n-dimensional torus $\mathbf{T}^n$. We identify $\mathbf{T}^n$ with $(\mathbf{R} \cup \{\infty\})^n$. For a basis $B = (v_1, \ldots, v_n)$ of $\mathbf{R}^n$, we define a $\Gamma_{n,k}$-action σ_B on $\mathbf{T}^n$ by

$$\begin{aligned} \sigma_B(a)(x_1, \ldots, x_n) &= (kx_1, \ldots, kx_n) \\ \sigma_B(b_i)(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, x_i + v_i, x_{i+1}, \ldots, x_n). \end{aligned}$$

By the same way as above, we can show a rigidity result for this action.

Theorem 2.3. *If a smooth $\Gamma_{n,k}$-action σ is sufficiently close to σ_B, then there exists a basis B' of $\mathbf{R}^n$ and a diffeomorphism h of $\mathbf{T}^n$ such that $\sigma(\gamma) = h \cdot \sigma_{B'}(\gamma) \cdot h^{-1}$.*

The second application is another proof of Ghys's local rigidity theorem on Fuchsian action on $\mathbf{R}P^1$. Let Γ be a cocompact lattice of $\mathrm{PSL}(2, \mathbf{R})$ Since $\mathrm{PSL}(2, \mathbf{R})$ acts on $\mathbf{R}P^1$ naturally, Γ acts on $\mathbf{R}P^1$ as a subgroup of $\mathrm{PSL}(2, \mathbf{R})$. We denote this action by ρ_Γ. More generally, when a homomorphism $\pi : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ is given, we can define a Γ-action ρ_π on $\mathbf{R}P^1$ by $\rho_\pi(\gamma)(x) = \pi(\gamma) \cdot x$.

Theorem 2.4 (Ghys [2])**.** *If a Γ-action ρ on $\mathbf{R}P^1$ is sufficiently close to ρ_Γ, then there exists an homomorphism $\pi : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ and a diffeomorphism h of S^1 such that $\rho(\gamma) = h \cdot \rho_\pi(\rho) \cdot h^{-1}$ for any $\gamma \in \Gamma$.*

All known proofs ([2, 3, 4]) use the Schwarzian derivative, but our proof does not. We use that fact that any $j^2 \operatorname{Diff}(\mathbf{R}, 0)$-cocycle can be extended to a cocycle valued in projective transformations of $(\mathbf{R}, 0)$

Outline of our proof. Let P be the subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that consists of lower triangular elements. It is generated by one-parameter subgroups $A = (a^t)_{t \in \mathbf{R}}$ and $N = (b^s)_{s \in \mathbf{R}}$ with a relation $a^t b^s a^{-t} = b^{s \exp t}$. Define a smooth right P-action ρ_P on $\Gamma \backslash \mathrm{PSL}(2, \mathbf{R})$ by $\rho_P(\Gamma g, p) = \Gamma(gp)$, and denote the orbit foliation of ρ_P by $\mathcal{F}_P$. As Ghys proved, it is sufficient to show that any foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_P$ admits a smooth transversely projective structure. Since the restriction of ρ_P to A is an Anosov flow and $\mathcal{F}_P$ is its unstable foliation, we can find a homeomorphism of M which sends each leaf of $\mathcal{F}_P$ to that of $\mathcal{F}$. This homeomorphism induces a continuous P-action ρ whose orbit foliation is $\mathcal{F}$. The holonomy map of $\mathcal{F}$ gives a $\mathrm{Diff}(\mathbf{R}, 0)$-valued cocycle $\bar{\alpha}$ over ρ.

The group P naturally contains $BS(1, k)$ as a subgroup. Let α be the restriction of the cocycle $\bar{\alpha}$ to $BS(1, k)$. To show that $\mathcal{F}$ is transversely projective, it is enough to see that $\bar{\alpha}$ is $\mathrm{Diff}(\mathbf{R}, 0)$-equivalent to a cocycle whose values are projective transformation. But, it is an easy consequence of the rigidity lemma. In fact, as mentioned above. any $j^2 \mathrm{Diff}(\mathbf{R}, 0)$-valued cocycle can be extended to a cocycle whose values are projective transformations in one-dimension (it is not true for higher dimension). So, the rigidity lemma implies that α is $G^{(2)}$-equivalent to such a cocycle. $\square$

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On Fatou-Julia decompositions of complex dynamical systems

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1. Introduction

A Fatou-Julia decomposition for transversely holomorphic, complex codimension-one foliations is introduced by Ghys, Gomez-Mont and Saludes [4] (and in [6]) in terms of deformations of holomorphic structures. Another decomposition is introduced in [2] in terms of normal families. These decompositions enjoy some properties similar to those of classical Fatou-Julia decomposition and also to the decomposition of the sphere into the domains of discontinuity and the limit sets (of Kleinian groups). In [3], a Fatou-Julia decomposition is introduced for pseudosemigroups. The decomposition is still difficult to study, however, it provides a natural unification of the notions of Fatou-Julia decomposition of mapping iterations, foliations and the decomposition of sphere with respect to the action of Kleinian groups. In this article, we will introduce pseudosemigroups and the Fatou-Julia decomposition, and explain how decompositions are unified (Theorem 2.16) after [2] and [3].

2. Pseudosemigroups and Fatou-Julia decompositions

We first introduce notions of pseudosemigroups and their Fatou-Julia decompositions. The notion of pseudosemigroups has already appeared (cf. [8], [11] and [7]). We will make use of a similar but different one.

In what follows, we consider holomorphic mappings unless otherwise mentioned, although pseudosemigroups can be considered in much more generalities.

In short, a pseudosemigroup is a pseudogroup but the inverse is not necessarily defined.

Definition 2.1. Let T be an open subset (not necessarily connected) of $\mathbb{C}^n$ and Γ be a family of mappings from open subsets of T into T (we call such mappings *local mappings*). Then, Γ is a (holomorphic) *pseudosemigroup* (psg for short) if the following conditions are satisfied.

1) $\mathrm{id}_T \in \Gamma$, where id_T denotes the identity map of T.

2) If $\gamma \in \Gamma$, then $\gamma|_U \in \Gamma$ for any open subset U of $\operatorname{dom}\gamma$.
3) If $\gamma_1, \gamma_2 \in \Gamma$ and $\operatorname{range}\gamma_1 \subset \operatorname{dom}\gamma_2$, then $\gamma_2 \circ \gamma_1 \in \Gamma$, where $\operatorname{dom}\gamma$ and $\operatorname{range}\gamma$ denotes the domain and the range of γ, respectively.
4) Let U be an open subset of T and γ continuous mapping defined on U. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to Γ, then $\gamma \in \Gamma$.

If in addition Γ consists of local homeomorphisms, namely, homeomorphisms from domains to ranges, then Γ is a *pseudogroup* (pg for short) if Γ satisfies 1), 2), 3) and the following conditions.

4′) Let U be an open subset of T and γ a homeomorphism from U to $\gamma(U)$. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to Γ, then $\gamma \in \Gamma$.
5) If $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$.

If Γ is either a psg or pg, then we set for $x \in T$

$$\Gamma_x = \{\gamma_x \mid x \in \operatorname{dom}\gamma\}.$$

By abuse of notation, an element of Γ_x is considered as an element of Γ defined on a neighborhood of x.

One might expect that a pg is a psg but it is not always the case.

Example 2.2. Let $T = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and define an automorphism f of $\mathbb{C}P^1$ by $f(z) = -z$. We denote by Γ the pg generated by f, that is, the smallest pg which contains f. Let $U = \{z \in \mathbb{C} \mid |z-2| < 1\}$ and $V = f(U)$. If we set $\gamma = f|_V$, then $\gamma \cup \mathrm{id}\colon V \cup U \to U$ is not an element of Γ, because $\gamma \cup \mathrm{id}$ is not a homeomorphism. If Γ were a psg, then $\gamma \cup \mathrm{id} \in \Gamma$ by the condition 4).

Definition 2.3. We denote by $\Gamma_0^\times$ the subset of Γ which consists of invertible elements, namely,

$$\Gamma_0^\times = \{\gamma \in \Gamma \mid \gamma^{-1} \in \Gamma\}.$$

We denote by $\Gamma^\times$ the subset of Γ which consists of locally invertible elements, namely,

$$\Gamma^\times = \left\{\gamma \in \Gamma \,\middle|\, \begin{array}{l} \exists \text{ an open covering } \{U_\lambda\}_{\lambda\in\Lambda} \text{ of } \operatorname{dom} f \\ \text{such that } (\gamma|_{U_\lambda})^{-1} \in \Gamma \end{array}\right\}.$$

Note that $\Gamma_0^\times$ is a pseudogroup.

Definition 2.4. Let (Γ, T) be a psg. We denote by $\mathscr{T}$ the family of relatively compact open subsets of T. If $T' \in \mathscr{T}$, then the *restriction* of Γ

to T' is defined by

$$\Gamma_{T'} = \{\gamma \in \Gamma \mid \operatorname{dom}\gamma \subset T' \text{ and } \operatorname{range}\gamma \subset T'\}.$$

The notion of *compact generation* [6] is also significant for psg's. The notions of morphisms and equivalences are given as follows.

Definition 2.5. Let (Γ, T) and (Δ, S) be psg's. A (holomorphic) *morphism* $\Phi\colon \Gamma \to \Delta$ is a collection Φ of local mappings from T to S with the following properties.

1) $\{\operatorname{dom}\phi \mid \phi \in \Phi\}$ is an open covering of T.
2) If $\phi \in \Phi$, then any restriction of ϕ to an open set of $\operatorname{dom}\phi$ also belongs to Φ.
3) Let U be an open subset of T and ϕ a mapping from U to S. If for any $x \in U$, there exists an open neighborhood U_x of x such that $\phi|_{U_x} \in \Phi$, then $\phi \in \Phi$.
4) If $\phi \in \Phi$, $\gamma \in \Gamma^\times$ and $\delta \in \Delta^\times$, then $\delta \circ \phi \circ \gamma \in \Phi$.
5) Suppose that $\gamma \in \Gamma$ and $x \in \operatorname{dom}\gamma$. If $x \in \operatorname{dom}\phi$ and $\gamma(x) \in \operatorname{dom}\phi'$, where $\phi, \phi' \in \Phi$, then there is an element $\delta \in \Delta$ such that $\phi(x) \in \operatorname{dom}\delta$, and $\delta \circ \phi = \phi' \circ \gamma$ on a neighborhood of x.

A morphism from (Γ, T) to itself is called an *endomorphism* of (Γ, T).

Definition 2.6. Let (Γ, T) and (Δ, S) be psg's and Φ a morphism from Γ to Δ.

1) Φ is called an *étale morphism* if Φ consists of étale mappings, namely, mappings of which the restriction to sufficiently small open sets are homeomorphisms.
2) Suppose that Γ and Δ are psg's on complex one-dimensional manifolds. A morphism is said to be *ramified* if $\phi \in \Phi$ and $x \in \operatorname{dom}\phi$, then there exists an open neighborhood U_x of x such that $\phi|_{U_x}$ is the restriction of the composite of ramified coverings and holomorphic étale mappings.

Definition 2.7. Let (Γ, T) and (Δ, S) be psg's. A collection Φ of local homeomorphisms from T to S is an *étale morphism* of pg's if Φ satisfies the conditions in Definition 2.5 but 'a continuous map from U to S' in 3) is replaced by 'a local homeomorphism from T to S'.

Definition 2.8. Let A be a set which consists of local mappings on T. A psg Γ is said to be *generated* by A if Γ contains A and is the smallest with respect to inclusions. The psg generated by A is denoted by $\langle A\rangle$. Similarly, we consider morphisms generated by local mappings from T to S.

Definition 2.9. If $\Phi_1\colon \Gamma_1 \to \Gamma_2$ and $\Phi_2\colon \Gamma_2 \to \Gamma_3$ are morphisms of psg's, then the *composite* $\Phi_2 \circ \Phi_1$ is defined by

$$\Phi_2 \circ \Phi_1 = \langle \phi_2 \circ \phi_1 \,|\, \phi_1 \in \Phi_1,\ \phi_2 \in \Phi_2,\ \operatorname{range}\phi_1 \subset \operatorname{dom}\phi_2 \rangle.$$

Definition 2.10. An étale morphism $\Phi\colon \Gamma \to \Delta$ is an *equivalence* if there is an étale morphism $\Psi\colon \Delta \to \Gamma$ such that $\Psi \circ \Phi = \Gamma^\times$ and $\Phi \circ \Psi = \Delta^\times$. Such a Ψ is unique so that it is denoted by Φ^{-1}. We call Φ^{-1} the *inverse morphism* of Φ. An equivalence from (Γ, T) to itself is called *automorphism*.

If Φ_1 and Φ_2 are equivalences, then $\Phi_2 \circ \Phi_1$ is also an equivalence.

Definition 2.11. A psg (Γ, T) is *compactly generated* if there is a relatively compact open set T' in T, and a finite subset $\{\gamma_1, \dots, \gamma_r\}$ of Γ such that the domains and the ranges are contained in T' and that

1) if we denote by $\Gamma_{T'}$ the restriction of Γ to T', then $\Gamma_{T'}$ is generated by $\{\gamma_1, \dots, \gamma_r\}$,
2) for each γ_i, there exists an element $\widetilde{\gamma}_i$ of Γ such that $\operatorname{dom}\widetilde{\gamma}_i$ contains the closure of $\operatorname{dom}\gamma_i$, $\widetilde{\gamma}_i|_{\operatorname{dom}\gamma_i} = \gamma_i$ and that $\widetilde{\gamma}_i$ is étale on a neighborhood of $\operatorname{dom}\widetilde{\gamma}_i \setminus \operatorname{dom}\gamma_i$,
3) the inclusion of T' into T induces an equivalence from $\Gamma_{T'}$ to Γ.

Such a $(\Gamma_{T'}, T')$ is called a *reduction* of (Γ, T).

Remark 2.12. If Γ is a compactly generated psg on a one-dimensional complex manifold, then Γ is étale or ramified. In addition, the last condition in 2) is equivalent to $\operatorname{Sing}\widetilde{\gamma}_i = \operatorname{Sing}\gamma_i$.

For example, if (Γ, T) is generated by a holonomy pseudogroup of a foliation of a closed manifold, then (Γ, T) is compactly generated. We need to choose a complete transversal in order to define a holonomy pseudogroup. If we change the choice of complete transversals, then we obtain pseudogroups which are equivalent. Another source of compactly generated psg's are rational mappings on $\mathbb{C}P^1$. (Γ, T) is also compactly generated if $T = \mathbb{C}P^1$ and Γ is generated by a rational semigroup [10] which acts on $\mathbb{C}P^1$. See [3] for details.

Assumption 2.13. We assume that Γ is generated by local biholomorphic diffeomorphism of $\mathbb{C}^q$, $q > 1$, or by local biholomorphic diffeomorphisms of $\mathbb{C}$ or ramified coverings, where a holomorphic map, say f, from an open set of $\mathbb{C}$ to $\mathbb{C}$ is said to be ramified covering if there exist biholomorphic diffeomorphisms φ from $\operatorname{dom} f$ to a domain in $\mathbb{C}$ and ψ from $\operatorname{range} f$ to a domain in $\mathbb{C}$ such that $\psi \circ f \circ \varphi^{-1}(z) = z^n$ holds for some positive integer n, where $z \in \operatorname{range}\varphi$.

Note that under our assumption, Γ consists of holomorphic open mappings.

Definition 2.14. Let $T' \in \mathscr{T}$.

1) A connected open subset U of T' is a wF-open set (weak 'Fatou'-open set) if the following conditions are satisfied:
   i) If γ_x is the germ of an element of $\Gamma_{T'}$ at x, γ is defined on U as an element of Γ, where $(\Gamma_{T'}, T')$ is the restriction of Γ to T'.
   ii) Let Γ^U be the subset of Γ which consists of elements of Γ obtained as in (a). Then Γ^U is a normal family.
2) A connected open subset V of T' is an F-open set ('Fatou'-open set) if $\gamma \in \Gamma'$ and if $\operatorname{dom}\gamma \subset V$, then $\operatorname{range}\gamma$ is a union of wF-open sets.

Definition 2.15. Let (Γ, T) be a psg which fulfills Assumption 2.13. If $T' \in \mathscr{T}$, then let $F(\Gamma_{T'})$ be the union of F-open subsets of T'. Let $J(\Gamma_{T'}) = T' \setminus F(\Gamma_{T'})$, and $J_0(\Gamma) = \bigcup_{T' \in \mathscr{T}} J(\Gamma_{T'})$. Let $J(\Gamma)$ be the closure of $J_0(\Gamma)$ and $F(\Gamma) = T \setminus J(\Gamma)$. We call $F(\Gamma)$ and $J(\Gamma)$ the Fatou set and the Julia set of (Γ, T), respectively.

Roughly speaking, $J(\Gamma)$ is defined as follows. We regard $(\Gamma_{T'}, T')$ as an approximation of (Γ, T), and define $J(\Gamma_{T'})$. Indeed, it can be shown that if (Γ, T) is compactly generated, then $J(\Gamma_{T'}) = J(\Gamma) \cap T'$ holds for sufficiently large T'. If $T' \subset T''$, then $J(\Gamma_{T'}) \subset J(\Gamma_{T''}) \cap T'$ so that we take the union. Finally by taking the closure, we will obtain a set which consists of points where some 'complicated dynamics' occur in every neighborhood of that point.

Thus defined Julia sets have the following properties.

Theorem 2.16. *If Γ is a psg, then we denote by $J_{\mathrm{psg}}(\Gamma)$ its Julia set in the sense of Definition 2.15. Then we have the following.*

1) *If f is a rational mapping on $\mathbb{C}P^1$, then $J(f) = J_{\mathrm{psg}}(\langle f \rangle)$, where $\langle f \rangle$ denotes the pseudosemigroup generated by f. More generally, if $f_1, \dots, f_r$ are rational mappings on $\mathbb{C}P^1$ and if G is the semigroup generated by $f_1, \dots, f_r$, then $J(G) = J_{\mathrm{psg}}(\langle f_1, \dots, f_r \rangle)$, where $\langle f_1, \dots, f_r \rangle$ denotes the pseudosemigroup generated by $f_1, \dots, f_r$ (or by G).*
2) *If f is an entire function, then let $\langle f \rangle$ be the pseudosemigroup generated by f which acts on $\mathbb{C}P^1$, where* $\operatorname{dom} f$ *is considered to be $\mathbb{C}$. Then, $J(f) \cup \{\infty\} = J_{\mathrm{psg}}(\langle f \rangle)$.*
3) *If G is a finitely generated Kleinian group, then $\Lambda(G) = J_{\mathrm{psg}}(\Gamma)$, where Γ is the pseudosemigroup generated by G and $\Lambda(G)$ denotes the limit set of G.*

4) *If Γ is the holonomy pseudogroup of a complex codimension-one foliation of a closed manifold with respect to a complete transversal (it suffices to assume that Γ is a compactly generated pseudogroup of local biholomorphic diffeomorphisms on $\mathbb{C}$). If we denote by Γ_{psg} the smallest pseudosemigroup which contains Γ, then $J(\Gamma) = J_{\mathrm{psg}}(\Gamma_{\mathrm{psg}})$, where $J(\Gamma)$ is the Julia set of compactly generated pseudogroup in the sense of* [2].

Theorem 2.16 can be seen as a partial refinement of Sullivan's dictionary [9].

In the 4) of Theorem 2.16 the Julia set in the sense of Ghys, Gomez-Mont and Saludes is also defined [4]. The following is known.

Theorem 2.17. *Let Γ be a compactly generated pseudogroup of local biholomorphic diffeomorphisms on $\mathbb{C}$. If we denote by $J_{\mathrm{GGS}}(\Gamma)$ the Julia set of Γ in the sense of Ghys, Gomez-Mont and Saludes, then $J(\Gamma) \subset J_{\mathrm{GGS}}(\Gamma)$.*

There are examples where the inclusion is strict.

Remark 2.18. If we denote by $F_{\mathrm{GGS}}(\Gamma)$ the Fatou set of Γ in the sense of [4], there is a classification of the connected components of $F_{\mathrm{GGS}}(\Gamma)$. We have also a classification of $F(\Gamma)(\supset F_{\mathrm{GGS}}(\Gamma))$ of the same kind. We refer [2] and [1] for more properties of Fatou-Julia decompositions of compactly generated pg's.

Pseudosemigroups in Theorem 2.16 are compactly generated except the case 3). Other psg's which are not necessarily compactly generated are obtained by studying (transversely) holomorphic foliations of open manifolds, or singular holomorphic foliations. A Fatou–Julia decomposition of these foliations can be introduced by using the decomposition in the sense of Definition 2.15. In [3], some properties of such decompositions are studied.

Some of common properties of the Julia sets and the limit sets can be regarded as properties of Julia sets of compactly generated pseudosemigroups. For example, we have the following.

Lemma 2.19. *Let Γ be a compactly generated pseudosemigroup. If we denote by $F(\Gamma)$ and $J(\Gamma)$ Fatou and Julia sets of Γ, then we have the following.*

1) *$F(\Gamma)$ is forward Γ-invariant, i.e., $\Gamma(F(\Gamma)) = \Gamma$, where $\Gamma(F(\Gamma)) = \{x \in T \mid \exists\, \gamma \in \Gamma,\ \exists\, y \in F(\Gamma) \ \text{s.t.}\ x = \gamma(y)\}$.*
2) *$J(\Gamma)$ is backward Γ-invariant, i.e., $\Gamma^{-1}(J(\Gamma)) = J(\Gamma) = \{x \in T \mid \exists\, \gamma \in \Gamma \ \text{s.t.}\ \gamma(x) \in J(\Gamma)\}$.*

If (Γ, T) is a compactly generated pg, then there is a Hermitian metric on $F(\Gamma)$ invariant under Γ [2]. In this sense, the action of Γ is not quite wild on $F(\Gamma)$. If (Γ, T) is a psg, then invariant metrics need not exist in general. Indeed, if $z \in F(\Gamma)$, $\gamma \in \Gamma$ and $\gamma'_z = 0$, then $(\gamma^* g)_z = 0$ so that there is no Γ-invariant metric on $F(\Gamma)$, where γ'_z denotes the derivative of γ at z. For example, let $T = \mathbb{C}$ and define $f\colon T \to T$ by $f(z) = z^2$. Then, the open unit disc is a connected component of the Fatou set, however, f cannot be an isometry for any metric.

Inspired by the Schwarz lemma on the Poincaré disc, we introduce the notion of semi-invariant metrics as follows.

Definition 2.20. Let g_1 and g_2 be Hermitian metrics on $F(\Gamma)$. If $z \in F(\Gamma)$, then we denote by $(g_1)_z$ the metric on $T_z F(\Gamma)$. Suppose that we have $g_1 = f_1^2 g_0$ and $g_2 = f_2^2 g_0$ on a neighborhood of z, where g_0 denotes the standard Hermitian metric on $\mathbb{C}$. If $f_1(z) \leq f_2(z)$, then we write $(g_1)_z \leq (g_2)_z$. Note that this condition is independent of the choice of charts about z. If $(g_1)_z \leq (g_2)_z$ holds on $F(\Gamma)$, then we write $g_1 \leq g_2$.

Definition 2.21. Let g be a Hermitian metric on $F(\Gamma)$. The metric g is said to be *semi-invariant* if $z \in F(\Gamma)$ and if $\gamma \in \Gamma$ is defined on a neighborhood of z, then $\gamma^* g \leq g$ holds on $\operatorname{dom}\gamma$.

The following is known. See [3] and [2] for details.

Theorem 2.22. 1) *Suppose that (Γ, T) is compactly generated, then the metric g is finite and locally Lipschitz continuous on $F(\Gamma)$.*

2) *If $\Gamma^\times = \Gamma$, then $F(\Gamma)$ admits a Hermitian metric which is locally Lipschitz continuous and Γ-invariant.*

3) *If (Γ, T) is generated by a compactly generated pg, then $F(\Gamma)$ admits a Hermitian metric which is of class C^ω and Γ-semiinvariant.*

Example 2.23 ([3], Example 4.21). We define $\gamma\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = z^2$. Then, $J(\gamma) = \{|z| = 1\}$. If we set

$$f(z) = \begin{cases} 1 & \text{if } |z| \leq \frac{1}{2}, \\ 2^k |z|^{2^k - 1} & \text{if } 2^{-\frac{1}{2^{k-1}}} \leq |z| \leq 2^{-\frac{1}{2^k}}, \\ 2^k |z|^{-2^k - 1} & \text{if } 2^{\frac{1}{2^k}} \leq |z| \leq 2^{\frac{1}{2^{k-1}}}, \\ \frac{1}{|z|^2} & \text{if } |z| \geq 2, \end{cases}$$

then $g = f^2 |dz|^2$ gives a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is locally Lipschitz continuous and semi-invariant under the action of Γ, where $\Gamma = \langle \gamma \rangle$. On the other hand, if we consider the Poincaré metric on the unit disc, then γ is contracting by the Schwarz lemma. Hence the Poincaré metrics on the unit disc and $\mathbb{C}P^1 \setminus \{|z| \leq 1\}$ give rise to a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is of class C^ω and semi-invariant under the action of Γ. On the other hand, there is no Γ-invariant metric on $F(\Gamma)$. Indeed, $0 \in F(\Gamma)$ but $(\gamma^* g)_0 = 0$ for any metric g on $F(\Gamma)$.

Let $\widehat{\Gamma}$ be the psg generated by $\gamma|_{\mathbb{C}P^1\setminus\{0,\infty\}}$ and its local inverses. Then $F(\widehat{\Gamma}) = \mathbb{C}\setminus(S^1 \cup \{0\})$. An invariant metric on $F(\widehat{\Gamma})$ is given by $|dz|^2/(|z| \log|z|)^2$ on $\{0 < |z| < 1\}$. We can find on $\{1 < |z|\}$ a metric of the same kind.

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Contact structures, deformations and taut foliations

Jonathan BOWDEN

1. Introduction

In her PhD thesis H. Eynard-Bontemps proved the following theorem:

Theorem 1.1 (Eynard-Bontemps [5]). *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be smooth oriented taut foliations on a 3-manifold M whose tangent distributions are homotopic as (oriented) plane fields. Then $T\mathcal{F}_0$ and $T\mathcal{F}_1$ are smoothly homotopic through integrable plane fields.*

This raises the question of whether any two taut contact structures that are homotopic as plane fields are also homotopic as taut foliations. An interesting special case of this concerns the path connectedness of the space of horizontal foliations on S^1-bundles (i.e. those that are transverse to the fibers). We provide various examples which show that the answer to both of these questions is negative. One of the main tools are contact perturbations of foliations given by Eliashberg and Thurston [3].

This naturally leads to the problem, first raised by Eliashberg and Thurston (see also [4]), of which (universally tight) contact structures are perturbations of taut or Reebless foliations, which can be answered completely for Seifert fibered spaces over surfaces of genus at least one.

2. Main results

Let $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ denote the space of holonomy representations of smooth horizontal foliations on an oriented S^1-bundle of Euler class e over a closed, oriented surface Σ_g of genus g.

Theorem 2.1. *The space $Rep_e(\pi_1(\Sigma_g), Diff_+(S^1))$ with fixed Euler class $e \neq 0$ is in general not path connected.*

To prove this theorem one distinguishes path components of the space $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ using the isotopy class of contact perturbations approximating the associated suspension foliations. However, care must be taken as the isotopy class of a contact structure approximating a contact structure is in general not well-defined. On the other hand Vogel [8] has shown that the isotopy class of the approximating contact structure is well-defined for foliations without torus leaves, apart from a small list of special cases, although for our applications a relatively simple argument using linear deformations of foliations suffices.

Theorem 2.1 can also be shown using the following extension of a result of Ghys [7], which answers a question posed to us by Y. Mitsumatsu.

Theorem 2.2. *Any representation $\rho \in Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ that lies in the C^0-connected component of an Anosov representation ρ_{An} is itself Anosov. In particular, it is conjugate to a discrete subgroup of a finite covering of $PSL(2,\mathbb{R})$ and is injective.*

Similar ideas yield the following

Theorem 2.3. *There exist infinitely many examples of manifolds admitting taut foliations $\mathcal{F}_0, \mathcal{F}_1$ that are homotopic as foliations but not as taut foliations. Furthermore, the same result holds true for diffeomorphism classes of unoriented foliations.*

Concerning which contact structures can be realised as perturbations of Reeebless/taut foliations, we obtain a characterisation for a large class of Seifert fibered spaces. In order to state this result recall the notion of the *enroulement* (cf. [6]) or *twisting number* $t(\xi)$ of a contact structure ξ on a Seifert fibered space which is defined as the maximal Thurston-Bennequin number of a Legendrian knot that is isotopic to a regular fiber, where this is measured relative to the canonical framing coming from the base. Moreover, a *deformation* of a foliation $\mathcal{F}$ is a smooth family of 2-plane fields $\{\xi_t\}_{t\in[0,1]}$ so that $\xi_0 = T\mathcal{F}$ and ξ_t is a contact structure for $t > 0$.

Theorem 2.4. *Let ξ be a universally tight contact structure on a Seifert fibered space with infinite fundamental group and $t(\xi) \geq 0$, then ξ is isotopic to a deformation of a Reebless foliation. If $g > 0$ and $t(\xi) < 0$, then ξ is isotopic to a deformation of a taut foliation.*

3. Questions

Question 3.1. (1) Is the space $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1))$ of topological S^1-actions of fixed Euler class path connected? A related question is whether the image of

$$\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1)) \xrightarrow{e_b} H^2_b(\pi_1(\Sigma_g), \mathbb{R})$$

under the bounded Euler class is path connected (in the weak-* topology).

(2) Does any 3-manifold M with infinite fundamental group that admits universally tight contact structures for both orientations necessarily admit a smooth Reebless/taut foliation? (Note that the existence of universally tight contact structures for both orientations is a necessary condition by [2]).

(3) Are there examples of manifolds for which the space of taut foliations in a given homotopy class has infinitely manifold path components up to diffeomorphism and deformation?

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Hyperbolic Geometry and Homeomorphisms of Surfaces

John CANTWELL and Lawrence CONLON

1. Introduction

Let L be an arbitrary connected surface, compact or noncompact, with or without boundary and orientable or nonorientable. Let $f : L \to L$ be a homeomorphism. We discuss two topics which are related but perhaps, at first, not obviously so.

The first topic is the Handel-Miller theory of endperiodic maps of surfaces, never published even as an announcement, although it has been used by various authors in the study of foliated 3-manifolds. The second topic concerns the Epstein-Baer theorem that homotopic homeomorphisms of surfaces are isotopic. Both of these topics will be studied via a suitable hyperbolic metric on L (Definition 1.1).

For endperiodic maps, we will sketch the main points of the theory and announce new results. For homotopic homeomorphisms, we will outline a new line of proof of Epstein-Baer using hyperbolic geometry. This involves extending classical results about complete hyperbolic surfaces with finite area to complete hyperbolic surfaces with geodesic boundary and infinite Euler characteristic.

The Handel-Miller theory determines an endperiodic map $h : L \to L$, in the same isotopy class as f, which preserves a pair of transverse geodesic laminations and has, in a certain sense, the "tightest" dynamics in its isotopy class. This has obvious analogies with the Nielsen-Thurston theory of automorphisms of compact surfaces, but there are remarkable differences also. In proving that h is in the isotopy class of f, one is led to the second topic of this talk.

Definition 1.1. A hyperbolic metric on a surface L is "standard" if it is complete, makes ∂L geodesic and admits no isometrically imbedded hyperbolic half planes. A surface equipped with such a metric is called a standard hyperbolic surface. A surface which is homeomorphic to a standard hyperbolic surface will simply be called standard.

This is not a serious restriction topologically. Up to homeomorphism, there are exactly 13 nonstandard surfaces, none of them interesting for Handel-Miller theory.

2. Endperiodic Homeomorphisms

Let $\mathcal{E}(L)$ denote the set of ends of L, a compact, totally disconnected, metrizable space which compactifies L.

Definition 2.1. An end $e \in \mathcal{E}(L)$ is an attracting end if it admits a neighborhood $U_e \subset L$ such that, for a least integer $p_e \geq 0$,

$$U_e \supset f^{p_e}(U_e) \supset \cdots \supset f^{np_e}(U_e) \supset \cdots$$

and $\bigcap_{n=0}^{\infty} f^{np_e}(U_e) = \emptyset$. Repelling ends are defined similarly, using iterates of f^{-1}. The integer p_e is called the period of e.

Definition 2.2. A homeomorphism $f : L \to L$ is endperiodic if $\mathcal{E}(L)$ is finite and each end is either attracting or repelling..

Examples will be pictured in the talk. The definition of "endperiodic" can be extended to surfaces with infinite endset, even a Cantor set of ends, and this has important applications to foliations. But in this generalization, there will only be finitely many attracting and repelling ends, and one passes to the "soul" of L, an f-invariant subsurface with finitely many ends on which all of the interesting dynamics takes place. This effectively reduces us to the case considered by Handel and Miller.

Definition 2.3. An end e is simple if it is isolated and either annular or simply connected. Standard hyperbolic surfaces without simple ends are called "admissible" surfaces.

In the rest of this section we consider admissible surfaces L with finitely many ends and endperiodic homeomorphisms $f : L \to L$.

An attracting end e of period p_e has compact fundamental domains B_i such that $U_e = B_0 \cup B_1 \cup \cdots$ and $f^{p_e}(B_i) = B_{i+1}$, $0 \leq i < \infty$. There is a similar notion of fundamental domain for repelling ends. The intersection $B_i \cap B_{i+1}$ is called a positive juncture. It is a compact 1-manifold. The negative junctures are defined similarly in neighborhoods of repelling ends. Each juncture is the union of finitely many 2-sided, essential closed curves and/or properly imbedded arcs.

The Handel-Miller construction. Start applying powers of f^{-1} to positive junctures. The result is an infinite family of ultimately "distorted" junctures. Generally the distortions get enormous and the distorted junctures wrap around in L in increasingly complex ways. (In the talk, examples will be pictured to illustrate this.) It will be convenient also to call a distorted juncture by the name "juncture". In the homotopy class of each component of a juncture (endpoint preserving homotopy for properly imbedded arcs) there is a unique geodesic. This infinite family of geodesics accumulates exactly on a closed geodesic lamination Λ_- with complete, noncompact leaves (the absence of half planes is critical here). Every leaf of this lamination penetrates arbitrarily deeply into the neighborhoods of repelling ends, but the lamination is uniformly bounded away from the attracting ends. Using the junctures of negative ends, one similarly defines the geodesic lamination Λ_+, transverse to Λ_-, which penetrates arbitrarily deeply into the attracting ends but is uniformly bounded away from the repelling ends. The final step is to define an endperiodic homeomorphism $h : L \to L$ which preserves these laminations and is isotopic to f. The dynamics of h is "tightest possible" in its isotopy class, in the sense that h has the smallest possible invariant set $\mathcal{I}$ and the dynamics of $h|\mathcal{I}$ is Markov.

Definition 2.4. A pseudo-geodesic σ in L is a continuous, imbedded curve, any lift of which to the universal cover $\widetilde{L}$ (viewed as a surface in the Poincaré disk) has well defined endpoints on the circle at infinity.

We have axiomatized the Handel-Miller theory to allow the laminations to be pseudo-geodesic. Again the endperiodic homeomorphism preserving the laminations is isotopic to f. This generalization is quite useful in applications to foliation theory. We have developed an extensive structure theory for the laminations, based on the axioms, which reveals many surprising features.

Here are two new results.

Theorem 2.5. *The pseudo-geodesic laminations of the axiomatized Handel-Miller theory are simultaneously ambient isotopic to the geodesic laminations described above.*

Generally, h is not smooth, even in the case that the laminations are geodesic. One advantage to relaxing the geodesic condition is the following.

Theorem 2.6. *There is a choice of Handel-Miller map h, corresponding to pseudo-geodesic laminations, which is a diffeomorphism except, perhaps, at finitely many p-pronged singularities.*

3. Homotopic Homeomorphisms

The fact that the Handel-Miller endperiodic map h is isotopic to the original endperiodic map f from which it is derived needs to be proven using the ideas in this section.

The surface L is now any standard one. Give it a standard hyperbolic metric. Denote by Δ the open unit disk with the Poincaré metric. Then either $\partial L = \emptyset$ and the universal covering space is $\widetilde{L} = \Delta$, or $\widetilde{L} \subsetneq \Delta$. The completion $\widehat{L}$ is the closure of $\widetilde{L}$ in the closed disk $\overline{\Delta}$. We denote by E the "ideal boundary" of $\widetilde{L}$, namely $E = S^1 \cap \widehat{L}$. The following is well known for complete hyperbolic surfaces of finite area. For standard hyperbolic surfaces, we can find no proof in the literature.

Theorem 3.1. *Any lift $\widetilde{h} : \widetilde{L} \to \widetilde{L}$ of a homeomorphism $h : L \to L$ extends canonically to a homeomorphism $\widehat{h} : \widehat{L} \to \widehat{L}$.*

The following is also known for compact hyperbolic surfaces.

Theorem 3.2. *If $h : L \to L$ is a homeomorphism having a lift such that $\widehat{h}|E = \mathrm{id}_E$, then h is isotopic to id_L.*

In particular, if f, g are two homeomorphisms of L with lifts such that $\widehat{f}|E = \widehat{g}|E$, then f and g are isotopic. In the Handel-Miller theory, one easily verifies this condition for f and h, hence h represents the isotopy class of f.

The following is an easy corollary of Theorem 3.2.

Theorem 3.3 (Epstein-Baer)**.** *If L is a standard surface, then homotopic homeomorphisms of L are isotopic.*

In proving this, Epstein put no restriction on L, but required the homotopy to respect ∂L and, if there were noncompact boundary components, required the homotopy to be proper. The first requirement is only needed for 4 of the 13 nonstandard surfaces. The second requirement is inconvenient in applications and is not needed at all in our proof.

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Connectedness of the space of smooth $\mathbb{Z}^2$ actions on $[0, 1]$

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(joint work with C. Bonatti)

1. Introduction

Our interest in $\mathbb{Z}^2$-actions on $[0, 1]$ stems from the general study of the set $\mathcal{F}ol(M)$ of all smooth codimension one cooriented foliations on a given closed oriented 3-manifold. By identifying every such foliation with its tangent plane field, one can think of $\mathcal{F}ol(M)$ as a subspace of the space $\mathcal{P}(M)$ of smooth plane fields on M, endowed with the usual C^∞ topology.

The inclusion $\mathcal{F}ol(M) \overset{\iota}{\hookrightarrow} \mathcal{P}(M)$ is strict. In fact, most plane fields are *not* tangent to foliations (or, in other words, are not *integrable*): one can easily see that $\mathcal{F}ol(M)$ is a closed subset of $\mathcal{P}(M)$ *with empty interior*. On the other hand, it has been known since the late sixties that $\mathcal{F}ol(M)$ is nonempty, and even that every plane field can be deformed into a (plane field tangent to a) foliation (see [13]). In other words, the map $\pi_0\mathcal{F}ol(M) \xrightarrow{\iota_*} \pi_0\mathcal{P}(M)$ induced by the inclusion is *surjective*. It is then natural to wonder whether this map is also injective, i.e:

QUESTION 1.1. If two foliations have homotopic tangent plane fields, are they connected by a path of foliations?

Larcanché [7] gave a positive answer for foliations transverse to the fibers of a circle bundle over a closed surface, and for pairs of taut foliations sufficiently close to each other. In [4], we extended this result to any pair of taut foliations homotopic as plane fields. Note that the foliations of the connecting path are not necessarily taut, and recent works by T. Vogel [12] and J. Bowden [3] actually show that the space of taut foliations in a given homotopy class is in general not path-connected. As for non-taut foliations, we reduced Question 1.1 to the particular case of "horizontal" foliations on the thick torus:

QUESTION 1.2. Consider a foliation τ on $\mathbb{T}^2 \times [0, 1]$ tangent to the boundary and transverse to the direction $[0, 1]$. As a plane field, τ is homotopic to the trivial foliation by $\mathbb{T}^2 \times \{.\}$ (rel. to the boundary) since both are transverse to $[0,1]$. But are they connected by a path of smooth foliations transverse to $[0,1]$?

This question has a translation in terms of holonomy. A foliation τ as above has a so-called *holonomy representation* which is a homomorphism $\rho(\tau) : \pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2 \to \mathrm{Diff}^\infty_+[0,1]$. Let us denote by $\mathcal{R}$ the set of all such homomorphisms. Since such a map is completely determined by the images of the standard generators of $\mathbb{Z}^2$, $\mathcal{R}$ can be thought of as the space of pairs of commuting elements of $\mathrm{Diff}^\infty_+[0,1]$, endowed with the usual C^∞ topology. One can then show that question 1.2 is equivalent to:

Question 1.3. Is the space $\mathcal{R}$ path-connected?

Our aim here is to present the following partial answer obtained in collaboration with C. Bonatti in [2].

2. Main result

Theorem 2.1 (Bonatti, E-B.)**.** *The space $\mathcal{R}$ of smooth orientation preserving $\mathbb{Z}^2$-actions on $[0,1]$ is connected. More precisely, the path-connected component $\mathcal{C}_{\mathrm{id}}$ of $(\mathrm{id},\mathrm{id})$ is dense in $\mathcal{R}$.*

Combined with [4], this yields the following:

Theorem 2.2. *For any closed 3-manifold M, the inclusion of $\mathcal{F}ol(M)$ into $\mathcal{P}(M)$ induces a bijection between the connected components of those two spaces.*

The analogous question for *path*-connected components however remains open (for foliations as well as for $\mathbb{Z}^2$ actions). One of our aims here will be to highlight the gap between connectedness and path-connectedness. But let us make a few remarks beforehand.

First of all, why isn't the answer to Question 1.2 obvious? Indeed, the space $\mathrm{Diff}^\infty_+[0,1]$ is contractible, so one can easily deform any given pair $(f,g) \in (\mathrm{Diff}^\infty_+[0,1])^2$ into any other. But this forgets about the commutativity condition, which is a huge constraint. It is not the only source of trouble though. Regularity is another. Indeed if we consider the same question for homeomorphisms of $[0,1]$ instead of smooth diffeomorphisms, we can easily see using some kind of "Alexander trick" that the space of orientation preserving C^0-actions of $\mathbb{Z}^2$ on $[0,1]$ is *contractible*. But such a "brutal" method is bound to fail in the C^∞ setting. The C^1 case is still different and was solved by A. Navas in [8] using completely different tools.

Outline of proof. As we already mentioned, deforming a given pair of diffeomorphisms (f, g) into another one is not difficult if one forgets about the commutativity condition, but this constraint adds a lot of rigidity to the problem. Namely, if we restrict to the case of diffeomorphisms f, g which are nowhere infinitely tangent to the identity in $(0, 1)$ (such pairs will be referred to as "nondegenerate"), classical results by N. Kopell [6], G. Szekeres [10] and F. Takens [11] imply that f and g belong either to a common infinite cyclic group generated by some C^∞ diffeomorphism h of $[0, 1]$ or to a common C^1 flow (C^∞ on $(0, 1)$ but in general not C^2 on $[0, 1]$). Then, our strategy is as follows.

- In the first case, any isotopy $t \in [0, 1] \mapsto h_t$ from id to h yields a path $t \mapsto (h_t^p, h_t^q)$ of commuting C^∞-diffeomorphisms from $(\mathrm{id}, \mathrm{id})$ to $(f = h^p, g = h^q)$, so (f, g) is actually *in* the path-connected component of (id, id) ($\mathcal{C}_{\mathrm{id}}$) and we have nothing to do.
- In the second case, however, extra-work is called for. If f and g are the time-α and β maps of a C^1 vector field ξ (C^∞ on $(0, 1)$), the idea is to construct a C^∞ vector field $\tilde{\xi}$ whose time-α and β maps φ^α and φ^β are arbitrarily C^∞ close to f and g respectively. The pair $(\varphi^\alpha, \varphi^\beta)$ is then easily connected to $(\mathrm{id}, \mathrm{id})$ by a continuous path of pairs of commuting C^∞ diffeomorphisms $t \in [0, 1] \mapsto (\varphi^{t\alpha}, \varphi^{t\beta})$. One can then conclude that (f, g) belongs to the closure of $\mathcal{C}_{\mathrm{id}}$.

In other words, what we show is that, among "nondegenerate" pairs, those made of iterates of the same smooth diffeomorphism or of elements of the same smooth flow form a *dense* and path-connected subset. Then, deriving the general result from the restricted ("nondegenerate") one we just mentioned is elementary.

The strategy seems very simple. But let us stress that, in the second case above, a random smoothing of the vector field ξ near the boundary won't do in general, for the resulting flow would be no more than C^1 close to that of ξ. So first, one needs to derive some nice estimates on ξ from the knowledge that *some* times of its flow are C^∞. More precisely, if ξ is not C^∞ near a point of the boundary, say 0, according to Takens [11], f and g are necessarily infinitely tangent to the identity at that point. What we show in that case is that, though the derivatives of ξ of order ≥ 2 globally diverge when one approaches 0, arbitrarily close to 0, one can find whole fundamental intervals of f and g where these derivatives are arbitrarily small. These estimates are a generalization of those obtained by F. Sergeraert in [9] for diffeomorphisms without fixed points in $(0, 1)$. Then the rough idea to construct $\tilde{\xi}$ is simply to replace ξ between 0 and such a "nice interval" by something smooth *and* "C^∞-small" (the latter being made possible precisely by the estimates on ξ in the "nice interval"), leaving it unchanged outside this small region. Then the time-α and β

maps of the new vector field $\tilde{\xi}$ basically coincide with f and g away from the boundary and are very close to the identity there, as are f and g !

Note, to conclude, that what our strategy provides in the situation above is an *approximation* of (f, g) by elements of $\mathcal{C}_{\mathrm{id}}$, not a *continuous deformation*, simply because between the "nice intervals" which are essential to our construction lie "nasty" ones. Precisely there lies the gap between connectedness and *path*-connectedness. $\square$

3. (Other) questions

Question 3.1. It has been a longstanding open question whether the space of smooth orientation preserving $\mathbb{Z}^2$ actions on the *circle* is (locally) connected. It follows from Theorem 2.1 that the subspace made of non-free actions (or equivalently, of pairs of commuting diffeomorphisms with rationally dependent rotation numbers) is connected. For commuting diffeomorphisms f and g with rationally independant rotation numbers $\rho(f)$ and $\rho(g)$, on the other hand, here is what is known:

- if $\rho(f)$ and $\rho(g)$ satisfy a joint diophantine condition, B. Fayad and K. Khanin [5] proved that f and g are simultaneously conjugate to the rotations of angle $\rho(f)$ and $\rho(g)$, denoted by $R_{\rho(f)}$ and $R_{\rho(g)}$ respectively, by an element φ of $\mathrm{Diff}^{\infty}_{+}\mathbb{S}^1$. The pair (f, g) is thus connected to $(\mathrm{id}, \mathrm{id})$ by the path $t \in [0,1] \mapsto (\varphi^{-1} \circ R_{t\rho(f)} \circ \varphi, \varphi^{-1} \circ R_{t\rho(g)} \circ \varphi)$ of smooth commuting diffeomorphisms.
- if $\rho(f)$ and $\rho(g)$ do not satisfy such a condition, (f, g) is not necessarily smoothly conjugate to $(R_{\rho(f)}, R_{\rho(g)})$. Nevertheless, according to M. Benhenda [1], there exists a Baire-dense subset B of $\mathbb{S}^1$ such that, if $\rho(f)$ *or* $\rho(g)$ belongs to B, (f, g) can be *approached* by pairs which *are* smoothly conjugate to $(R_{\rho(f)}, R_{\rho(g)})$. Thus (f, g) belongs to the *closure* of the path-connected component of $(\mathrm{id}, \mathrm{id})$.

It is not known, however, whether this last fact holds for *any* pair $(\rho(f), \rho(g)) \in (\mathbb{R} \setminus \mathbb{Q})^2$. A positive answer would imply the connectedness of the whole space of $\mathbb{Z}^2$-actions on the circle.

Question 3.2. How about smooth actions of other surface groups on $[0, 1]$? On the circle? Some progress has recently been made by J. Bowden [3] on this last subject.

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Classification and rigidity of totally periodic pseudo-Anosov flows in graph manifolds

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This is joint work with Thierry Barbot.

Pseudo-Anosov flows are extremely common amongst three manifolds, for example 1) Suspension pseudo-Anosov flows [Th1, Th2, Th3]; 2) Geodesic flows in the unit tangent bundle of negatively curved surfaces [An]; 3) Certain flows transverse to foliations in closed atoroidal manifolds [Mo, Cal1, Cal2, Cal3, Fe]; flows obtained from these by either 4) Dehn surgery on a closed orbit of the pseudo-Anosov flow [Go, Fr]; or 5) Shearing along tori [Ha-Th]; 6) Non transitive Anosov flows [Fr-Wi] and flows with transverse tori [Bo-La].

We consider the following question: how many different pseudo-Anosov flows are there in a manifold up to topological conjugacy? *Topological conjugacy* means that there is a homeomorphism between the manifolds which sends orbits of the first flow to orbits of the second flow. We also consider the notion of *isotopic equivalence*, i.e. a topological conjugacy induced by an isotopy, that is, a homeomorphism isotopic to the identity.

Here we consider only closed, orientable, toroidal manifolds. They have incompressible tori and and also since they support a pseudo-Anosov flow they are irreducible. Therefore the manifolds are Haken manifolds. We recently proved that if M is Seifert fibered, then the flow is up to finite covers topologically conjugate to a geodesic flow in the unit tangent bundle of a closed hyperbolic surface [Ba-Fe1]. We also proved that if the ambient manifold is a solvable three manifold, then the flow is topologically conjugate to a suspension Anosov flow [Ba-Fe1]. We stress that in both cases the results imply that the flow does not have singularities, that is, the type of the manifold strongly restricts the type of pseudo-Anosov that it can admit. This is in contrast with the strong flexibility in the construction of pseudo-Anosov flows − that is because many flows are constructed in atoroidal manifolds or are obtained by flow Dehn surgery on the pseudo-Anosov flow, which changes the topological type of the manifold. Therefore in many constructions one cannot expect the underlying manifold to be toroidal.

Here we consider pseudo-Anosov flows in graph manifolds. A *graph manifold* is an irreducible three manifold which is a union of Seifert fibered pieces. In [Ba-Fe1] we produced a large new class of examples in graph manifolds. These flows are *totally periodic*. This means that each Seifert piece of the torus decomposition of the graph manifold is *periodic*, that is, up to finite powers, a regular fiber is freely homotopic to a closed orbit of the flow. More recently, Russ Waller [Wa] has been studying how common these examples are, that is, the existence question for these type of flows. He showed that these flows are as common as they could be (modulo the necessary conditions).

Here we analyse the question of the classification and rigidity of such flows. To do that we introduce *Birkhoff annuli.* A Birkhoff annulus is an a priori only immersed annulus, so that the boundary is a union of closed orbits of the flow and the interior of the annulus is transverse to the flow. For example consider the geodesic flow of a closed, orientable hyperbolic surface. The ambient manifold is the unit tangent bundle of the surface. Let α be an oriented closed geodesic - a closed orbit of the flow - and consider a homotopy that turns the angle along α by π. The image of the homotopy from α to the same geodesic with opposite orientation is a Birkhoff annulus for the flow in the unit tangent bundle. If α is not embedded then the Birkhoff annulus is not embedded. In general Birkhoff annuli are not embedded, particularly in the boundary.

In [Ba-Fe1] we proved the following basic result about the relationship of a pseudo-Anosov flow and a periodic Seifert piece P: there is a spine Z for P which is a connected union of finitely many elementary Birkhoff annuli. In addition the union of the interiors of the Birkhoff annuli is embedded and also disjoint from the closed orbits in Z. These closed orbits, boundaries of the Birkhoff annuli in Z, are called *vertical periodic orbits.* The set Z is a deformation retract of P, so P is isotopic to a small compact neighborhood $N(Z)$ of Z.

The first theorems (Theorem A and B) are valid for general Seifert fibered pieces in any closed orientable manifold M, not necessarily a graph manifold.

Theorem A ([Ba-Fe2]). *Let Φ be a pseudo-Anosov flow in M^3. If $\{P_i\}$ is the (possibly empty) collection of periodic Seifert pieces of the torus decomposition of M, then the spines Z_i and neighborhoods $N(Z_i)$ can be chosen to be pairwise disjoint.*

The next result shows that the boundary of the pieces can be put in good position with respect to the flow:

Theorem B ([Ba-Fe2]). *Let Φ be a pseudo-Anosov flow and P_i, P_j be periodic Seifert pieces with a common boundary torus T. Then T can be isotoped to a torus transverse to the flow.*

Theorem C ([Ba-Fe2]). *Let Φ be a totally periodic pseudo-Anosov flow with periodic Seifert pieces $\{P_i\}$. Then neighborhoods $\{N(Z_i)\}$ of the spines*

$\{Z_i\}$ can be chosen so that their union is M and they have pairwise disjoint interiors. In addition each boundary component of every $N(Z_i)$ is transverse to the flow. Each $N(Z_i)$ is flow isotopic to an arbitrarily small neighborhood of Z_i.

We stress that for general periodic pieces it is not true that the boundary of $N(Z_i)$ can be isotoped to be transverse to the flow. There are some simple examples as constructed in [Ba-Fe1]. The point here is that we assume that *all* pieces of the JSJ decomposition are periodic Seifert pieces.

Hence, according to Theorem C, totally periodic pseudo-Anosov flow are obtained by glueing along the bondary a collection of small neighborhoods $N(Z_i)$ of the spines. There are several ways to perform this glueing which lead to pseudo-Anosov flows. The main result is that the resulting pseudo-Anosov flows are all topologically conjugate to each other:

Theorem D ([Ba-Fe2])**.** *Let Φ, Ψ be two totally periodic pseudo-Anosov flows on the same orientable graph manifold M. Let P_i be the Seifert pieces of M, and let $Z_i(\Phi)$, $Z_i(\Psi)$ be spines of Φ, Ψ in P_i. Then, Φ and Ψ are topologically conjugate if and only if there is a homeomorphism of M mapping the collection of spines $\{Z_i(\Phi)\}$ onto the collection $\{Z_i(\Psi)\}$ and preserving the orientations of the vertical periodic orbits induced by the flows.*

Finally we show that for any totally periodic pseudo-Anosov flow Φ there is a model pseudo-Anosov flow as constructed in [Ba-Fe1] which has precisely the same data Z_i, $N(Z_i)$ that Φ has. This proves the following:

Main theorem ([Ba-Fe2])**.** *Let Φ be a totally periodic pseudo-Anosov flow in a graph manifold M. Then Φ is topologically equivalent to a model pseudo-Anosov flow.*

Model pseudo-Anosov flows are defined by some combinatorial data (essentially, the data of some fat graphs and Dehn surgery coefficients) and some parameter λ. A nice corollary of these results is that, up to topological conjugation the model flows actually do not depend on the choice of λ, nor on the choice of the selection of the particular glueing map between the model periodic pieces.

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Thurston and foliation theory, some personal reminiscences

André HAEFLIGER

I plan to evoke my personal contacts with Thurston from 1972, when I met him for the first time in the Swiss alps, during the academic year 1972–1973 that we spent together at the Institute for Advanced Study at Princeton, and in the summer 1976, in Warwick and Varenna in Italy.

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Dynamical Lagrangian Foliations: Essential nonsmoothness and Godbillon–Vey classes

Patrick FOULON and Boris HASSELBLATT

1. Introduction

We present 2 results about foliations arising as stable and unstable foliations for a contact Anosov flow. The first gives Lagrangian foliations on 3-manifolds that can not be smoothed in the following sense: They are preserved by a contact Anosov flow and there is no topologically equivalent contact Anosov flow with C^2 stable and unstable foliations. The second, in early development, gives a representation of Godbillon–Vey classes for the invariant foliations of a contact Anosov flow and has promise for alternate proofs of pertinent results.

2. Nonsmooth foliations

There are contact Anosov flows on 3-manifolds whose Anosov splitting is not C^2 and such that the same holds for any topologically equivalent contact Anosov flow. In this sense, then, the invariant (and necessarily Lagragian) foliations cannot be smoothed out. These Anosov flows turn out to have a remarkable range of unconventional properties.

For a contact Anosov flow on a 3-manifold, the invariant (stable and unstable) foliations are $C^{1+\text{Zygmund}}$, i.e., differentiable with Zygmund-regular derivative. Indeed, this holds for the weak-stable and weak-unstable foliations of volume-preserving Anosov flows on 3-manifolds [10].

Definition 2.1. A continuous function $f: U \to L$ on an open set $U \subset L'$ in a normed linear space to a normed linear space is said to be *Zygmund-regular* if there is $Z > 0$ such that $\|f(x+h)+f(x-h)-2f(x)\| \le Z\|h\|$ for all $x \in U$ and sufficiently small $\|h\|$. It is said to be *"little Zygmund"* (or "zygmund") if $\|f(x+h)+f(x-h)-2f(x)\| = o(\|h\|)$. For maps between manifolds these definitions are applied in smooth local coordinates.

The "nonsmooth" in the section title actually refers to "not $C^{1+\text{zygmund}}$," i.e., no more regular than is always known to be the case. For our purposes the following rigidity result by Hurder and others is central.

Theorem 2.2 ([10, 8]). *If a volume-preserving Anosov flow on a 3-manifold has $C^{1+zygmund}$ Anosov splitting, then it is smoothly conjugate to a geodesic flow (or a suspension).*

To produce examples of contact Anosov flows whose invariant foliations are not $C^{1+\mathrm{zygmund}}$ and such that the same holds for any topologically equivalent contact Anosov flow, it thus suffices to construct contact Anosov flows that are not topologically equivalent to any geodesic flow.

The novelty is that these are *contact* flows, and the novelty of the method (due to Foulon) is to refine previous surgery methods to preserve the existence of a contact structure. The surgery is a Dehn surgery in a knot neighborhood, and in our context the knot should be of the following type.

DEFINITION 2.3. A *Legendrian curve* in a contact manifold is a curve tangent to the contact structure at every point. In the presence of a contact Anosov flow, a Legendrian curve (which is by construction transverse to the flow) is said to be *E-transverse* if it is also transverse to both the strong stable and strong unstable subbundles E^- and E^+ of the flow.

Our main result has a rather long statement because these flows have a host of interesting properties, as do the manifolds we obtain.

Theorem 2.4. *A contact Anosov flow φ on a 3-manifold M with an E-transverse Legendrian knot K admits smooth Dehn surgeries that produce new contact Anosov flows. If φ is the geodesic flow on the unit tangent bundle of a negatively curved surface, then these surgeries include the Handel–Thurston surgery [9], in which case the resulting flow has the following properties:*

1. *It acts on a manifold that is not a unit tangent bundle.*
2. *It is not topologically orbit equivalent to an algebraic flow.*
3. *Its weak stable foliation is not transversely projective [1, Théorème A].*
4. *Its Anosov splitting $TM = E^\varphi \oplus E^+ \oplus E^-$ does not have "little Zygmund" (hence not Lipschitz-continuous) derivative (Theorem 2.2).*
5. *Its topological and volume entropies differ, or, equivalently, the measure of maximal entropy is always singular (otherwise it would be up to finite covers smoothly conjugate to a geodesic flow of constant curvature [7]).*

Moreover, there are contact Anosov flows on hyperbolic manifolds: If $M \smallsetminus K$ is a hyperbolic manifold, then all but finitely many of our Dehn surgeries produce a hyperbolic manifold. The resulting contact Anosov flow (and any contact Anosov flow topologically orbit equivalent to it) has the following additional properties.

6. *It is associated with a new example of a quasigeodesic pseudo-Anosov flow (see Definition 2.5, [6], [12, Section 5]).*
7. *It is not quasigeodesic (Definition 2.5).*
8. *Its orbits are geodesics for suitable Riemannian metrics on M.*
9. *Each closed orbit is isotopic to infinitely many others*[1] *[4, Theorem A], [2, Remark 5.1.16, Theorem 5.3.3], [3].*
10. *Only finitely many pairs of closed orbits bound an* embedded *cylinder*[2] *[3].*

DEFINITION 2.5. A *quasigeodesic* curve is one that is efficient, up to a bounded multiplicative distortion, in measuring distances in relative homotopy classes, and a flow is said to be quasigeodesic if all flow lines are quasigeodesics [5].

3. Godbillon–Vey classes for Legendrian foliations

Consider a contact Anosov flow φ^t on a $2m+1$-dimensional manifold (M, A) with invariant splitting $\mathbb{R}X \oplus E^+ \oplus E^-$. We can take $A(X) \equiv 1$, and $A \restriction_{E^+ \oplus E^-} = 0$. Then $i_X dA = 0$ on $E^- \cup E^+$ and $dA \restriction_{\mathbb{R}X \oplus E^-} = 0$. E^+ has dimension m and has an unstable volume a. The *normal n-bundle* of a subbundle F of TM is

$$\mathcal{N}_n(F) := \{\omega \in \bigwedge^n (T^*M) \mid \omega(u_1, \dots, u_n) = 0 \text{ whenever } u_i \in F \text{ for any } i\}.$$

For an unstable volume $a\colon M \to \bigwedge^m(E^+)$ define $\alpha \in \mathcal{N}_m(\mathbb{R}X \oplus E^-)$ by $\alpha \restriction_{E^+} = a$.

Proposition 3.1. *If α is C^1, then there is a 1-form β such that $d\alpha = \beta \wedge \alpha$.*

β is as regular as the foliations. If $\beta = 0$ on E^+, then $i_X d\alpha = \beta(X)\alpha$, i.e., $\beta(X)$ is the infinitesimal relative change of the unstable volume under the flow.

DEFINITION 3.2 (Godbillon–Vey classes). Suppose (M, A) is a contact manifold of dimension $2m + 1$. For an A-preserving Anosov flow $\varphi^t\colon M \to$

[1] For algebraic flows, free homotopy (hence isotopy) classes of closed orbits have cardinality at most 2.

[2] This relation is neither transitive nor reflexive. For comparison, isotopy is the equivalence relation of being the boundary components of an *immersed* cylinder.

M with C^2 Anosov splitting, we define the *Godbillon–Vey classes* by $GV_0 = \int_M A \wedge dA^m$,

$$GV_1 = \int_M \beta \wedge dA^m$$
$$GV_2 = \int_M \beta \wedge d\beta \wedge dA^{m-1}$$
$$\vdots$$
$$GV_{m+1} = \int_M \beta \wedge d\beta^m$$

REMARK 3.3. We will show that the C^2 assumption is not needed.

Lemma 3.4. *The Godbillon–Vey classes are well-defined, independently of the choices of a and β.*

Theorem 3.5. *GV_0 is the contact (or Liouville) volume. GV_1 is the Liouville entropy ($\beta(X)$ measures the relative rate of change of unstable volume, and the time average (hence by ergodicity, the space average) of this is the sum of the positive Lyapunov exponents, which by the Pesin Entropy Formula is the Liouville entropy), and for geodesic flows of surfaces, GV_2 is the usual Godbillon–Vey class (we derive the Mitsumatsu formula).*

We can apply these classes to geometric rigidity of geodesic flows on surfaces. Analogously to a result of Mitsumatsu [11] we have:

Proposition 3.6. $\dfrac{GV_0 GV_2}{(GV_1)^2} \geq 1$ *with equality iff M has constant curvature.*

Proof. We have $\dim E^- = \dim E^+ = 1$. Denote the standard vertical vector field by Y and the standard horizontal vector field by h to get

$$[X, Y] = -h, \qquad [Y, h] = -X, \qquad [X, h] = RY,$$

where R is the curvature. We write the unstable and stable vector fields as $\xi^\pm = u^\pm Y \pm h$, where $\dot{u}^\pm + {u^\pm}^2 + R = 0$ (Riccati equation). With $u := -u^-$ we have

$$GV_0 = \int_M A \wedge dA, \quad GV_1 = \int_M uA \wedge dA, \quad GV_2 = \int_M u^2 + 3(\mathcal{L}_Y u)^2 A \wedge dA,$$

so the Cauchy–Schwarz inequality

$$\int_M uA \wedge dA \leq \Big(\int_M u^2 A \wedge dA \Big)^{1/2} \Big(\int_M A \wedge dA \Big)^{1/2}$$

gives

$$GV_1 \leq (GV_2)^{1/2}(GV_0)^{1/2}$$

with equality only if $u \equiv \text{const}$ (and, redundantly, $\mathcal{L}_Y u \equiv 0$), which in turn happens iff M has constant curvature. □

This easily recovers a rigidity result of Hurder and Katok.

Theorem 3.7. *Suppose φ^t and ψ^t are geodesic flows for Riemannian surfaces M and S, respectively, and S has constant curvature -1. If F is a conjugacy that sends the contact form A for φ^t to that for ψ^t, and if the Godbillon–Vey classes match up, i.e., $GV_i = GV_i'$ for $i = 0, 1, 2$, then M and S are isometric.*

Proof. For the constantly curved manifold we have $GV_0' = GV_1' = GV_2' = \operatorname{vol}(S)$, so $\dfrac{GV_0 GV_2}{(GV_1)^2} = 1$, and Theorem 3.6 implies that M has constant curvature. □

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Minimal sets for foliations

Steven HURDER

1. Introduction

In this talk, we will discuss recent results in the program to understand the exceptional minimal sets for foliations of codimension $q \geq 1$. The outline of this program is discussed below.

Let $\mathcal{F}$ be a C^r-foliation of a compact connected Riemannian manifold M, for $r \geq 0$. The leaves of $\mathcal{F}$ are then smoothly immersed submanifolds in M of codimension $q \geq 1$, and each leaf inherits a natural quasi-isometry class of Riemannian metrics.

A closed subset $\mathfrak{M} \subset M$ is *minimal* for $\mathcal{F}$ if for each $x \in \mathfrak{M}$ the leaf $L_x \subset \mathfrak{M}$, and the closure $\overline{L_x} = \mathfrak{M}$. Moreover, if for each transversal $\mathcal{T}_x \subset M$ to $\mathcal{F}$, the intersection $\mathcal{T}_x \cap \mathfrak{M}$ is a totally disconnected set, hence is homeomorphic to a Cantor set as $\mathfrak{M}$ is minimal, then we say that $\mathfrak{M}$ is an *exceptional minimal set*. Here is the general problem.

Problem 1.1. Classify the exceptional minimal sets for C^r-foliations, up to homeomorphism (or possibly orbit equivalence), where $q \geq 1$ and $r \geq 0$.

The approach we take to this very broad problem, is to consider an exceptional minimal set $\mathfrak{M} \subset M$ as a smooth foliated space in the sense of [21], or Candel and Conlon [3, Chapter 11], with additional properties.

Definition 1.2. A *matchbox manifold* is a smooth foliated space $\mathfrak{M}$, whose transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ for the foliation charts are totally disconnected compact metric spaces. If every leaf of the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$ is dense, we say that $\mathfrak{M}$ is *minimal*, and then each transversal space $\mathfrak{X}_i$ is a clopen set in some Cantor set model $\mathfrak{X}$.

Definition 1.3. A matchbox manifold $\mathfrak{M}$ is *Lipshitz*, if the holonomy transformations defined by parallel transport along paths in the leaves of $\mathcal{F}_{\mathfrak{M}}$ are Lipshitz homeomorphisms with respect to the given metrics on the models spaces $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$.

With this definition, Problem 1.1 can be stated as:

Problem 1.4. Let $\mathfrak{M}$ be a Lipshitz minimal matchbox manifold. Given $r \geq 0$ and $q \geq 1$, when does there exists a compact Riemannian manifold M with C^r-foliation of codimension q, and a leafwise smooth embedding $\iota_{\mathfrak{M}} \colon \mathfrak{M} \to M$ so that the image is a minimal set for $\mathcal{F}$?

Observe that if such an embedding $\iota_{\mathfrak{M}} \colon \mathfrak{M} \to M$ exists, then every leaf of $\mathcal{F}_{\mathfrak{M}}$ is realized by a leaf of $\mathcal{F}$ in the same quasi-isometry class of leafwise metrics. Thus, a solution to Problem 1.4 implies a solution to the question posed by Cass in [4]. This problem can also be considered as a generalization of the problem posed by McDuff in [20].

2. Existence results

There are a wide variety of constructions of classes of minimal matchbox manifolds, and a vast literature on the study of these special classes. For example, the tiling space Ω of a tiling of $\mathbb{R}^n$ is defined as the closure of the space of tilings obtained via the translation action of $\mathbb{R}^n$, in a suitable metric topology. The assumption that the tiling is repetitive, aperiodic, and has finite local complexity implies that Ω is locally homeomorphic to a disk in $\mathbb{R}^n$ times a Cantor set [22], and thus is a matchbox manifold. The *Pisot Conjecture* for tilings essentially asks when a particular class of tilings embeds into a generalized Denjoy C^1-foliation.

Weak solenoids were introduced by McCord in [19] and Schori in [23], which generalize the classical case of Vietoris solenoids, which fiber over $B = \mathbb{S}^1$. All weak solenoids are matchbox manifolds with leaves of dimension n. Their transverse dynamics are always equicontinuous, and for a base manifold B of dimension $n \geq 2$, there are many subtleties.

The Williams solenoids introduced in [25], which are expanding attractors for Axiom A dynamical systems, are defined as the inverse limit of an expanding map on a branched manifold of dimension n. The leaves of the expanding foliation defines a matchbox manifold structure for these.

The Ghys-Kenyon construction in [13, 2] yields the *graph matchbox manifolds*, which have many remarkable properties as a class of examples [17]. Lozano-Rojo and Lukina show in [18] that each *generalized Bernoulli shift* yields a graph matchbox manifold with leaves of dimension 2.

Finally, Chapter 11 of the text by Candel and Conlon [3] contains many constructions of foliated spaces, many of which have totally disconnected transverse models, so are matchbox manifolds.

3. Non-embedding results

There are two types of non-embedding results for matchbox manifolds. Note that an embedding of $\mathfrak{M}$ as a minimal set for a foliation of a compact manifold M is a fortiori an embedding of $\mathfrak{M}$ into M. Clark and Fokkink prove in [5] the following:

Theorem 3.1. *Suppose that $\mathfrak{M}$ is homeomorphic to a weak solenoid with leaves of dimension 1, and the Čech cohomology of $\mathfrak{M}$ is not finite dimensional, then $\mathfrak{M}$ cannot be embedded in a compact manifold M of dimension $n+1$. In particular, such $\mathfrak{M}$ cannot be homeomorphic to a minimal set in a codimension-one foliation.*

For higher codimension, obstructions to embedding a continuum such as $\mathfrak{M}$ into a compact manifold M are more delicate, and do not hold in such generality as above; see the discussion in [6]. The known obstructions to a solution to Problem 1.4 in higher codimensions use properties of the dynamics of $\mathfrak{M}$.

The work [1] by Attie and Hurder introduces the notion of the *leaf entropy* for a leaf of a C^0-foliation, whose definition extends naturally to the leaves of a foliated space. The work [16] by Hurder and Lukina use the methods of Lukina in [17] to construct examples of graph matchbox manifolds whose leaves have infinite leaf entropy, which yields:

Theorem 3.2. *There exists graph matchbox manifolds $\mathfrak{M}$ which cannot be embedded as a minimal set for any C^1-foliation of a compact manifold.*

If $\mathfrak{M}$ is a minimal matchbox manifold which embeds as a minimal set of a C^1-foliation of a compact manifold, then there exists a metric on the transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ for $\mathcal{F}_{\mathfrak{M}}$ such that the holonomy of $\mathcal{F}_{\mathfrak{M}}$ is Lipshitz. In the work [16] we show:

Theorem 3.3. *There exists a minimal matchbox manifold $\mathfrak{M}$ for which there does not exist a metric on the transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ such that the holonomy of $\mathcal{F}_{\mathfrak{M}}$ is Lipshitz. Thus, each such example cannot be embedded as a minimal set for any C^1-foliation.*

Given a finitely-generated, torsion-free group Γ, and a minimal action by homeomorphisms $\varphi\colon \Gamma\mathfrak{X} \to \mathfrak{X}$ on a Cantor set $\mathfrak{X}$, then the suspension construction yields a minimal matchbox manifold $\mathfrak{M}$ whose transverse holonomy groupoid is given by the action φ. The results in [5] show that such matchbox manifolds always admit an embedding into a C^0-foliation with codimension 2.

Problem 3.4. Let $\varphi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be a minimal action on a Cantor set $\mathfrak{X}$. Find invariants of the action which are obstructions to embedding a matchbox manifold $\mathfrak{M}$ obtained from a suspension of φ, into a C^r-foliation of a compact manifold, for $r \geq 1$.

4. Embedding results

The problem of embedding a 1-dimensional matchbox manifold $\mathfrak{M}$ in a C^r-flow has two forms. If the dynamics of the flow restricted to $\mathfrak{M}$ are equicontinuous, or equivalently the flow is almost periodic on $\mathfrak{M}$, then $\mathfrak{M}$ must a Vietoris solenoid [24]. The realization of solenoids as minimal sets for flows has an extensive literature (see [7] for a discussion and references).

The other possibility in the 1-dimensional case, is that the dynamics of $\mathfrak{M}$ are transversally expansive. In this case, $\mathfrak{M}$ has a presentation as an inverse limit of branched 1-manifolds, which can be used to give effective criteria for embedding into punctured surfaces, for example as considered in [14].

The case where $\mathfrak{M}$ is minimal with leaf dimension $n \geq 2$ is much more difficult, and few results are known except when such an embedding is part of the data in the construction, such as for the action of a rank-one lattice in a Lie group, acting on its boundary when it is totally disconnected.

In the work [7], the authors' studied the embedding problem for the base $\mathbb{T}^n$, and developed criteria for when $\mathfrak{M}$ has a smooth embedding.

Theorem 4.1. *Let $\mathfrak{M}$ be a weak solenoid with base manifold $\mathbb{T}^n$. Then there exists a C^0-foliation $\mathcal{F}$ of codimension-$2n$ on a compact manifold with minimal set $\mathfrak{M}$. If a mild restriction of the model of $\mathfrak{M}$ by compact tori is assumed, then it can be realized by a C^1-foliation $\mathcal{F}$ of codimension-$2n$.*

5. Classification

The study of the exceptional minimal sets for foliations also includes the problem of classification of minimal matchbox manifolds, up to homeomorphism and orbit equivalence. This is work in progress [8, 9, 10, 11, 15].

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On codimension two contact embeddings

Naohiko KASUYA

1. Introduction and the statements of the results

We study codimension two contact embeddings in the odd dimensional Euclidean space. Let (M^{2n-1}, ξ) be a closed contact manifold and (N^{2m-1}, η) be a co-oriented contact manifold. An embedding $f : M^{2n-1} \to N^{2m-1}$ is said to be a contact embedding if $f_*(TM^{2n-1}) \cap \eta|_{f(M^{2n-1})} = f_*\xi$. Note that ξ must be co-orientable since $f^*\beta$ is a global defining 1-form of ξ, where β is a global defining 1-form of η. For given (M^{2n-1}, ξ), we would like to know whether there exists a contact embedding of (M^{2n-1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$, where η_0 is the standard contact structure on $\mathbb{R}^{2n+1}$. It is equivalent to the existence of contact embeddings of (M^{2n-1}, ξ) in the $(2n+1)$-sphere with the standard contact structure. We see that the first Chern class is an obstruction for the existence of such an embedding.

Theorem 1.1. *If a closed contact manifold (M^{2n-1}, ξ) is a contact submanifold of a co-oriented contact manifold (N^{2n+1}, η) satisfying the condition $H^2(N^{2n+1}; \mathbb{Z}) = 0$, then the first Chern class $c_1(\xi)$ vanishes.*

In particular, there are infinitely many contact 3-manifolds which cannot be embedded in $(\mathbb{R}^5, \eta_0)$ as contact submanifolds. We note that any 3-manifold can be embedded in $\mathbb{R}^5$ by Wall's theorem[16]. We also note that A.Mori[10] constructed a contact immersion of any closed co-orientable contact 3-manifold in $(\mathbb{R}^5, \eta_0)$ and D.Martinez[9] proved that any closed co-orientable contact $(2n+1)$-manifold can be embedded in $(\mathbb{R}^{4n+3}, \eta_0)$ as a contact submanifold. For the existence of contact embeddings of contact 3-manifolds in $(\mathbb{R}^5, \eta_0)$, there are several known examples. Some of them are links of isolated complex surface singularities in $\mathbb{C}^3$. The canonical contact structure on a link is given by the complex tangency, and it is a contact submanifold of (S^5, η_{std}), where η_{std} is the standard contact structure on S^5. Though it is difficult to determine the structure on a link in general, it is done in the cases of the quasi-homogeneous singularities[13] and the cusp singularities[4],[11],[13]. In these cases, the link is the quotient of a cocompact lattice of a Lie group G and the contact structure is invariant under the action of G. Another example is given by A.Mori[12] and Niederkrüger-Presas[14]. They independently constructed a contact embedding of the overtwisted contact structure on S^3 associated to the negative Hopf band in (S^5, η_{std}). In spite of these examples, we do not know whether every contact 3-manifold with $c_1(\xi) = 0$ can be embedded in $(\mathbb{R}^5, \eta_0)$ as a contact submanifold. By Gromov's h-principle, however, we can show the following result.

Theorem 1.2. *If $c_1(\xi) = 0$, we can embed (M^3, ξ) in $\mathbb{R}^5$ as a contact submanifold for some contact structure on $\mathbb{R}^5$.*

2. Preliminary

2.1. The Chern classes of a co-oriented contact structure

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a co-oriented contact structure. Since the 2-form $d\alpha$ induces a symplectic structure on ξ, $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle over M^{2n-1}. Since the conformal class of the symplectic bundle structure does not depend on the choice of α, we define the Chern classes of ξ as the Chern classes of this symplectic vector bundle.

2.2. The conformal symplectic normal bundle of a contact submanifold

Let $(M, \eta_M) \subset (N, \eta = \ker \beta)$ be a contact submanifold. The vector bundle η splits along M into the Whitney sum of the two subbundles

$$\eta|_M = \eta_M \oplus (\eta_M)^\perp,$$

where η_M is the contact plane bundle on M given by $\eta_M = TM \cap \eta|_M$ and $(\eta_M)^\perp$ is the symplectic orthogonal of η_M in $\eta|_M$ with respect to the form $d\beta$. We can identify $(\eta_M)^\perp$ with the normal bundle νM. Moreover, $d\beta$ induces a conformal symplectic structure on $(\eta_M)^\perp$. We call $(\eta_M)^\perp$ the conformal symplectic normal bundle of M in N.

2.3. The Euler class of the normal bundle of an embedding

Let K^k be a closed orientable k-manifold, L^l an orientable l-manifold and $f\colon K^k \to L^l$ an embedding.

Theorem 2.1. *If $H^{l-k}(L^l; \mathbb{Z}) = 0$, the Euler class of the normal bundle of f vanishes.*

Proof. By Theorem 11.3 of [7], the Euler class of the normal bundle of f is the image of the dual cohomology class of K^k by the homomorphism $f^* : H^{l-k}(L^l; \mathbb{Z}) \to H^{l-k}(K^k; \mathbb{Z})$. Thus, if $H^{l-k}(L^l; \mathbb{Z}) = 0$, it vanishes. $\square$

In particular, when $l = k + 2$, the normal bundle is a 2-dimensional trivial vector bundle.

3. Proof of Theorem 1.1

Proof. Let $f : M^{2n-1} \to N^{2n+1}$ be an embedding such that

$$f_*(TM^{2n-1}) \cap \eta|_{f(M^{2n-1})} = f_*\xi.$$

Since $H^2(N^{2n+1}; \mathbb{Z}) = 0$ and the normal bundle of f is 2-dimensional, it is topologically trivial by Theorem 2.1. Since the conformal symplectic structure on 2-dimensional trivial vector bundle is unique, the normal bundle of $f(M^{2n-1})$ is also trivial as a conformal symplectic vector bundle. That is, the vector bundle η splits along $f(M^{2n-1})$ such that

$$\eta|_{f(M^{2n-1})} = \eta_{f(M^{2n-1})} \oplus (\eta_{f(M^{2n-1})})^{\perp},$$

where $\eta_{f(M^{2n-1})} = f_*\xi$ and $(\eta_{f(M^{2n-1})})^{\perp}$ is a trivial symplectic bundle. By the naturality of the first Chern class and the condition $H^2(N^{2n+1}; \mathbb{Z}) = 0$, it follows that $c_1(\eta|_{f(M^{2n-1})}) = f^*c_1(\eta) = 0$. On the other hand, taking the Whitney sum with a trivial symplectic bundle does not change the first Chern class. Thus, $c_1(\eta|_{f(M^{2n-1})}) = c_1(\xi)$ holds. It follws that $c_1(\xi) = 0$. $\square$

4. Proof of Theorem 1.2

4.1. h-principle

We review Gromov's h-principle and prove Propositon 4.4 as a preliminary for the proof of Theorem 1.2.

DEFINITION 4.1. Let N^{2n+1} be an oriented manifold. An almost contact structure on N^{2n+1} is a pair (β_1, β_2) consisting of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$.

REMARK 4.2. There is another definition. We can define an almost contact structure on N^{2n+1} as a reduction of the structure group of TN^{2n+1} from $\mathrm{SO}(2n+1)$ to $\mathrm{U}(n)$. Since a pair (β_1, β_2) satisfying $\beta_1 \wedge \beta_2^n \neq 0$ can be seen as the cooriented hyperplane field $\ker \beta_1$ with an almost complex structure compatible with the symplectic structure $\beta_2|_{\ker \beta_1}$, the two definitions are equivalent up to homotopy.

Theorem 4.3 (Gromov[2], Eliashberg-Mishachev[1]). *Suppose N^{2n+1} is an open manifold. If there exists an almost contact structure over N^{2n+1}, then there exists a contact structure on N^{2n+1} in the same homotopy class* *of almost contact structures. Moreover if the almost contact structure is already a contact structure on a neighborhood of a compact submanifold $M^m \subset N^{2n+1}$ with $m < 2n$, then we can get a contact structure on N^{2n+1} which coincides with the original one on a small neighborhood of M^m.*

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a closed cooriented contact manifold and M^{2n-1} be embedded in $\mathbb{R}^{2n+1}$. By Theorem 2.1, there exists an embedding

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}.$$

The form $\alpha + r^2 d\theta$ induces a contact form β on $U = F(M^{2n-1} \times D^2)$. By Theorem 4.3, in order to extend given contact structure, it is enough to extend it as an almost contact structure. Almost contact structures on N^{2n+1} correspond to sections of the principal $SO(2n+1)/U(n)$ bundle associated with the tangent bundle TN^{2n+1}. In particular, almost contact structures on $\mathbb{R}^{2n+1}$ correspond to smooth maps

$$\mathbb{R}^{2n+1} \to SO(2n+1)/U(n).$$

Thus we get the following proposition.

Proposition 4.4. *We can embed (M^{2n-1}, ξ) in $\mathbb{R}^{2n+1}$ as a contact submanifold for some contact structure, if and only if there exists an embedding $F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}$ such that the map $g : M^{2n-1} \to SO(2n+1)/U(n)$ induced by the underlying almost contact structure of $(M^{2n-1} \times D^2, \alpha + r^2 d\theta)$ is contractible.*

Proof. The underlying almost contact structure of $(U, \beta) \subset \mathbb{R}^{2n+1}$ is identified with the map $\tilde{g} : U \to SO(2n+1)/U(n)$ whose restriction to M^{2n-1} is g. We can take an extension of $\tilde{g}$ over $\mathbb{R}^{2n+1}$ if and only if g is contractible. □

4.2. Proof of Theorem 1.2

Proof. There exists an embedding $f\colon M^3 \to \mathbb{R}^5$ [16], and the normal bundle of f is trivial. Thus we can take an embedding $F\colon M^3 \times D^2 \to \mathbb{R}^5$. By Proposition 4.4, it is enough to prove that if $c_1(\xi) = 0$, then there exists an embedding F such that the map $g\colon M^3 \to SO(5)/U(2)$ induced by F is contractible. Let us take a triangulation of M^3 and $M^{(l)}$ be its l dimensional skeleton, i.e.,

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3.$$

The condition $c_1(\xi) = 0$ is equivalent to that ξ is a trivial plane bundle over M^3. Hence a trivialization τ of ξ and the Reeb vector field R of α give a trivialization of TM^3. This trivialization of TM^3 and a trivialization ν of the normal bundle νM^3 form a map

$$h\colon M^3 \to SO(5).$$

In other words, h is a trivialization of $T\mathbb{R}^5 \mid_{M^3}$ consisting of R, τ and ν. Composing with the projection $\pi\colon SO(5) \to SO(5)/U(2)$, it induces the map $g = \pi \circ h\colon M^3 \to SO(5)/U(2)$. Thus h is a lift of g. Now we consider whether h is null-homotopic over $M^{(1)}$. In other words, we consider the difference between the spin structures on $T\mathbb{R}^5 \mid_{M^3}$ induced by h and the constant map I_5. Then the obstruction is the Wu invariant $c(f) \in \Gamma_2(M^3)$, where $\Gamma_2(M^3) = \{C \in H^2(M^3;\mathbb{Z}) \mid 2C = 0\}$. The following explanation of the Wu invariant is due to [15]. The Wu invariant is defined for an immersion of the parallelized 3-manifold with trivial normal bundle. A normal trivialization ν of f and the tangent trivialization define a map $\pi_1(M^3) \to \pi_1(SO(5))$, namely an element $\tilde{c}_f$ in $H^1(M^3;\mathbb{Z}_2)$. If we change ν by an element $z \in [M^3, SO(2)] = H^1(M^3;\mathbb{Z})$, then the class $\tilde{c}_f$ changes by $\rho(z)$, where ρ is the mod 2 reduction map $H^1(M^3;\mathbb{Z}) \to H^1(M^3;\mathbb{Z}_2)$. Hence the coset of $\tilde{c}_f$ in $H^1(M^3;\mathbb{Z}_2)/\rho(H^1(M^3;\mathbb{Z}))$ does not depend on ν. The cokernel of ρ is identified with $\Gamma_2(M^3)$ by the canonical map induce by the Bockstein homomorphism. Under this identification, the coset of $\tilde{c}_f$ corresponds to the Wu invariant $c(f) \in \Gamma_2(M^3)$. Now we fix the trivialization of TM^3 formed by τ and R. By Theorem 3.8 of [15], there exists an embedding $f : M^3 \to \mathbb{R}^5$ such that $c(f) = 0$. Moreover, there exists a normal trivialization ν of f such that $\tilde{c}_f = 0 \in H^1(M^3;\mathbb{Z}_2)$. With the embedding f and the normal trivialization ν, the map h is null-homotopic over $M^{(1)}$. Since $\pi_2(SO(5)) = 0$, it is also null-homotopic over $M^{(2)}$ and so is the map $g = \pi \circ h : M^3 \to SO(5)/U(2)$. Since $\pi_3(SO(5)/U(2)) = 0$, g is contractible. This completes the proof of Theorem 1.2. $\square$

5. Examples of codimension 2 contact submanifolds

5.1. Singularity links

Let X be a complex algebraic surface in $\mathbb{C}^3$ with an isolated singularity at the origin 0. The intersection L^3 of X and a sufficiently small sphere S^5_ε is called the link of $(X, 0)$. The canonical contact structure ξ on L^3 is given by $\xi = TL^3 \cap JTL^3$, where J is the standard complex structure on $\mathbb{C}^3$. It is obviously a contact submanifold of (S^5, η_{std}). In the case of quasi-homogeneous singularity and cusp singularity, Neumann[13] showed that there is a one-one correspondence between geometric structures on L^3 and complex analytic structures on $(X, 0)$.

Example 5.1 (Brieskorn singularity). Let $X = \{x^p + y^q + z^r = 0\}$. The link L^3 is a quotient of the Lie group $G = SU(2), Nil^3$ or $\widetilde{SL}(2;\mathbb{R})$, according as the rational number $p^{-1}+q^{-1}+r^{-1}-1$ is positive, zero or negative [8]. Since the canonical contact structure ξ on L^3 is invariant under the action of G, ξ is determined[13].

Example 5.2 (Cusp singularity). Let $X = \{x^p + y^q + z^r + xyz = 0\}$ with $p^{-1}+q^{-1}+r^{-1} < 1$. This singularity is analytically equivalent to a Hilbert modular cusp associated with a quadratic field over $\mathbb{Q}$ [3],[5],[6]. Thus the link L^3 is a hyperbolic mapping torus and has a geometry of the Lie group $G = Sol^3$. ξ is the positive contact structure associated with the Anosov flow on L^3 [4],[11],[13].

5.2. Other examples

Let $(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3)$ be the polar coordinates on $S^5 \subset \mathbb{C}^3$, where

$$(z_1, z_2, z_3) = (r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2}, r_3 e^{2\pi i\theta_3}) \in \mathbb{C}^3,\ S^5 = \left\{r_1^2 + r_2^2 + r_3^2 = 1\right\}.$$

The standard contact form on S^5 is $\alpha_0 = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3$. Let $\phi\colon S^5 \to \mathbb{R}^3$ be the projection, where $\phi(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = (r_1^2, r_2^2, r_3^2)$. Then the image $\phi(S^5) = \{x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ is a regular triangle in $\mathbb{R}^3$. It is called the moment polytope $\triangle$. Note that π is a T^3-fibration over Int$\triangle$ and is a T^2-fibration over $\partial\triangle$ except on the three vertices. Choosing a curve c on $\triangle$ and a section over c appropriately, one can get an embedding of a 3-manifold in S^5.

Example 5.3 (Mori's example). Let (S^3, η_{neg}) be the negative overtwisted contact structure associated with the negative Hopf link. Using the moment polytope, A.Mori constructed a deformation of embedded standard contact 3-sphere to (S^3, η_{neg}) in (S^5, ξ_{std}), via the Reeb foliation on S^3 foliated by immersed Legendrian submanifolds of S^5 [12]. Slightly changing this example, we can also see that tight contact structures on the 3-torus can be embedded in (S^5, η_{std}) as contact submanifolds.

Example 5.4 (Furukawa's example). In a similar way, R.Furukawa constructed the contact embeddings of universally tight contact structures on some T^2 bundles over S^1. His examples cover the link of cusp singularities and Brieskorn Nil singularities.

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Towards the classification theorem for one-dimensional pseudogroups

Victor KLEPTSYN

My talk will be devoted to a joint project with B. Deroin, D. Filimonov and A. Navas, that has as its ultimate goal to understand and classify real one-dimensional finitely generated pseudo-group actions. This naturally includes group actions on the circle and real codimension one foliations of compact manifolds.

Our study was motivated by following questions, going back to 1980's, that were asked by D. Sullivan, E. Ghys, and G. Hector; we're stating them both in the group actions and foliations setting:

Question 0.1 (Ghys, Sullivan). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle minimally. Is this action necessarily ergodic with respect to the Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold, which is minimal. Is it necessarily ergodic with respect to the Lebesgue measure?

Question 0.2 (Ghys, Sullivan). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Is K necessarily of zero Lebesgue measure?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold, having an exceptional minimal set $\mathcal{K}$. Is it necessarily of zero Lebesgue measure?

Question 0.3 (Hector). Let G be a finitely generated group of (C^2-)smooth circle diffeomorphisms, acting on the circle with a Cantor miminal set K. Does the action of G on the connected components of $S^1 \setminus K$ necessarily have but a finite number of orbits?

Let $\mathcal{F}$ be a transversely (C^2-)smooth foliation of a compact manifold M, having an exceptional minimal set $\mathcal{K}$. Does the complement $M \setminus \mathcal{K}$ necessarily have at most finite number of connected components?

Our results partially answer these questions; what is even more important, some general paradigm seems to turn up. Namely, it *seems* that (up

Talk on a joint project with Bertrand Deroin, Dmitry Filimonov, Andrés Navas

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to some modifications[1]) the following general paradigm takes place:

Paradigm. *For a finitely-generated pseudogroup of transformations of a real line, the following dichotomy holds:*

- *Either it has local flows in its local closure,*
- *Or it admits a Markov partition (of the minimal set).*

This is closely related to what was done and suggested as a generic behavior for the case of an exceptional minimal set by Cantwell and Conlon in [1, 2]. Though, for the case of an exceptional minimal set we expect that Markov partition always exists, as the sense in which we understand the Markov partition is slightly weaker than the one of Cantwell–Conlon (and this covers also the type of behavior mentioned in [2, §7] that did not fit in their definition).

A road towards this paradigm lies through the local discreteness, which makes the dichotomy between the two cases above. Namely, *if* a (pseudo)group is not locally discrete and its action does not preserve a measure, the arguments of Scherbakov-Nakai-Loray-Rebelo ([5, 12, 11, 14]) imply that it contains local flows in its local closure. Roughly speaking, due to the absence of a preserved measure there is a map with a hyperbolic fixed point; expanding the sequence of maps (C^1-)convergent to the identity with help of this map, one finds local flows.

The case to consider is then the one of locally discrete groups, and here the *non-expandable points* come into play. Namely, there Sullivan's exponential expansion strategy allows to expand arbitrarily small neighborhoods of points of the minimal set with a uniform control on the distortion, provided that for any point of the minimal set there is a map that expands linearly at this point. This allow to obtain, under this assumption, positive answers to the Questions 0.1–0.3, and it is quite likely to provide a Markov partition (with the same mechanism as the one used in [6]: under the expansion the maps stay uniformly close to the identity, and thus there should be a repetition between the expanded images).

The non-expandable points that we mentioned earlier are the obstacles to the application of Sullivan's expansion strategy:

Definition 0.4. A point x of a minimal set is *non-expandable* for the action of a (pseudo)group $\mathcal{G}$, if for any $g \in \mathcal{G}$ (defined in x) one has $|g'(x)| \leq 1$.

Note, that their presence in a minimal set immediately implies the local discreteness of the group: otherwise, local vector flows would allow minimal-

[1] For instance, for the local flows one shouldn't consider glueing or too large domains of definition: otherwise, for the standard Thomson group action, generated by the doubling map, one would both have a Markov partition and a flow.

ity with a uniform control on the derivative, and bringing a non-expandable point close to a hyperbolic repelling one would imply a contradiction. Thus, in view of the paradigm above, we should try showing that in this case there exists a Markov partition.

Though the non-expandable points are an obstacle to the "fast" expansion procedure, an additional assumption allows to handle this difficulty:

Definition 0.5 ([3]). A minimal action of a finitely generated (pseudo)group $\mathcal{G}$ has property $(\star)$, if any non-expandable fixed point is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.

An action of a finitely generated (pseudo)group $\mathcal{G}$ with a Cantor minimal set K has property $(\Lambda\star)$, if any non-expandable fixed point $x \in K$ is right- and left- isolated fixed point for some maps $g_+, g_- \in \mathcal{G}$.

When this assumption is satisfied, one can modify the Sullivan's exponential expansion strategy by a "slow" expansion near the non-expandable points, by iterating the $g_\pm$ (or their inverses) till the point leaves the neighborhood of a non-expandable point. Such a modification have allowed us in [3] to obtain *under this assumption* the positive answers to Questions 0.1–0.3.

This also allows to describe the structure of a (pseudo)group: it turns out (see [6, 7]) that if this assumption is satisfied, and there actually is at least one non-expandable point, then the dynamics indeed admits some kind of Markov partition. Also, the (pseudo)group is in a sense Thomson-like: for the piecewise-nonstrictly expanding map R, associated to this partition, the maps from the (pseudo)group locally are composition of its iterations, the branches of its inverse, and an intermediate map chosen from a finite set.

What is left for establishing the paradigm and for answering Questions 0.1–0.3 is thus to prove that the property $(\star)$ (or $(\Lambda\star)$) always holds. And two our recent works make an advancement towards it:

Finally, for some cases, the property $(\star)$ (or $(\Lambda\star)$) can be shown to hold:

Theorem 0.6 ([4]). *Let G be a (virtually) free finitely generated subgroup of the group of analytic circle diffeomorphisms, such that the action G does not have finite orbits. Then, G satisfies property $(\star)$ or $(\Lambda\star)$ (depending on whether the action is minimal or possesses an exceptional minimal set).*

Remark 0.7. Recall, that due to a result by Ghys [10], a finitely generated group of analytic circle diffeomorphisms, acting with a Cantor minimal set, is *always* virtually free. Hence, Theorem 0.6 implies positive answers for Questions 0.2 and 0.3 for the case of an analytic group action on the circle.

Theorem 0.8 ([8]). *Let G be a finitely generated subgroup of the group of analytic circle diffeomorphisms, acting minimally, that has one end, is finitely presented, and in which one cannot find elements of arbitrarily large finite order. Then, G satisfies property ($\star$).*

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Foliations of $\mathbb{S}^3$ by cyclides

Rémi LANGEVIN

1. Introduction

Throughout last 2-3 decades, there was great interest in extrinsic geometry of foliated Riemannian manifolds (see [As], [B-L-R] and [Ze]).

One approach is to build examples of foliations with reasonably simple singularities with leaves admitting some very restrictive geometric condition. After considering foliations of $\mathbb{S}^3$ by totally geodesic of totally umbilical leaves with isolated singularities, totally geodesic foliations of $\mathbb{H}^2$ or $\mathbb{H}^3$, [La-Si] provide families of foliations of $\mathbb{S}^3$ by Dupin cyclides with only one smooth curve of singularities. Quadrics and other families of cyclides like Darboux cyclides provide other examples. In all cases the results are obtained considering an auxiliary space associated to the geometry imposed to our leaves, the space of spheres, of lines, of circles for the examples mentioned above.

Another motivation for our construction is the use of cyclides in computer graphics, see for example [Po-Li-Sko].

The results mentioned in this conference come from a joint work with Jean-Claude Sifre.

2. The spaces of lines, spheres, and circles

2.1. The space of lines

The set of affine lines of $\mathbb{R}^3$ is a vector bundle of base $\mathbb{P}^2$ and fiber $\mathbb{R}^2$ of dimension 6. The projective space $\mathbb{RP}^3$ completes $\mathbb{R}^3$. The set of projective lines of $\mathbb{RP}^3$ is isomorphic to the Grassmann manifold $G(4,2)$ of planes of $\mathbb{R}^4$.

Let us first show how, using Plücker coordinates, $G(4,2)$ can be seen as a quadric $\Pi \subset \mathbb{RP}^5$.

The condition that a vector U of $\bigwedge^2(\mathbb{R}^4)$ is pure, that is of the form $u \wedge v$, $u \in \mathbb{R}^4$, $v \in \mathbb{R}^4$ writes $U \wedge U = 0$; this provides a quadratic form, called the *Plücker form* which defines the *Plücker cone.*

The incidence relation of two lines corresponding to the 2-vectors U and V obtained checking that the corresponding 2-planes of $\mathbb{R}^4$ generate a subspace of dimension at most 3; it writes $U \wedge V = 0$.

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A pencil of lines ℓ is the projective image of a totally isotropic plane of $\bigwedge^2(\mathbb{R}^4)$ for the Plücker form; it is of index $(3,3)$. We call the corresponding projective line a *projective light-ray*.

Geometrically, a pencil of lines of $\mathbb{P}^3$ correspond to a *contact condition*, that is a pair (m,P), $m \in P \subset \mathbb{P}^3$.

2.2. The space of spheres

It will be convenient for us to realize both our ambient space $\mathbb{S}^3$ and the set of oriented spheres as subsets of the Lorentz space $\mathbb{R}^5_1$, that is $\mathbb{R}^5$ endowed with the Lorentz quadratic form $\mathcal{L}(x) = \mathcal{L}(x_0,x_1,x_2,x_3,x_4) = -x_0^2 + \sum_{i=1}^4 x_i^2$.

The *light-cone* $\mathcal{L}i$ is the set $\mathcal{L}(x) = 0$. Its generatrices are called *light-rays*. We also call affine lines parallel to a generatrix of the light-cone light-rays.

The light-cone separates vectors of $\mathbb{R}^5 \setminus \mathcal{L}i$ in two types: space-like vectors, such that $\mathcal{L}(v) > 0$ and time-like vectors, such that $\mathcal{L}(v) < 0$. A plane will be called space-like if it contains only space-like (non-zero) vectors. It is called time-like if it contains non zero time-like vectors (then it contains vectors of the three types). It is called light-like is it contains non-zero light-like vectors but no time-like vector.

The space of oriented 2-dimensional spheres in $\mathbb{S}^3$ may be parameterized by the *de Sitter quadric* $\Lambda^4 \subset \mathbb{R}^5_1$ defined as the set of points $\sigma = (x_0,x_1,x_2,x_3,x_4)$ such that $\mathcal{L}(\sigma) = 1$, in the following way. The hyperplane $\sigma^\perp$ orthogonal to σ (for the Lorentz quadratic form $\mathcal{L}$) cuts the affine hyperplane $H_0 = \{x_0 = 1\}$ along a 3-dimensional oriented affine hyperplane, which cuts the unit sphere $\mathbb{S}^3 \subset H_0$ along a 2-dimensional sphere Σ. Let us orient the sphere Σ as boundary of the ball $B_\sigma = \mathbb{S}^3 \cap \{\mathcal{L}(x,\sigma) \geq 0\}$.

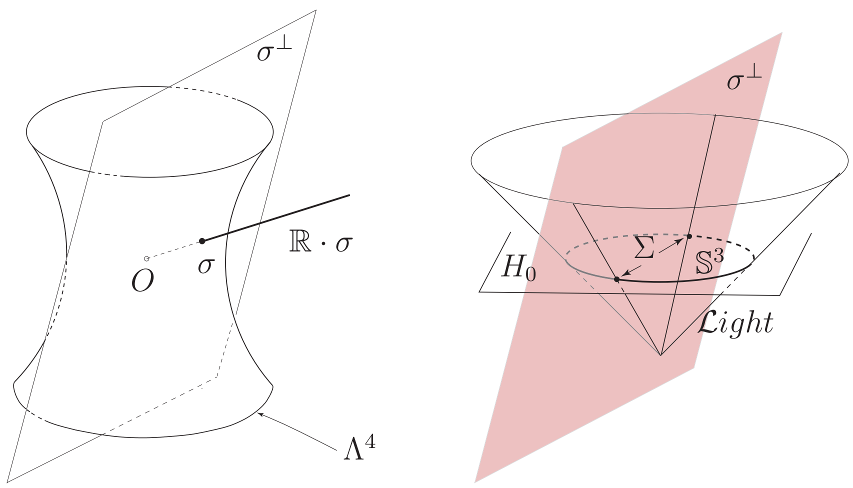


Figure 1: The correspondence between points of Λ^4 and spheres in $\mathbb{S}^3$.

This correspondence between points σ of Λ^4 and oriented spheres $\Sigma \subset \mathbb{S}^3 \subset H_0$ is bijective.

Geometric properties of spheres have a counterpart in Λ^4. For example, two oriented spheres Σ and Σ' in $\mathbb{S}^3$ are positively (i.e. respecting the orientation) tangent if and only if the corresponding points σ and σ' in Λ^4 verify: $\mathcal{L}(\sigma, \sigma') = 1$. In that case, the points σ and σ' are joined by a segment of light-ray contained in Λ^4. In fact the oriented spheres tangent to Σ correspond to the points of the 3-dimensional cone $T_\sigma\Lambda^4 \cap \Lambda^4$ which is a union of (affine) light-rays.

The tangent space $T_\sigma\Lambda^4$ is parallel to the hyperplane $(\mathbb{R} \cdot \sigma)^\perp$. It is therefore of index $(3,1)$. This means it contains space-like, time like and light-like vectors.

A *contact element* (or simply a *contact*) in $\mathbb{S}^3$ is a pair (m, h), where $m \in \mathbb{S}^3$ and h is a vector plane $h \subset T_m\mathbb{S}^3$. The set of contact conditions is of dimension 5. To each contact element (m, h) corresponds a pencil of spheres tangent to h at m. Orienting $h \subset T_m\mathbb{S}^3$ allows to orient the spheres of the pencil, and distinguishes one of the light-rays of Λ^4 corresponding to spheres of the pencil.

Reciprocally, each light-ray contained in Λ^4 defines a *contact element* in $\mathbb{S}^3$. Precisely, the intersection of the direction of the light-ray ℓ with H_0 is a point m_ℓ of $\mathbb{S}^3$ and the spheres Σ associated to the points $\sigma \in \ell$ are the spheres having a common oriented contact $h \subset m_\ell$ at the point m_ℓ. We can now observe that the quadric Λ^4 is ruled by a 5-dimensional family of (affine) light-rays.

Pencils of spheres can be of the types: pencils of tangent spheres, pencils of spheres with a base circle and pencils of spheres with limit points. The corresponding points of Λ^4 are respectively two parallel light-rays, the intersection of Λ^4 with a space-like vectorial plane and the intersection of Λ^4 with a time-like plane.

2.3. The space of circles

A circle $\Gamma \subset \mathbb{S}^3$ is the axis of a pencil of spheres. This pencil corresponds to the points of intersection of the quadric $\Lambda^4 \subset \mathbb{R}^5_1$ and a space-like vectorial plane. Therefore the space of circles $\mathcal{C}$ can be seen as a subset of the set of lines of the cone $\mathcal{P} \subset \bigwedge^2(\mathbb{R}^5)$ given by the Plücker relations defining pure 2-vectors.

The wedge product defines a bilinear form $\mathcal{Q}_\mathcal{C} : \bigwedge^2(\mathbb{R}^5) \times \bigwedge^2(\mathbb{R}^5) \to \bigwedge^4(\mathbb{R}^5)$. The condition $\mathcal{Q}_\mathcal{C}(U, U) = 0$ gives 5 quadratic equations. They are not independent. One can prove that the equality $\mathcal{Q}_\mathcal{C}(U, U) = 0$ defines a 7-dimensional cone $\mathcal{P}$. We will soon see that the set of lines corresponding to circles is open in $\mathbb{P}(\mathcal{P})$. We could have checked directly that the set of oriented circles $\mathcal{C}$ is a 6-dimensional space.

Let now U_{Γ_1} and U_{Γ_2} be two pure vectors corresponding to the two circles Γ_1 and Γ_2. The condition $0 = \mathcal{Q}_{\mathcal{C}}(U_{\Gamma_1}, U_{\Gamma_2} = U_{\Gamma_1} \wedge U_{\Gamma_2}$ is equivalent to $dim(p_{\Gamma_1} + p_{\Gamma_2}) \leq 3$, that is to say $\exists \sigma \in p_{\gamma_1} \cap p_{\gamma_2} \cap \Lambda^4$. In other terms, the two circles Γ_1 and Γ_2 belong to the same sphere Σ if and only if corresponding 2-vector U_{Γ_1} and U_{Γ_2} satisfy $U_{\gamma_1} \wedge U_{Gamma_2} = 0$.

The condition is satisfied in particular when the two circles intersect at two distinct points or are tangent.

2.3.1. Plücker and Lorentz quadratic forms

It is natural to consider on $\bigwedge^2(\mathbb{R}^5)$ a quadratic form coming from the Lorentz quadratic form $\mathcal{L}$ on $\mathbb{R}^5$ defined by $\mathcal{L}(x) = -x_0^2 + x_1^2 + \cdots + x_4^2$. Consider on $\mathbb{R}^5$ the basis $e_0, e_1, \cdots e_4$; the 10 2-vectors $e_i \wedge e_j$, $i < j$ form a basis of $\bigwedge^2(\mathbb{R}^5)$. the quadratic form $\mathbb{L}$ on $\bigwedge^2(\mathbb{R}^5)$ is defined by $\mathbb{L}(e_1 \wedge e_j) = +1$ if $i \geq 1$, $\mathbb{L}(e_1 \wedge e_j) = -1$ if $i = 0$. The signature of $\mathbb{L}$ is therefore $(6, 4)$.

The light-cone of $\mathbb{L}$ contains the lines generated by wedge of vectors of $\mathbb{R}^5$ contained in a 2-plane tangent to the light-cone of $\mathcal{L}$.

One may visualize the set of “true” oriented circles of $\mathbb{S}^3$ as the intersection $\mathcal{C}$ of the Plücker cone of “pure” 2-vectors, defined by the equations $u \wedge u = 0$, and the quadric of equation $\mathbb{L}(x) = 1$.

Using on $\mathcal{C}$ the pseudo-metric induced from $\mathbb{L}$, we get a pseudo metric of signature $(4, 2)$. We admit that this pseudo-metric does not depend on the choice of the orthonormal basis (for $\mathcal{L}$) of $\mathbb{R}^5$. A way to visualize orthogonal directions for $\mathbb{L}$ in $T_\gamma \mathcal{C}$ is explained in [La-O'H].

3. The d'Alembert property

Cyclides are surfaces of $\mathbb{S}^3$ which are, at least in two different ways, union of one-parameter families of circles. We will here accept lines are particular circles. Many interesting examples are proposed in [Po-Li-Sko].

Definition 3.0.1. Two one-parameter families of circles $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the d'Alembert property if any two circles $\Gamma_1 \in \mathcal{C}_1$ and $\Gamma_2 \in \mathcal{C}_2$ are contained in a sphere $\Sigma_{1,2}$. We will call a cyclide, union of the circles of two families satisfying the d'Alembert property a d'Alembert cyclide.

Remark. – d'Alembert observed that ellipsoids admit two families of circles which are the intersection of the ellipsoid with planes parallel to the tangent planes at the umbilics (see [d'A] and Figure 2). Two circles, one in each family are always contained in a common sphere (this is clear, from topological reasons, when the two circles intersect).

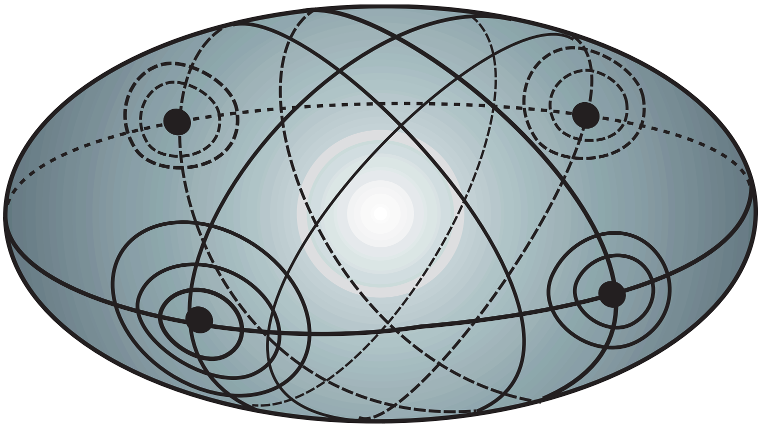


Figure 2: Two families of circles on an ellipsoid.

– Whereas a quadric can be described as the zero-set of second order polynomial in Cartesian coordinates (x_1, x_2, x_3), a large family of cyclide, called the *Darboux cyclides*, is given by the zero-set of a second order polynomial in (x_1, x_2, x_3, r^2), where $r^2 = x_1^2 + x_2^2 + x_3^2$. Thus they are quartic surfaces in Cartesian coordinates, with an equation of the form:

$$Ar^4 + 2r^2 \sum_{i=1}^{3} B_i x_i + \sum_{i,j=1}^{3} Q_{ij} x_i x_j + 2 \sum_{i=1}^{3} C_i x_i + a$$

where Q is a 3×3 matrix, B_i are a 3-dimensional vectors, and A and a are constants [Ta].

They are d'Alembert cyclides, and have been classified by Takeushi [Ta]. We hope to know soon wether all d'Alembert cyclides are Darboux or not.

Proposition 3.0.2. *The points of Λ^4 corresponding to spheres which contain a pair of circles, one in each family, of a d'Alembert cyclide, are contained in a 4 dimensional subspace of $\mathbb{R}^5_1$.*

Proof. Let us chose two circles γ_1, γ_2 of the first family, they are the axis of two pencils of spheres P_{γ_1} and P_{γ_2}. The points corresponding to the spheres of these pencils are intersection of Λ^4 with the planes p_1 and p_2. A circle τ of the second family is the axis of the pencil P_τ. The definition of a d'Alembert cyclide implies that a sphere Σ_1 of $P_{\gamma_1} \cap P_\tau$ contains γ_1 and τ and that a sphere Σ_2 of $P_{\gamma_2} \cap P_\tau$ contains γ_2 and τ. This implies that τ is the pencil generated by Σ_1 and Σ_2. The spheres of this pencil correspond to points of Λ^4 contained in the 4-dimensional subspace $p_1 \oplus p_2 \subset \mathbb{R}^5_1$. It is now enough to use the d'Alembert condition satisfied by a circle of the first family and to given circles of the second family to obtain a proof of the proposition. $\square$

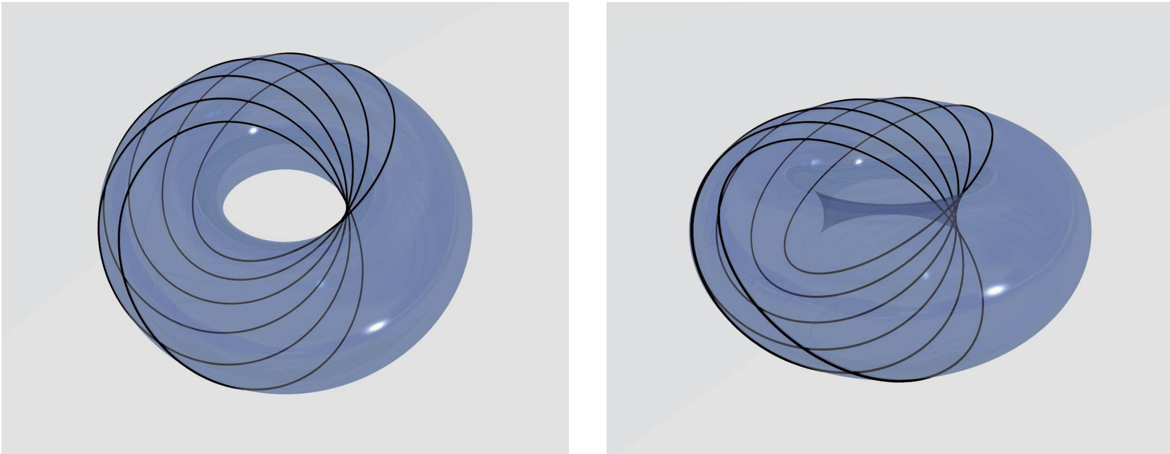


Figure 3: Villarceau circles on a torus

Remark. A regular Dupin cyclide, that is an embedded torus which is in two different ways the envelope of a one-parameter family of spheres is also a d'Alembert cyclide, as the two families of Villarceau circles satisfy the d'Alembert condition. The d'Alembert property reflects on the two curves of the space of circles corresponding to the circles of the two families.

In Proposition 3.0.3 the notions of time-like, space-like and light-like refer to the Lorentz quadratic form $\mathcal{L}(x) = -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^4$.

Proposition 3.0.3. *To each 4-dimensional subspace $\mathcal{H} \subset \mathbb{R}^5_1$ corresponds a 9-dimensional family of d'Alembert cyclides $\mathcal{A}_{\mathcal{H}}$.*

1) *If $\mathcal{H}$ is space-like, there exist a metric of $\mathbb{S}^3$ of constant curvature* 1 *such that all the circles of the two families are geodesics.*
2) *If $\mathcal{H}$ is light-like, that is tangent to the light-cone along a light-ray $\mathbb{R} \cdot m$, then, choosing m as the point at infinity, the cyclide becomes a ruled quadric of $\mathbb{R}^3 \simeq \mathbb{S}^3 \setminus m$.*
3) *If $\mathcal{H}$ is time-like, then all the circles of the cyclide are orthogonal to the sphere Σ corresponding to the two points of $\mathcal{H}^\perp \cap \Lambda^4$.*

From now on, we will use the quadratic form defined on $\bigwedge^2(\mathcal{H})$ by $Plu(U,U) = U \wedge U$. It is of index $(3,3)$. The totally isotropic subspaces of $\bigwedge^2(\mathcal{H})$ will be called like-like subspaces. It is convenient , instead of dealing with planes, 3-dimensional subspaces and the Plücker cone of $\mathbb{R}^6 \simeq \bigwedge^2(\mathcal{H})$ to work in the projective space $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2(\mathcal{H}))$. The Plucker quadric π is the image of the Plücker cone of equation (only one in $\bigwedge^2(\mathcal{H})$) $Plu(U,U) = 0$. A projective light-ray is the image of a totally isotropic plane and two orthogonal 3-dimensional subspaces provide two conjugate projective planes.

Theorem 3.0.4. *The two families of circles of a d'Alembert cyclide form two conics, intersection of the Plücker quadric $\pi\mathbb{P}^5$ with conjugate projective planes.*

The proof is quite similar to the analogous result obtained in [La-Si-Dru-Gar-Pa] for Dupin cyclides.

4. Cyclides, contact conditions and foliations

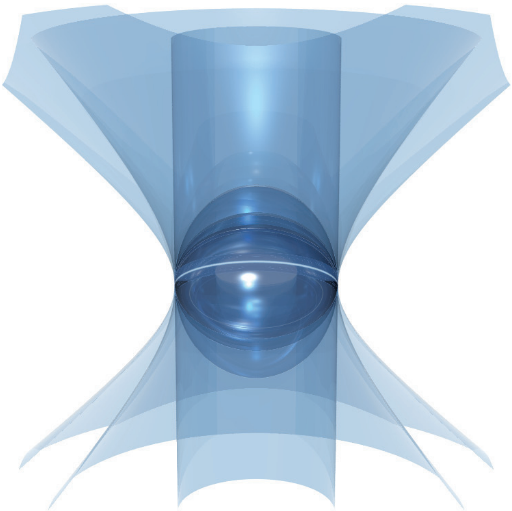
In [La-Si-Dru-Gar-Pa] the authors studied the existence of Dupin cyclides satisfying three contact conditions, that is tangent to three planes at three points. The solutions, when they exist, form a foliation of $\mathbb{S}^3$ with a singular locus which is a curve where all the solutions are tangent (see [La-Si].

Propositions 3.0.2 and 3.0.3 let us hope for a similar result for each family of d'Alembert cyclide.

The proofs will use a dynamical construction using three projective light-rays of $\mathbb{P}^5 = \mathbb{P}(\bigwedge^2(\mathcal{H}))$ corresponding to the three d'Alembert pairs, two circles contained in a common sphere. When cyclides containing the three d'Alembert pairs exist, they are tangent along a curve and form a foliation of $\mathbb{S}^3$ in case 1) of Proposition 3.0.3, and other wise a foliation of a simple domain of ${}_s s^3$ that can be used as a building block.

Remark. Algebraic geometers would give a proof of theorem 3.0.4 using linear families. In particular, the three contact problem in case 2) of Proposition 3.0.3 can be reduced to Brianchon theorem for conics.

5. Examples of foliations by d'Alembert cyclides and tangent d'Alembert cyclides



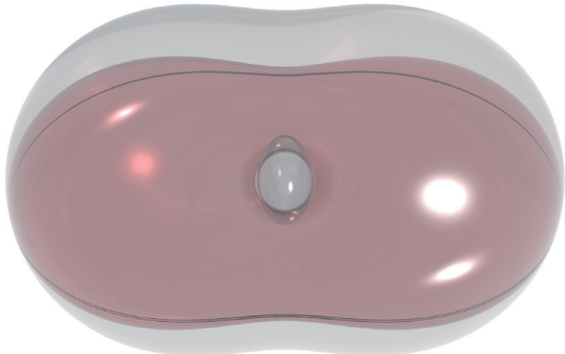


Figure 4: Foliation of $\mathbb{R}^3$ by quadrics and Darboux cyclides tangent along a curve

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Geometry and Foliations 2013
Komaba, Tokyo, Japan

Codimension one foliations calibrated by non-degenerate closed 2-forms

David MARTÍNEZ TORRES

1. Introduction

In this talk we will be presenting some fundamental results about a class of codimension one foliations which generalises 3-dimensional taut foliations.

Definition 1.1. A codimension one foliation $\mathcal{F}$ of M^{2n+1} is said to be 2-calibrated, if there exists a closed 2-form ω such that the restriction of ω^n to the leaves of $\mathcal{F}$ is no-where vanishing.

A 2-calibrated foliation $(M, \mathcal{F}, \omega)$ (M always closed) is an object which essentially belongs to symplectic geometry, as the following fact illustrates: the flow along the kernel of ω induces on each small open subset of each leaf a Poincaré return map which is a symplectomorphism.

1.1. Examples

There are three elementary families of 2-calibrated foliations: products, cosymplectic foliations and symplectic bundle foliations.

A **product** is the result of crossing a 2-calibrated foliation, typically a 3-dimensional taut foliation, with a (non-trivial) symplectic manifold, and putting the product foliation and the obvious closed 2-form.

A **cosymplectic foliation** is a triple (M, α, ω), where α is a no-where vanishing closed 1-form and $(M, \ker\alpha, \omega)$ is a 2-calibrated foliation. If α has rational periods then the cosymplectic foliation is a symplectic mapping torus, i.e., a mapping torus with fibre a symplectic manifold and return map a symplectomorphisms.

A bundle foliation with fibre S^1 is by definition an S^1-fibre bundle $\pi\colon M \to X$ endowed with a codimension one foliation $\mathcal{F}$ transverse to the fibres. If the base space admits a symplectic form σ, then $(M, \mathcal{F}, \pi^*\sigma)$ is a 2-calibrated foliation which we refer to as a **symplectic bundle foliation**.

2. Surgeries

It is possible to build new examples out of old ones: via a surgery with generalises the **normal connected sum** of symplectic manifolds [2], one can construct 2-calibrated foliations which belong to none of the three elementary classes [4].

There exist a second surgery in which a neighbourhood of a Lagrangian sphere is separated into 2 copies, and glued back by a generalised Dehn twist (a symplectic generalisation to any dimension of a 2-dimensional Dehn twist around a closed curve). **Generalised Dehn surgery** has an alternative presentation, in which the original and the resulting 2-calibrated foliations are the boundary of certain elementary symplectic cobordism [4]; this is related to the fact that a Lagrangian sphere in a leaf of a 2-calibrated foliation determined a canonical framing, and therefore an elementary cobordism.

3. Submanifolds and transverse geometry

Up to date, there are no differential geometric conditions on a foliation which guarantee the existence of submanifolds everywhere transverse to the leaves.

A **2-calibrated submanifold** is an embedded submanifold $j\colon W \hookrightarrow M$ everywhere transverse to $\mathcal{F}$ and intersecting each leaf in a symplectic submanifold. This means that W inherits a 2-calibrated foliation $(\mathcal{F}_W, \omega_W)$.

Methods of symplectic geometry developed by Donaldson [1] can be used to prove the following essential property:

Theorem 3.1. *[3] A 2-calibrated foliation $(M^{2n+1}, \mathcal{F}, \omega)$ has 2-calibrated* submanifolds *of any even codimension. In particular $(M^{2n+1}, \mathcal{F}, \omega)$ $(2n+1 \geq 5)$ contains W^3 a 3-dimensional manifold inheriting a taut foliation.*

3.1. Transverse geometry

Following Haefliger's viewpoint, the transverse geometry of a foliation $(M, \mathcal{F})$ is captured by the group-like structures in which the holonomy parallel transport is encoded. These are either the holonomy pseudogroup or the holonomy groupoid.

Here is our main result formulated in the framework of holonomy groupoids.

Theorem 3.2. *[5] Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. There exist $W \hookrightarrow M$ 3-dimensional submanifold $W \pitchfork \mathcal{F}$, which inherits a taut foliation $\mathcal{F}_W$ from $\mathcal{F}$ with the following property: the map induced by the inclusion* *between holonomy groupoids*

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F}). \tag{3.3}$$

is an essential equivalence.

The interpretation of theorem 3.2 is as follows: the taut foliation $(W, \mathcal{F}_W)$ and the 2-calibrated foliation $(M, \mathcal{F}, \omega)$ have equivalent transverse geometry.

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Rotation number and actions of the modular group on the circle

Yoshifumi MATSUDA

1. Introduction

Let Σ be a connected and oriented two dimensional orbifold with empty boundary and negative Euler characteristic $\chi(\Sigma) < 0$. We consider the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$ of homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{Homeo}_+(\mathrm{S}^1)$ with the compact-open topology. Let $\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$.

When Σ is a closed surface, we have the Euler number $\mathrm{eu}(\phi) \in \mathbb{Z}$ of ϕ and Milnor-Wood inequality ([7], [10])

$$|\,\mathrm{eu}(\phi)| \leq |\chi(\Sigma)|$$

holds. Matsumoto [6] showed that $|\,\mathrm{eu}(\phi)| = |\chi(\Sigma)|$ if and only if ϕ is semi-conjugate to an injective homomorphism onto a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Homeo}_+(\mathrm{S}^1)$, which is the holonomy representation of a hyperbolic structure on Σ (we call such a homomorphism a hyperbolization of Σ).

When Minakawa [8] dealt with the case where Σ is compact and has cone points. He defined the Euler number $\mathrm{eu}(\phi) \in \mathbb{Q}$ of ϕ by

$$\mathrm{eu}(\phi) = \frac{\mathrm{eu}(\phi|_\Gamma)}{[\pi_1(\Sigma) \colon \Gamma]},$$

where Γ is a torsion-free subgroup of $\pi_1(\Sigma)$ of finite index, and generalized the above results.

For the case where Σ is a noncompact surface of finite type. Burger, Iozzi and Wienhard [1] introduced the bounded Euler number $\mathrm{eu}^b(\phi) \in \mathbb{R}$ of ϕ by using bounded cohomology and generalized Milnor-Wood inequality and the above result of Matsumoto.

In this talk we deal with the case where Σ is noncompact and has cone points. In particular, we consider Milnor-Wood type inequality on each connected component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$.

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2. Bounded Euler number

Let Σ be a noncompact, connected and oriented two dimensional orbifold with cone points. For $\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$, we define the bounded Euler number $\mathrm{eu}^b(\phi) \in \mathbb{R}$ of ϕ by

$$\mathrm{eu}^b(\phi) = \frac{\mathrm{eu}^b(\phi|_\Gamma)}{[\pi_1(\Sigma)\colon \Gamma]},$$

where Γ is a torsion-free subgroup of $\pi_1(\Sigma)$ of finite index. The bounded Euler number has the following properties.

Proposition 2.1. *(1) We have*

$$\chi(\Sigma) \le \mathrm{eu}^b(\phi) \le -\chi(\Sigma). \tag{2.2}$$

Furthermore $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma)$ *if and only if* ϕ *is semi-conjugate to a hyperbolization of* Σ.
(2) Suppose that $\Sigma = \Sigma_{g,n}(q_1 \dots, q_m)$, *an orbifold whose underlying space is a surface of genus* g *with* p *punctures with* m *cone points of order* $q_1, \dots, q_m$. *Then under the presentation*

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n, d_1, \dots, d_m \colon \\ d_k^{q_k}, k = 1, \dots, m, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n} c_j \prod_{k=1}^{m} d_k\rangle,$$

we have

$$\mathrm{eu}^b(\phi) = \widetilde{\mathrm{rot}}(\prod_{i=1}^{g}[\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}] \prod_{j=1}^{n} \widetilde{\phi(c_j)} \prod_{k=1}^{m} \widetilde{\phi(d_k)}) \\ - \sum_{j=1}^{n} \widetilde{\mathrm{rot}}(\widetilde{\phi(c_j)}) - \sum_{k=1}^{m} \widetilde{\mathrm{rot}}(\widetilde{\phi(d_k)}),$$

where $\tilde{g} \in \widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1)$ *is a lift of* $g \in \mathrm{Homeo}_+(\mathrm{S}^1)$ *and* $\widetilde{\mathrm{rot}}\colon \widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1) \to \mathbb{R}$ *is the translation number.*

Remark 2.3. We make several remarks on the case where $\Sigma = \Sigma_{0,1}(q_1, q_2)$ with $\dfrac{1}{q_1} + \dfrac{1}{q_2} < 1$.
(1) The equality $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma_{0,1}(q_1, q_2))$ can be characterized by rotation numbers without translation numbers. Indeed $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma_{0,1}(q_1, q_2))$ if and only if $(\mathrm{rot}(\phi(c_1)), \mathrm{rot}(\phi(d_1)), \mathrm{rot}(\phi(d_2))) = \left(0, \pm\dfrac{1}{q_1}, \pm\dfrac{1}{q_2}\right)$.

(2) There exists $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(q_1, q_2)), \mathrm{Diff}^\omega_+(\mathrm{S}^1))$ such that $\phi(c)$ is topologically conjugate to a parabolic Möbius transformation and ϕ has an exceptional minimal set. This makes a contrast to the case of closed surface groups [3]. Such a homomorphism is obtained by taking ϕ so that $\phi([a, b])$ has more than two fixed points. If ϕ were minimal, then it is topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area and hence for every $g \in \pi_1(\Sigma_{0,1}(2, 3))$, $\phi(g)$ has at most two fixed points.

(3) There exists $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(2, 3)), \mathrm{Diff}^\omega_+(\mathrm{S}^1))$ such that ϕ is topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area but they are not C^1-conjugate. Note that a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area is unique up to conjugate in $\mathrm{PSL}(2, \mathbb{R})$. This also makes a contrast to the case of closed surface groups [4]. Existence of such a homomorphism is established by checking that we can deform $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(2, 3)), \mathrm{Diff}^\omega_+(\mathrm{S}^1))$ so that ϕ is kept topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area and the derivative of $\phi([a, b])$ at the attracting fixed point varies.

3. Extremals on connected components

Let $m, n \geq 1$ and $\Sigma = \Sigma_{g,n}(q_1, \ldots, q_m)$. For integers $p_1, \ldots, p_m$, we put

$$\begin{aligned} &H_{g,n}\left(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}\right) \\ &= \left\{\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))\colon\ \mathrm{rot}(\phi(d_k)) = \frac{p_k}{q_k}, k = 1, \ldots, m\right\}. \end{aligned}$$

Since $n \geq 1$, the subset $H_{g,n}(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m})$ is a connected component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$. The inequality (2.2) is not optimal on each connected component $H_{g,n}\left(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}\right)$. We can obtain the optimal inequality by Proposition 2.1 (2) and results of Jankins, Neumann [5] and Naimi [9] (see also [2] for more general study). For example, when $\Sigma = \Sigma_{0,1}(2, 3)$, we have

$$\frac{1}{5}\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\chi(\Sigma)$$

on $H_{0,1}\left(\frac{1}{2}, \frac{1}{3}\right)$ and

$$\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\frac{1}{5}\chi(\Sigma)$$

on $H_{0,1}\left(\frac{1}{2},-\frac{1}{3}\right)$. Note that $\phi \in H\left(\frac{1}{2},\pm\frac{1}{3}\right)$ satisfies $\mathrm{eu}^b(\phi)=\pm\frac{1}{5}\chi(\Sigma)$ if and only if $\mathrm{rot}(c_1)=\pm\frac{1}{5}$. In this case, we have the following result.

Theorem 3.1. *If $\Sigma=\Sigma_{0,1}(2,3)$ and $\phi \in H_{0,1}\left(\frac{1}{2},\pm\frac{1}{3}\right)$ satisfies $\mathrm{eu}^b(\phi)=\pm\frac{1}{5}\chi(\Sigma)$, then ϕ is semi-conjugate to a 5-fold covering of a hyperbolization of Σ.*

Remark 3.2. Theorem 3.1 cannot be generalized straightforward when we change Σ and $(p_1,\dots,p_m)$. For example, when $\Sigma=\Sigma_{0,1}(2,7)$, we have

$$\chi(\Sigma)\le \mathrm{eu}^b(\phi)\le -\frac{3}{25}\chi(\Sigma)$$

on $H_{0,1}(\frac{1}{2},\frac{1}{7})$ and $\phi \in H_{0,1}(\frac{1}{2},\frac{1}{7})$ with $\mathrm{eu}^b(\phi)=-\frac{3}{25}\chi(\Sigma)$ is not semi-conjugate to a finite covering of a hyperbolization of Σ.

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Conitinuous leafwise harmonic functions on codimension one transversely isometric foliations

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1. Introduction

Let M be a closed C^2 manifold, and let $\mathcal{F}$ be a continuous leafwise C^2 foliation on M. This means that M is covered by a finite union of continuous foliation charts and the transition functions are continuous, together with their leafwise partial derivatives up to order 2. Let g be a continuous leafwise[1] C^2 leafwise Riemannian metric. In this talk, such a triplet $(M, \mathcal{F}, g)$ is simply refered to as a *leafwise C^2 foliations.* For simplicity, we assume throughout that the manifold M and the foliation $\mathcal{F}$ are oriented. For a continuous leafwise[2] C^2 real valued function h on M, the leafwise Laplacian Δh is defined by $\Delta h = *d * d\,h$, where $*$ is the leafwise Hodge operator induced by the leafwise metric g.

Definition 1.1. A continuous leafwise C^2 function h is called *leafwise harmonic* if $\Delta h = 0$.

Definition 1.2. A leafwise C^2 foliation $(M, \mathcal{F}, g)$ is called *Liouville* if any continuous leafwise harmonic function is leafwise constant.

As an example, if $\mathcal{F}$ is a foliation by compact leaves, then $(M, \mathcal{F}, g)$ is Liouville. Moreover there is an easy observation:

Proposition 1.3. *If $\mathcal{F}$ admits a unique minimal set, then $(M, \mathcal{F}, g)$ is Liouville.*

This can be seen as follows. Let m_1 (resp. m_2) be the maximum (resp. minimum) value of the continuous leafwise harmonic function h on M. Assume h takes the maximum value m_1 at $x \in M$. Then by the maximum principle, $h = m_1$ on the leaf F_x which passes through x. Now the closure of F_x contains the unique minimal set X. Therefore $h = m_1$ on X. The

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[1]This means that the leafwise partial derivatives up to order 2 of the components of g in each foliation chart are continuous in the chart.

[2]The leafwise partial derivatives of h up to order 2 in each foliation chart are continuous in the chart.

same argument shows that $h = m_2$ on X. That is, $m_1 = m_2$, showing that h is constant on M.

A first example of non-Liouville foliations is obtained by R. Feres and A. Zeghib in a simple and beautiful construction [FZ]. It is a foliated S^2-bundle over a hyperbolic surface, with two compact leaves. There are also examples in codimension one. B. Deroin and V. Kleptsyn [DK] have shown that a codimension one foliation $\mathcal{F}$ is non-Louville if $\mathcal{F}$ is transversely C^1, admits no transverse invariant measure and possesses more than one minimal sets, and they have constructed such a foliation.

A codimension one foliation $\mathcal{F}$ is called $\mathbb{R}$-*covered* if the leaf space of its lift to the universal covering space is homeomorphic to $\mathbb{R}$. See [F] or [FFP]. It is shown in [F] and [DKNP] that an $\mathbb{R}$-covered foliation without compact leaves admits a unique minimal set. Therefore the above example of a codimension one non-Liouville foliation is not $\mathbb{R}$-covered. This led the authors of [FFP] to the study of Liouville property for $\mathbb{R}$-covered foliations. The main purpose of the present talk is to generalize a result of [FFP].

Definition 1.4. A codimension one leafwise C^2 foliation $(M, \mathcal{F}, g)$ is called *transversely isometric* if there is a continuous dimension one foliation ϕ transverse to $\mathcal{F}$ such that the holonomy map of ϕ sending a (part of a) leaf of $\mathcal{F}$ to another leaf is C^2 and preserves the leafwise metric g.

Notice that a transversely isometric foliation is $\mathbb{R}$-covered. Our main result is the following.

Theorem 1.5. *A leafwise C^2 transversely isometric codimension one foliation is Liouville.*

In [FFP], the above theorem is proved in the case where the leafwise Riemannian metric is negatively curved. Undoubtedly this is the most important case. But the general case may equally be of interest.

If a transversely isometric foliation $\mathcal{F}$ does not admit a compact leaf, then, being $\mathbb{R}$-covered, it admits a unique minimal set, and Theorem 1.5 holds true by Proposition 1.3. Therefore we only consider the case where $\mathcal{F}$ admits a compact leaf. In this case the union X of compact leaves is closed. Let U be a connected component of $M \setminus X$, and let N be the metric completion of U. Then N is a foliated interval bundle, since the one dimensional transverse foliation ϕ is Riemannian.

Therefore we are led to consider the following situation. Let K be a closed C^2 manifold of dimension ≥ 2, equipped with a C^2 Riemannian metric g_K. Let $N = K \times I$, where I is the interval $[0, 1]$. Denote by $\pi : N \to K$ the canonical projection. Consider a continuous foliation $\mathcal{L}$ which is transverse to the fibers $\pi^{-1}(y)$, $\forall y \in K$. Although $\mathcal{L}$ is only continuous, its leaf has a C^2 differentiable structure as a covering space of

K by the restriction of π. Also $\mathcal{L}$ admits a leafwise Riemannian metric g obtained as the lift of g_K to each leaf by π. Such a triplet $(N, \mathcal{L}, g)$ is called a *leafwise C^2 foliated I-bundle* in this talk. Now Theorem 1.5 reduces to the following theorem.

Theorem 1.6. *Assume a leafwise C^2 foliated I-bundle $(N, \mathcal{L}, g)$ does not admit a compact leaf in the interior* $\mathrm{I}nt(N)$. *Then any continuous leafwise harmonic function is constant on N.*

An analogous result for random discrete group actions on the interval was obtained in [FR].

2. Outline of the proof of Theorem 1.6

The proof is by absurdity. Let $(N, \mathcal{L}, g)$ be a leafwise C^2 foliated I-bundle without interior compact leaves, and we assume that there is a continuous leafwise harmonic function f such that $f(K \times \{i\}) = i$, $i = 0, 1$.

A probability measure μ on N is called *stationary* if $\langle \mu, \Delta h \rangle = 0$ for any continuous leafwise C^2 function h.

Proposition 2.1. *There does not exist a stationary measure μ such that*

$$\mu(\mathrm{I}nt(N)) > 0.$$

This can be shown as follows. Denote by X the union of leaves on which f is constant. The subset X is closed in N. L. Garnett [G] has shown that $\mu(X) = 1$ for any stationary measure μ. Therefore if $\mu(\mathrm{I}nt(N)) > 0$, there is a leaf L in $\mathrm{I}nt(N)$ on which f is constant. But since we are assuming that there is no interior compact leaves, the closure of L must contain both boundary components of N. A contradiction to the continuity of f.

The proof of Theorem 1.6 is obtained by studying leafwise Brownian motions. Let us denote by Ω the space of continuous leafwise paths $\omega : [0, \infty) \to N$. For any $t \geq 0$, a random variable $X_t : \Omega \to N$ is defined by $X_t(\omega) = \omega(t)$. For any point $x \in N$, the Wiener probability measure P^x is defined using the leafwise Riemannian metric g. Notice that $P^x\{X_0 = x\} = 1$.

Given $0 < \alpha < 1$, let $V = K \times (\alpha, 1]$, and define a subset Ω_V of Ω by

$$\Omega_V = \{X_{t_i} \in V, \ \exists t_i \to \infty\}.$$

Clearly Ω_V is invariant by the shift map. Then, as is well known, the function $p : M \to [0, 1]$ defined by $p(x) = P^x(\Omega_V)$ is leafwise harmonic. Another important feature of the function p is that p is nondecreasing along the fiber $\pi^{-1}(y)$, $\forall y \in K$, since our leafwise Brownian motion is synchronized, i. e, it is the lift of the Brownian motion on K. The key fact for the proof is the following:

The function p is constant on $\mathrm{I}nt(N)$.

This follows from Proposition 2.1. That is, if we assume p nonconstant, then we can construct a stationary measure μ such that $\mu(\mathrm{I}nt(N)) > 0$. Next an easy observation shows the following:

The function p is 1 on $\mathrm{I}nt(N)$.

This implies that $\limsup_{t\to\infty} f(X_t) = 1$, P^x-almost surely, since the neighbourhood V can be arbitrary. Likewise considering neighbourhoods of $K \times \{0\}$, we have $\liminf_{t\to\infty} f(X_t) = 0$.

But since f is leafwise harmonic, the family $\{f(X_t)\}$ is a P^x-martingale, and the martingale convergence theorem asserts that there exist $\lim_{t\to\infty} f(X_t)$, P^x-almost surely. The contradiction shows Theorem 1.6.

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The normal h-principle for foliations and Mather-Thurston homology equivalence

Gaël MEIGNIEZ

This talk introduces a *normal* h-principle for foliations, which is a refinement of Thurston's h-principle. "Normal" means that we prescribe a plane field to which the foliation to be built will be normal, except in some parts of the ambiant manifold, the so-called holes. The interesting case is when this plane field is tangential to the fibres of a fibration. We get:

Theorem 0.1. *Let* $f : M^{p+q} \to B^p$ *be a fibration between closed manifolds,* $q \geq 2$ *, and let* ξ *be a* Γ_q*-structure on* M *, whose normal bundle is isomorphic to* $\ker Df$ *. Then, there is a foliation* $\mathcal{F}$ *of codimension* q *on* M *s.t.*

- $\mathcal{F}$ *is homotopic to* ξ *as a* Γ_q*-structure;*
- $\tau\mathcal{F}$ *is a limit of* p*-plane fields transverse to the fibres.*

More precisely, $\mathcal{F}$ is transverse to the fibres, except along a submanifold of codimension 1 , union of compact leaves bounding some kind of vertical Reeb components, given by Thurston's method to fill holes in codimension larger than 1 . The theorem also holds true when $q = 1$, $p = 2$, and when f is a Seifert fibration.

The same method also gives a new proof of the Mather-Thurston homology equivalence in all codimensions.

The h-principle for foliations is a powerful tool to build foliations on a given manifold M , and to classify them up to concordance. Due to Haefliger for M open, and to Thurston for M compact, it says that every *formal foliation* on M is homotopic to some genuine foliation.

We are interested in the case where M is closed. Recall that a Γ_q-structure on M is a pair $\xi = (\nu\xi, \mathcal{X})$ where

- $\nu\xi$ is a real linear bundle of rank q over M , the *normal bundle,* or *microbundle*;
- $\mathcal{X}$ is a foliation of codimension q on the total space $\nu\xi$, transverse to the fibres (in fact, the germ of such a foliation along the null section).

A formal foliation is a pair (ξ, j) , where ξ is a Γ_q-structure on M , and where $j : \nu\xi \to \tau M$ is a linear bundle monomorphism.

Write $\bar{M} := M \times [0,1]$ and $\bar{M}_i := M \times i$ $(i = 0,1)$. A *homotopy* between two formal foliations (ξ_i, j_i) on M , sharing the same normal bundle, is a Γ_q-structure ξ on $\bar{M}$ s.t. $\xi|\bar{M}_i = \xi_i$ $(i = 0,1)$, together with a continuous homotopy of linear bundle monomorphisms $j_t : \nu\xi_0 \to \tau M$ $(t \in [0,1])$.

Recall the sketch of Thurston's proof (skipping technicallities about "good position" and "civilization"). We start from a closed manifold M on which are given a Γ_q-structure ξ and a linear bundle monomorphism $j : \nu\xi \to \tau M$ from the microbundle of ξ into τM . One easily translates these data into an embedding of $\bar{M}$ into a large-dimensional open manifold E and a plane field F of codimension q on E s.t.

- On some neighborhood of $\bar{M}_0$, the field F is integrable, and this foliation induces ξ by restriction to $\bar{M}_0$;
- Along $\bar{M}_1$, the field F is transverse to $j(\nu\xi)$.

Then, $\bar{M}$ is finely triangulated, and jiggled in E , giving a PL submanifold $\bar{M}' \subset E$, C^0-close to $\bar{M}$, s.t. the field F in transverse to $\bar{M}'$, which means by definition, transverse to every simplex of $\bar{M}'$. Write $\bar{M}'_i \subset \bar{M}'$ the jiggled image of $\bar{M}_i$ $(i = 0,1)$. Thanks to the transversality of F and $\bar{M}_1$, one arranges that $\bar{M}_1$ is globally invariant by the jiggling, and thus $\bar{M}'_1 = \bar{M}_1$ remains a smooth submanifold in E .

Thurston's method consists in applying a homotopy to F , in a small neighborhood of $\bar{M}'$, relative to $\bar{M}'_0$, among the plane fields transverse to $\bar{M}'$, to make F integrable in a neighborhood of $\bar{M}'$. Then, F will induce on $\bar{M}_1$ the seeked foliation.

The homotopy is realized simplex after simplex, climbing up a collapsing of $\bar{M}'$ onto $\bar{M}'_0$ ("inflation") . The heart of the construction is the inflation process: given a $(q+k)$-simplex α of $\bar{M}'$ and a "free" hyperface $\beta \subset \alpha$ s.t. F is already integrable in a neighborhood of $\lambda := (\partial\alpha) \setminus \beta$, one homotopes F relatively to λ , to an integrable field in the whole of α . In a first time, this leaves in the interior of α an unfoliated subset ("hole") diffeomorphic to $\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{D}^q$. This hole is filled in a second time. (For $q = 1$, things are a little more complicated: the hole needs to be extended before we fill it).

To prove the *normal* h-principle, write $N := j(\nu\xi)$, a q-plane field on M . Consider a tubular neighborhood T of $\bar{M}_0$ in E . Write $\pi : T \to M$ the projection, $\ker\pi$ the plane field in T tangential to the fibres, and consider $\pi^*(N)$, a foliation on T .

In the beginning, T is a small tubular neighborhood of $\bar{M}_0$ in E . As the inflation process goes, we extend T by successive isotopies of embeddings in E s.t. T remains a small neighborhood of the union of the simplicies where F has been made integrable; and moreover:

- $\ker\pi$ and $\pi^*(N)$ are transverse to $\bar{M}'$;

- $\ker \pi \subset F$ at every point of every free simplex of $\bar{M}'$;
- $\pi^*(N) \pitchfork F$ except in the holes;
- $\pi^*(N) \pitchfork (\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times t)$ in every hole, and for every $t \in \boldsymbol{D}^q$.

It is better not to fill the holes during the inflation. The holes propagate. At the end, F defines on $\bar{M}'$ a Γ_q-structure with holes. Each hole has the form

$$\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{D}^{n+1-k} \times \boldsymbol{D}^q$$

where $n := \dim M$, and meets $\bar{M}_1$ on

$$\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{S}^{n-k} \times \boldsymbol{D}^q$$

In restriction to $\bar{M}_1$, outside the holes, F defines a foliation normal to $\pi^*(N)$. The projection of $F|\bar{M}_1$ through π is a foliation on M transverse to N , except in the holes.

Then, we can fill the holes by the classical way, obtaining a foliation on M , normal to N but in the holes: this is the normal h-principle.

In case N is an *integrable* q-plane field on M , we can arrange that in each hole, N coincides with the $\boldsymbol{D}^q$-fibres.

Assume moreover that N is tangential to the fibres of a fibration $M \to B$. Then, we can fill each hole by some suspension, at the price of a surgery on B : this leads to some *bordism equivalence* between the classifying space $BDiff_c(\boldsymbol{R}^q)$ for foliated bundles, and the Thom space of $B\bar{\Gamma}_q$; and then, through the Atiyah-Hirzebruch spectral sequence, to the Mather-Thurston homology equivalence.

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Genus one Birkhoff sections for suspension Anosov flows

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1. Introduction

There are two fundamental examples of Anosov flows of clesed connected 3-manifolds. One is the suspension flow $\phi_{\overline{A}}$ of a hyperbolic toral automorphism $\overline{A}$ induced by an element $A \in SL(2, \mathbb{Z})$ with $trace(A) > 2$ and the other is the geodesic flow ϕ_{Σ_g} of a closed hyperbolic surface Σ_g of genus g.

Two flows ϕ of M and ϕ' of M' are said to be *topologically equivalent* if there exists a homeomorphism $h : M \to M'$ which maps ϕ-orbits to ϕ'-orbits preserving their orientation. No suspension Anosov flow is topologically equivalent to a geodesic flow. Two flows ϕ, ϕ' are said to be *topologically almost equivalent* if there exists a finite union Γ (resp. Γ') of periodic orbits of ϕ (resp. ϕ') such that $\phi|_{M \setminus \Gamma}$ is topologically equivalent to $\phi'|_{M' \setminus \Gamma'}$. Then, each geodesic flow is topologically almost equivalent to some suspension Anosov flow. This is proved by constructing genus one Birkhoff sections for geodesic flows. A *Birkhoff section* for a flow ψ of a closed connected 3-manifold M is defined to be the pair (S, ι) of a compact connected surface S with the boundary ∂S and an immersion $\iota : S \to M$ such that (1) $\iota|_{Int(S)}$ is an embedding transverse to the flow, (2) each component of the boundary ∂S covers a periodic orbit of ψ by ι, and (3) every orbit starting from any point of M meets S in a uniformly bounded time. The image $\iota(S)$ is also called a Birkhoff section. These observations may lead one to ask the following question.

Question 1.1. Is any geodesic flow topologically almost equivalent to any suspension Anosov flow?

The aim of this article is to give a positive answer to this question.

2. First return maps of Birkhoff sections

Let (S, ι) be a Birkhoff section of an Anosov flow ϕ of a closed 3-manifold M. Let Γ be a finite union of periodic orbits of ϕ. For any orbit $\gamma \subset \Gamma$, take a tubular neighborhood $N(\gamma) = D^2 \times S^1$. Let M^Γ be the 3-manifold with boundary obtained from M by blowing up each point $p \in \Gamma$ using a polar coordinate of D^2. There uniquely exists a flow ϕ^Γ of M^Γ such that $\phi|_{M\setminus\Gamma} = \phi^\Gamma|_{Int(M^\Gamma)}$. The surface $\iota(S)$ gives rise to an embedded surface $S^\Gamma \subset M^\Gamma$ transverse to ϕ^Γ, which is really a cross section of ϕ^Γ. Let $r : S^\Gamma \to S^\Gamma$ be the first return map of the flow ϕ^Γ. Fried has shown that the return map r is topologically conjugate to a pseudo-Anosov diffeomorphism h of S^Γ ([2, Theorem 3]). Let $r_S : \hat{S}^\Gamma \to \hat{S}^\Gamma$ be the homeomorphism obtained from $r : S^\Gamma \to S^\Gamma$ by collapsing each component of ∂S^Γ to a point. If S is of genus one, r_S is topologically conjugate to a toral automorphism $\overline{B}$ for some hyperbolic element $B \in SL(2,\mathbb{Z})$(see [4, Lemma 1 in Section 3]).

3. Main results

Let A be an element of $SL(2,\mathbb{Z})$ with $trace(A) > 2$. Then, A is conjugate, in $GL(2,\mathbb{Z})$, to a matrix of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_2} \cdots\cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_{2n-1}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_{2n}} \qquad (\ a_i \geq 1)$$

(see [3, Section 1]).

Suppose $B_1, B_2 \in SL(2,\mathbb{Z})$ are given. If B_1 is conjugate to B_2 in $GL(2,\mathbb{Z})$, the induced flow $\phi_{\overline{B_1}}$ is topologically equivalent to $\phi_{\overline{B_2}}$. So we may assume that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \geq b \geq d$, and $a \geq c \geq d$.

Theorem 3.1. *If $trace(A) > 3$, there exists a genus one Birkhoff section (S,ι) of $\phi_{\overline{A}}$ such that $\sharp Fix(r_S) < \sharp Fix(\overline{A})$.*

Here, $Fix(f)$ denotes the fixed point set of a map $f : X \to X$.

By arguments in Section 2, the map r_S obtained in Theorem 3.1 is topologically conjugate to a hyperbolic automorphism $\overline{B}$ determined by a matrix $B \in SL(2,\mathbb{Z})$. Then we have

$$\begin{aligned} trace(B) - 2 &= \#Fix(\overline{B}) \\ &= \#Fix(r_S) \\ &< \#Fix(\overline{A}) \\ &= trace(A) - 2. \end{aligned}$$

These observations lead one to the following theorem which gives one a positive answer to Question 1.1.

Theorem 3.2. *Let A be an element of $SL(2,\mathbb{Z})$ with $trace(A) > 3$. Then, there exists $B \in SL(2,\mathbb{Z})$ such that (1) $\phi_{\overline{A}}$ is topologically almost equivalent to $\phi_{\overline{B}}$, and (2) $2 < trace(B) < trace(A)$.*

4. Outline of the proof Theorem 3.1

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $SL(2,\mathbb{Z})$ with $a \geq b \geq d$ and $a \geq c \geq d$. Let $M_A = T^2 \times [0,1]/(p,1) \sim (\overline{A}(p),0)$. recall that the Anosov flow $\phi_{\overline{A}}$ is a suspension flow of M_A. A key point of the construction of a genus one Birkhoff section of $\phi_{\overline{A}}$ is to find a suitable rectangle $\square P_1P_2P_3P_4$ in $T^2 = T^2 \times \{0\} \subset M_A$ such that

1. $\phi_{\overline{A}}(\overline{P_1P_2}) = \overline{P_3P_2}$ and $\phi_{\overline{A}}(\overline{P_3P_4}) = \overline{P_1P_4}$, or
2. $\phi_{\overline{A}}(\overline{P_3P_2}) = \overline{P_1P_2}$ and $\phi_{\overline{A}}(\overline{P_1P_4}) = \overline{P_3P_4}$,

where $\overline{PQ}$ denote the edge of a rectangle connecting a vertex P with a vertex Q. Such a rectangle, together with two flow bands connecting $\overline{P_1P_2}$ with $\overline{P_3P_2}$ and $\overline{P_3P_4}$ with $\overline{P_1P_4}$ respectively, gives rise to an immersed surface S_1 in $M_{\overline{A}}$ such that 1) S_1 is homeomorphic to a 2-sphere minus three disks, 2) $int(S_1)$ is embedded in $M_{\overline{A}}$ and is transverse to $\phi_{\overline{A}}$, 3) each component of ∂S_1 covers a periodic orbit of $\phi_{\overline{A}}$(see [2, Section 2]). For any $0 < \epsilon < 1$, $S_1 \cup (T^2 \times \{\epsilon\})$ is a singular surface with double point curves. If you cut and paste the surface $S_1 \cup (T^2 \times \{\epsilon\})$ along the double curves to obtain a required genus one Birkhoff section (S,ι) of $\phi_{\overline{A}}$. In order to complete the proof of Theorem 3.1, it suffeces to find a rectangle mentioned above. To this end, take $P_1 = (0,0)$, $P_2 = (1/c, 0)$, $P_3 = (0,1)$, and $P_4 = (-1/c, 1)$ if $b = 1$, and take $P_1 = (1,0)$, $P_2 = \dfrac{1}{a+d-2}(a-1, c)$, $P_3 = (0,0)$, and $P_4 = \dfrac{1}{a+d-2}(d-1, -c)$ if $b \neq 1$. Then the rectangle $\square P_1P_2P_3P_4$ in $\mathbb{R}^2$ gives rise to a required rectangle in T^2.

5. Question

Given $A \in SL(2,\mathbb{Z})$ with $trace(A) < -3$, you can also make a Birkhoff section (S,ι) of $\phi_{\overline{A}}$ as in Section 4. In this case, the induced homeomorphism r_S is not an Anosov homeomorphism. Then you have the following question.

Question 5.1. Does there exist a pair of matrices A, B in $SL(2,\mathbb{Z})$ such that (1) $\phi_{\overline{A}}$ is topologically almost equivalent to $\phi_{\overline{B}}$, and (2) $trace(A) < trace(B) < -2$.

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Leafwise symplectic structures on codimension 1 foliations on S^5

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1. Introduction

In this talk, we present a framework for the existence of leafwise symplectic structures on a codimension 1 foliation associated with a Milnor fibration or with an open book decomposition supporting a contact structure.

Definition 1.1. A *leafwise symplectic structure* on a foliatited manifold $(M, \mathcal{F})$ is a smooth 2-form ω which restricts to a symplectic form on each leaf.

Our previous result was the following.

Theorem 1.2 ([Mi]). *The natural codimension 1 spinnable foliation $\mathcal{F}_k$ on S^5 associated with the simple elliptic hypersurface singularity $\widetilde{E}_k$, admits a leafwise symplectic structure for $k = 6, 7, 8$.*

Corollary 1.3. *There exist regular Poisson stuctures on S^5 whose symplectic dimension is 4.*

In particular for $k = 6$, the associated foliation is so called *Lawson's foliation* which is the first codimension one foliation found on S^5 ([L]).

The three deformation classes $\widetilde{E}_l$ $(l = 6, 7, 8)$ of simple elliptic hypersurface singularities are given by the following polyomials.

$$f_{\widetilde{E}_6} = Z_0^3 + Z_1^3 + Z_2^3 \ \ (+\lambda Z_0 Z_1 Z_2)$$
$$f_{\widetilde{E}_7} = Z_0^4 + Z_1^4 + Z_2^2 \ \ (+\lambda Z_0 Z_1 Z_2)$$
$$f_{\widetilde{E}_8} = Z_0^6 + Z_1^3 + Z_2^2 \ \ (+\lambda Z_0 Z_1 Z_2)$$

As the smooth topology of these objects does not depend on the choice of the constant λ, while λ should avoid finitely many exceptional values, we can ignore it and take it to be 0. In the case of $\widetilde{E}_6$ the objects are homogeneous of degree 3 and the hypersurfaces $f^{-1}(w)$ are all easily dealt with in a geometric sense. For $\widetilde{E}_7$ and $\widetilde{E}_8$, they are quasi homogeneous

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and still easy to handle. In particular we easily see that their links are ismorphic to 3-dimensional nil manifolds $Nil^3(-3)$, $Nil^3(-2)$, and $Nil^3(-1)$ respectively.

The next main target for us is the case of cusp singularities which is defined by the following polynomials.

$$f_{p,q,r} = Z_0^p + Z_1^q + Z_2^r + Z_0Z_1Z_2\,, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

whose links are known to be solv manifolds, *i.e.*, T^2-bundles over S^1 with monodoromies hyperblic toral auto's. If we try to generalize our consruction to a wider class of Milnor fibrations or open books supporting contact structures including the cusp singularities case, we need to formalize and brush up the construction in [Mi].

For basic facts about hypersurface singularities, the readers may refer to Milnor's seminal text book [M]. For the treatment of cusp singularities in this context, see [H] and [K].

Let (M, K, Φ) be a symplectic open book decomposition of M^{2n+1} supporting a contact structure ξ on M or a Milnor fibration on $M = S^{2n+1}$ associated with a singularity of $n+1$ variables with the link K. In the first case ξ restricts to $\xi_K = \xi|_K$ and in the latter case the ambient complex strucrue of $\mathbb{C}^{n+1}$ determines the standard contact structure $\xi = \xi_0$ on S^{2n+1} which restricts to a natural contact structure $\xi_K = \xi_0|_K$. Let F_θ denote the page over $\theta \in S^1$. The end of F_θ is diffeomorphic to $K \times \mathbb{R}_+$.

Now our main result is stated as follows.

Theorem 1.4. *In the above situation, supppose the following conditions are satisfied. Then M admits a natural 'spinnable' codimension 1 foliation $\mathcal{F}$ with a leafwise symplectic structure and an isotopic family of contact structures starting from the given one ξ on M which converges to $\mathcal{F}$ as a family of almost contact structures.*

(1) *The link K^{2n-1} fibres over S^1 with a symplectic sructure ω_Σ on the fibre Σ^{2n-2} which is invariant under the monodromy.*[1]

(2) *The restriction $H^2_{dR}(F_\theta) \to H^2_{dR}(K \times \mathbb{R}_+) \cong H^2_{dR}(K)$ hits $[\omega_\Sigma]$.*

(3) *ξ_K admits a contact form whose Reeb vector field X_K is tangent to the fibration in (1). For $n > 2$ we need extra quantitative conditions.*[2]

Corollary 1.5. *Associated with any of the cusp singularities of the form $f_{p,q,r} = 0$ at the origin ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$), the natural spinnable foliation on S^5 admits a leafwise symplectic structure.*[3]

[1]In the case $n = 2$, this simply meens K fibres over S^1.

[2]For many cases this extra condtion seems to be easily checked.

[3]This foliation has a unique compact leaf diffeomorphic to a solv manifold.

2. Mori's convergence criterion

Atsuhide Mori's criterions on a symplectic open book supporting a contact structure for the existence of a natural codimension 1 spinnable foliation or for the isotopic convergence of the contact structures to the foliation ([Mo1], [Mo2], and [Mo3]). These criterions are the prototypes of our conditions in the main theorem. Let (M, K, Φ), ξ, ξ_K,and X_K be as in the previous setion.

Theorem 2.1 (Mori)**.** *Let us consider the following two conditions.*

(I) *K admits a Riemannian foliation $\mathcal{G}$ of codimension* 1*.*

(II) *X_K is tangent to $\mathcal{G}$.*

1) *If* (I) *is satisfied, there exists a natural 'spinnable' foliation $\mathcal{F}$ on M which restricts to $\mathcal{G} = \mathcal{F}|_K$ on K.*
2) *Moreover if* (II) *is also satisfied, there exists an isotopic family of ξ which converges to $\mathcal{F}$.*

1) is applicable for any open book decomposition, because the condition (I) comes from Kopell's lemma and Thurston-Rosenberg ([RT]) showed that then $K \times D^2$ can be smoothly foliated. Topologically, (I) implies that K fibres over S^1 (Tischler's theorem). The coincidence of (II) with our condition (3) is rather surprising.

Mori's criterions implies the covergence of contact structures to a spinnable foliation is quite different in dimension 5 or higher than in dimension 3.

3. Further discussions

Naturally next main target for us is a construction of a codimension 1 leafwise symplectic foliation on S^7, while the Milnor fibre is simply connected for usual isolated singularities of 4 or more variables. Therefore already the condition (1) in our theorem fails.

Problem 3.1. 1) Does there exist mixed function with an isolated singularity to which we can apply the very basic theory of Milnor fibrations and contact structures which exhibit different features on the topologies of link and Milnor fibres?

2) In particular for 4 or more complex variables whose link K admits a symplectic fibre in the sense of (2) in our main Theorem ?

On the other hand, what is really missing is the search for *impossibility*.

Problem 3.2. 1) Find non-trivial obstructions for codimension 1 foliations to admit a leafwise symplectic structures.

2) Find further obstuctions for Stein manifolds to admit an end-periodic symplectic structures. The 2nd foliated cohomology does not seem to be a good candidate.

3) Does there exist a 2-calibrated codimension 1 foliation on S^5? Here, '2-calibrated' means 'equipped with a *global closed* 2-form which restricts to a symplectic form on each leaf'.

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Problems on characteristic classes of foliations

Shigeyuki MORITA

1. Introduction

The theory of characteristic classes of foliations was initiated by discovery of the Godbillon-Vey class of codimension 1 foliations [14] and a groundbreaking work of Thurston [35] proving that it can vary continuously. Soon after this, Bott and Haefliger [5], and also Bernstein and Rozenfeld [3] presented a general framework for this theory and during the 1970's, it has been developed extensively by many people including Heitsch [17] and Hurder [19]. There also appeared closely related theory of Gelfand and Fuks [11] and that of Chern and Simons [8]. The notions of Γ-structures and their classifying spaces due to Haefliger [18] played a crucial role in this theory and Mather [26] and Thurston [36] obtained many fundamental results by using them.

However there remain many important problems to be solved in future. In this talk, we would like to focus on the following two major problems both of which turn out to be extremely difficult. One is the determination of the homotopy type of the classifying space $\mathrm{B}\Gamma_1$ of Γ_1-structures in the C^∞-category. The other is development of characteristic classes of *transversely symplectic* foliations.

2. Homotopy type of $\mathrm{B}\Gamma_1$

The following is one of the major open problems in foliation theory.

Problem 2.1. Determine the homotopy type of $\mathrm{B}\Gamma_1$. More precisely, determine whether the classifying map

$$\mathrm{GV} : \mathrm{B}\overline{\Gamma}_1 \to K(\mathbb{R}, 3)$$

induced by the Godbillon-Vey class, is a homotopy equivalence or not. Here $\mathrm{B}\overline{\Gamma}_1$ denotes the homotopy fiber of the natural map $w_1 : \mathrm{B}\Gamma_1 \to \mathrm{BGL}(1, \mathbb{R}) = K(\mathbb{Z}/2, 1)$.

In [28], we introduced the concept of *discontinuous invariants* of foliations. One possible approach to the above problem would be the following.

PROBLEM 2.2. Determine whether the homomorphism

$$\mathrm{GV}_k : H_{3k}(\mathrm{B}\overline{\Gamma}_1, \mathbb{Z}) \to \wedge^k_{\mathbb{Z}} \mathbb{R} \ (\cong H_k(\mathbb{R}^\delta; \mathbb{Z}))$$

induced by the discontinuous invariants associated with the Godbillon-Vey class is non-trivial or not.

Let $\mathrm{B}\overline{\Gamma}^\omega_1$ denote the classifying space of transversely oriented *real analytic* Γ_1-structures. Haefliger [18] proved that $\mathrm{B}\overline{\Gamma}^\omega_1$ is a $K(\pi, 1)$ space for certain *perfect* group π.

PROBLEM 2.3. Determine whether the natural map

$$(\mathrm{B}\overline{\Gamma}^\omega_1)^+ \to \mathrm{B}\overline{\Gamma}_1$$

is a homotopy equivalence or not, where $^+$ denotes Quillen's plus construction.

Recall here that Thurston constructed a family of real analytic codimension 1 foliations on a certain 3-manifold by making use of the group

$$\mathrm{SL}(2, \mathbb{R}) *_{\mathrm{SO}(2)} \widetilde{\mathrm{SL}}(2, \mathbb{R})_n \subset \mathrm{Diff}^\omega_+ S^1$$

thereby proving that the homomorphism

$$\mathrm{GV} : H_3(\mathrm{B}\overline{\Gamma}^\omega_1; \mathbb{Z}) \to \mathbb{R}$$

is *surjective*. Here $\widetilde{\mathrm{SL}}(2, \mathbb{R})_n$ denotes the n-fold covering group of $\mathrm{SL}(2, \mathbb{R})$.

In the case of piecewise linear (PL for short) category, Greenberg [15] showed that there is a weak homotopy equivalence

$$\mathrm{B}\overline{\Gamma}^{\mathrm{PL}}_1 \sim \mathrm{B}\mathbb{R}^\delta * \mathrm{B}\mathbb{R}^\delta$$

where the right hand side represents the join of two copies of $\mathrm{B}\mathbb{R}^\delta$. It follows that $\mathrm{B}\overline{\Gamma}^{\mathrm{PL}}_1$ is 2-connected and he described the integral homology group of $\mathrm{B}\overline{\Gamma}^{\mathrm{PL}}_1$ completely. It also follows that the higher homotopy groups of this space is highly non-trivial.

By making use of this result, Tsuboi [33] showed that all the discontinuous invariants of $\mathrm{B}\overline{\Gamma}^{\mathrm{PL}}_1$ associated with the discrete Godbillon-Vey class $\in H^3(\mathrm{B}\overline{\Gamma}^{\mathrm{PL}}_1, \mathbb{R})$, defined by Ghys and Sergiescu [13], vanishes.

On the other hand, in a certain case of low differentiability (Lipschitz with bounded variation of derivatives), Tsuboi [34] proved that the second discontinuous invariant

$$\mathrm{GV}_2 : H_6(\mathrm{B}\overline{\Gamma}^{\mathrm{Lip,bdd}}_1, \mathbb{Z}) \to \wedge^2_{\mathbb{Z}} \mathbb{R}$$

is highly non-trivial (in fact its cockernel is a torsion group) where GV is the one he extended to this case.

The Godbillon-Vey class can be defined for transversely holomorphic foliations with trivialized normal bundles and Bott [4] proved that the homomorphism

$$\mathrm{GV}^{\mathbb{C}} : \pi_3(\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}}) \to \mathbb{C}$$

is surjective.

Problem 2.4. Determine the homotopy type of $\mathrm{B}\Gamma_1^{\mathbb{C}}$. More precisely, determine whether the classifying map

$$\mathrm{GV}^{\mathbb{C}} : \mathrm{B}\overline{\Gamma}_1^{\mathbb{C}} \to K(\mathbb{C}, 3)$$

induced by the complex Godbillon-Vey class, is a homotopy equivalence or not.

We refer to a book [1] by Asuke for a recent study of $\mathrm{GV}^{\mathbb{C}}$.

Finally we recall a closely related problem. Let $\mathcal{M}^h(3)$ denote the set of orientation preserving diffeomorphism classes of closed oriented *hyperbolic* 3-manifolds. For any such manifold M, we have its volume $\mathrm{vol}(M)$ and the η-invariant $\eta(M)$ of Atiyah-Patodi-Singer [2]. The combination $\eta + i\,\mathrm{vol}$ gives rise to a mapping

$$\eta + i\,\mathrm{vol} : \mathcal{M}^h(3) \to \mathbb{C}.$$

Problem 2.5 (Thurston ([37], Questions 22, 23). Study the above map. In particular, determine whether the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-subspace of $i\mathbb{R}$ generated by the second component of the image of the above map is infinite or not.

Recall that any such M defines a homology class $[M] \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ and we have the following closely related problem.

Problem 2.6. Determine the image of the map

$$\mathcal{M}^h(3) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z}) \xrightarrow{(\mathrm{CS}, i\mathrm{vol})} \mathbb{C}/\mathbb{Z}.$$

Problem 2.7. Study the discontinuous invariants of the group $\mathrm{PSL}(2,\mathbb{C})^\delta$ associated with the above classes. In particular, determine the value of the *total Chern Simons invariant* introduced in Dupont [9].

3. Characteristic classes of transversely symplectic foliations

One surprising feature of the Gelfand-Fuks cohomology theory was that

$$\dim H_c^*(\mathfrak{a}_n) < \infty$$

where $\mathfrak{a}_n$ denotes the Lie algebra consisting of all the formal vector fields on $\mathbb{R}^n$. The associated characteristic homomorphism

$$\Phi : H_c^*(\mathfrak{a}_n) \to H^*(\mathrm{B}\overline{\Gamma}_n; \mathbb{R})$$

is now very well understood. In contrast with this, the case of all the *volume preserving* formal vector fields $\mathfrak{v}_n \subset \mathfrak{a}_n$ and that of all the *Hamiltonian* formal vector fields $\mathfrak{ham}_{2n} \subset \mathfrak{a}_{2n}$ are both far from being understood.

Problem 3.1. Compute

$$H_c^*(\mathfrak{v}_n), \quad H_c^*(\mathfrak{v}_n, \mathrm{O}(n)), \quad H_c^*(\mathfrak{ham}_{2n}), \quad H_c^*(\mathfrak{ham}_{2n}, \mathrm{U}(n)).$$

In particular, prove (or disprove) that

$$\dim H_c^*(\mathfrak{v}_n) = \infty, \quad \dim H_c^*(\mathfrak{ham}_{2n}) = \infty.$$

Recall here that there are very few known results concerning this problem. First, Gelfand, Kalinin and Fuks [12] found an *exotic* class

$$\text{GKF class} \in H_c^7(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_8$$

and later Metoki [27] found another exotic class

$$\text{Metoki class} \in H_c^9(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{14}.$$

On the other hand, Perchik [32] obtained a formula for the Euler characteristic and computed it up to certain degree. It suggests strongly that the cohomology would be infinite dimensional.

Let $\mathrm{B}\Gamma_{2n}^{\mathrm{symp}}$ denote the Haefliger classifying space of transversely symplectic foliations of codimension $2n$.

Problem 3.2. Prove that, under the homomorphism

$$\Phi : H_c^*(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R})) \to H^*(\mathrm{B}\Gamma_2^{\mathrm{symp}}; \mathbb{R})$$

the GKF class and the Metoki class survive as non-trivial characteristic classes.

Kontsevich [22] introduced a new viewpoint in this situation. He considered two Lie subalgebras

$$\mathfrak{ham}_{2g}^1 \subset \mathfrak{ham}_{2g}^0 \subset \mathfrak{ham}_{2g}$$

consisting of Hamiltonian formal vector fields without constant terms and without constant as well as linear terms, respectively. Then he constructed a homomorphism

$$\Phi : H_c^*(\mathfrak{ham}_{2g}^0, \mathrm{Sp}(2g,\mathbb{R})) \cong H_c^*(\mathfrak{ham}_{2g}^1)^{\mathrm{Sp}} \to H_{\mathcal{F}}^*(M)$$

for any transversely symplectic foliation $\mathcal{F}$ on a smooth manifold M of codimension $2n$, where $H_{\mathcal{F}}^*(M)$ denotes the foliated cohomology group. By using this viewpoint, in a joint work with Kotschick [24] we decomposed the Gelfand-Kalinin-Fuks class as a product

$$\text{GKF class} = \eta \wedge \omega$$

where $\eta \in H_c^5(\mathfrak{ham}_2^0, \mathrm{Sp}(2,\mathbb{R}))_{10}$ is a certain leaf cohomology class and ω denotes the transverse symplectic form.

Conjecture 3.3 (Kotschick-M. [24])**.** *The Metoki class can also be decomposed as a product $\eta' \wedge \omega$ for a certain class $\eta' \in H_c^7(\mathfrak{ham}_2^0, \mathrm{Sp}(2,\mathbb{R}))_{16}$.*

On the other hand, $\mathfrak{ham}_{2g}^0, \mathfrak{ham}_{2g}^1$ can be described as

$$\mathfrak{ham}_{2n}^0 = \widehat{\mathfrak{c}}_n \otimes \mathbb{R}, \quad \mathfrak{ham}_{2n}^1 = \widehat{\mathfrak{c}}_n^+ \otimes \mathbb{R}$$

where $\mathfrak{c}_n$ denotes one of the three Lie algebras (commutative one) in Kontsevich's theory [20][21] of graph homology and $\widehat{\mathfrak{c}}_n$ denotes its completion. Thus the above homomorphim Φ can be written as

$$\Phi : H_c^*(\widehat{\mathfrak{c}}_n^+)^{\mathrm{Sp}} \otimes \mathbb{R} \cong H_c^*(\mathfrak{ham}_{2n}^1)^{\mathrm{Sp}} \to H_{\mathcal{F}}^*(M).$$

Besides the theory of transversely symplectic foliations as above, the graph homology of $\mathfrak{c}_n$ has another deep connection with the theory of *finite type* invariants for homology 3-spheres initiated by Ohtsuki [31] which we briefly recall. Let $\mathcal{A}(\phi)$ denote the commutative algebra generated by vertex oriented connected trivalent graphs modulo the (AS) relation together with the (IHX) relation. This algebra plays a fundamental role in this theory. In fact, the completion $\widehat{\mathcal{A}}(\phi)$ of $\mathcal{A}(\phi)$ with respect to its gradings is the target of the LMO invariant [25].

By using a result of Garoufalidis and Nakamura [10], in a joint work with Sakasai and Suzuki [30] we constructed an injection

$$\mathcal{A}(\phi) \to H_*(\mathfrak{c}_\infty^+)^{\mathrm{Sp}}$$

and defined the "complementary" algebra $\mathcal{E}$ so as to obtain an isomorphism

$$H_*(\mathfrak{c}_\infty^+)^{\mathrm{Sp}} \cong \mathcal{A}(\phi) \otimes \mathcal{E}$$

of bigraded algebras. $\mathcal{E}$ can be interpreted as the dual of the space of all the *exotic* stable leaf cohomology classes for transversely symplectic foliations.

Problem 3.4 (cf. Sakasai-Suzuki-M. [30]). Study the structure of $\mathcal{E}$.

4. Homology of $\mathrm{Diff}^\delta M$ and $\mathrm{Symp}^\delta(M,\omega)$

In general, homology group of the diffeomorphism group $\mathrm{Diff}^\delta M$ of a closed C^∞ manifold M, considered as a discrete group, or that of the symplectomorphism group $\mathrm{Symp}^\delta(M,\omega)$ of a closed symplectic manifold (M,ω), again with the discrete topology, is a widely open research area. One can also consider the real analytic case. Here we present a few problems in the cases of the circle S^1 and closed surfaces.

It was proved in [29] that the natural homomorphism

$$\Phi : H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2))) \cong \mathbb{R}[\alpha,\chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^\delta S^1; \mathbb{R})$$

from the Gelfand-Fuks cohomology of S^1, relative to $\mathrm{SO}(2) \subset \mathrm{Diff}_+ S^1$, to the cohomology of $\mathrm{Diff}_+^\delta S^1$, is injective. Also there were given certain non-triviality results for the associated discontinuous invariants.

Problem 4.1. Prove (or disprove) that the homomorphism

$$\Phi : H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2)) \cong \mathbb{R}[\alpha,\chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^{\omega,\delta} S^1; \mathbb{R})$$

is injective, where $\mathrm{Diff}_+^{\omega,\delta} S^1$ denotes the real analytic diffeomorphism group of S^1 equipped with the discrete topology.

Problem 4.2. Determine whether the natural inclusion

$$\mathrm{Diff}_+^{\omega,\delta} S^1 \to \mathrm{Diff}_+^\delta S^1$$

induces an isomorphism in homology or not.

Of course one can consider the above problem for any closed manifold M.

Let Σ_g denote a closed oriented surface of genus g. Harer stability theorem [16] states that the homology group $H_k(\mathrm{BDiff}_+\Sigma_g)$ is independent of g in a certain stable range $k \ll g$ (see a survey paper [38] by Wahl for more details).

By applying a general method, we can define certain characteristic classes for foliated Σ_g-bundles. Also, in [23] certain characteristic classes for foliated Σ_g-bundles with area-preserving holonomy were defined by making use of the notion of the flux homomorphism. These classes are all *stable* with respect to the genus g and it seems reasonable to present the following.

Problem 4.3. Determine whether certain analogue of Harer stability theorem holds for the group $\mathrm{Diff}^\delta\Sigma_g$ and/or $\mathrm{Symp}^\delta\Sigma_g$.

Bowden [6][7] obtained some interesting results related to this problem.

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One dimensional dynamics in intermediate regularity: old and new

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1. Introduction

The study of (pseudo)-group actions on 1-dimensional manifolds by diffeomorphisms of class C^2 is a classical subject that goes back to Denjoy, Schwartz and Sacksteder. By many reasons, it is desirable to extend this to intermediate regularity (smaller than C^2 but larger than C^1 in the Hölder scale). T. Tsuboi and S. Hurder were among the first in being interested on this, both interested in the possibility of extending the Godbillon-Vey class to a certain critical regularity [4, 8]. This made natural several questions of a dynamical natural, formulated by Tsuboi as conjectures. Most of them were solved in collaboration with Deroin and Kleptsyn [2], as for example:

Theorem 1.1 (à la Denjoy). *Every free action of $\mathbb{Z}^d$ by circle diffeomorphisms of class $C^{1+\alpha}$, $\alpha > 1/d$, is minimal and topologically conjugated to a group of rotations.*

Theorem 1.2 (à la Kopell). *Let $f_1, \ldots, f_d, f_{d+1}$ be diffeomorphisms of the closed unit interval for which there exist subintervals $I_{i_1,\ldots,i_d,i_{d+1}}$ disposed lexicographically and such that $f_j(I_{i_1,\ldots,i_j,\ldots,i_{d+1}}) = I_{i_1,\ldots,i_j+1,\ldots,i_{d+1}}$ for all j. If the f_j's, with $j = 1, \ldots, d$, are all of class $C^{1+\alpha}$, where $\alpha > 1/d$, and commute (among them and) with f_{d+1}, then f_{d+1} cannot be of clas C^1.*

In both cases, counter-examples in class $C^{1+\alpha}$, $\alpha < 1/d$, were already been constructed by Tsuboi [9].

In this talk, I will concentrate on recent extensions of these kind of results to more genral groups/regularities. I start with a theorem obtained in collaboration with Kleptsyn [5] concerning the case where the regularities of the maps are different.

Theorem 1.3. *In both theorems above, one may suppose that the regularities of the f_j's are different, say f_j is of class $C^{1+\alpha_j}$, $j = 1, \ldots, d$, provided $\frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_d} > 1$. Moreover, for every combination of exponents satisfying the reverse (strict) inequality, one can construct counter-examples.*

The second extension is very recent and concerns the critical regularity $C^{1/d}$. Although this remains open for the case of the circle, for the interval is completely settled in [7].

Theorem 1.4. *Theorem 1.2 still holds in class $C^{1+1/d}$ (assuming f_{d+1} of class $C^{1+\alpha}$ for some positive α). This remains true for the extension to different regularities above.*

Finally, we consider the case of more complicated groups. We start with a result that completely solves the question concerning growth of groups of diffeomorphisms.

Theorem 1.5. *If Γ is a finitely generated subgroup of $\mathrm{Diff}_+^{1+\alpha}([0,1])$, then either Γ is almost nilpotent or it contains a free semigroup. Moreover, this is no longer true in class C^1.*

This theorem makes natural the study of nilpotent groups of diffeomorphisms of the interval. In this direction, the classical Kopell-Plante-Thurston theorem establishes that no such group can be contained in $\mathrm{Diff}_+^2([0,1])$ unless it is Abelian. However, it is a classical fact (going back to Malcev and Newmann) that every torsion-free, finitely-generated nilpotent group Γ is left-orderable, hence it acts on the interval. Actually, there is a very natural action, which was proven by B. Farb and J. Franks [3] to be smoothable to the class C^1. This action is constructed as follows: According to a theorem of Malcev, Γ embeds into a group N_d of $d \times d$ lower-triangular matrices with integer entries and 1 in the diagonal. This group acts on $\mathbb{Z}^d$, fixing the hyperplane $\{1\} \times \mathbb{Z}^{d-1}$ and respecting the lexicographic order therein. If we make correspond an interval $I_{i_1,\ldots,i_{d-1}}$ to all such points in $\mathbb{Z}^{d-1}$ and we dispose them on $[0,1]$ in an ordered way, this naturally induces an action of N_d (hence of Γ) on $[0.1]$ (just use infinitely many affine maps and paste them). The next result was shown in collaboration with E. Jorquera and G. Castro.

Theorem 1.6. *The Farb-Franks action of N_d above is not (semi-)conjugate to an action by $C^{1+\alpha}$ diffeomorphisms, where $\alpha \geq \frac{2}{(d-1)(d-2)}$. This is no longer true in lower regularity.*

This result concerns a single action, and it remains the question of determining the best regularity of arbitrary actions of nilpotent groups on the interval. This is work in progress (in collaboration with Castro, Jorquera, C. Rivas and R. Tessera).

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Rigidity and arithmeticity in Lie foliations

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1. Introduction

Background. For a Lie group G, a *G-Lie foliation* is a foliation transversely modeled on the G-action on G by left translation. Such foliations have been investigated being motivated by the classification of Riemannian foliations (see [6, 8]). The first example of Lie foliations is the following one, which is called *homogeneous*:

Example 1.1. Let G and H be connected Lie groups. Let K be a closed Lie subgroup of H, and Γ a torsion-free cocompact lattice of $H \times G$. Then we have a G-Lie foliation on $K\backslash H \times G/\Gamma$ induced from the product foliation $K\backslash H \times G = \sqcup_{g\in G} K\backslash H \times \{g\}$.

A number of examples of nonhomogeneous Lie foliations were constructed in [15, 16, 9]. On the other hand, under various conditions, minimal Lie foliations tend to be homogeneous or have rigidity which is quite useful for the classification: Caron-Carrière [3] showed that 1-dimensional Lie foliation is diffeomorphic to a linear flow on a torus. Matsumoto-Tsuchiya [14] proved that any 2-dimensional affine Lie foliation on closed 4-manifolds are homogeneous. Zimmer [24] proved that if a minimal G-Lie foliation admits a Riemannian metric such that each leaf is isometric to a product of symmetric space of noncompact type of rank greater than one, then the holonomy group is arithmetic.

Motivation. This work was motivated by the following two questions on rigid aspects of Lie foliations mentioned in the last paragraph.

Question 1.2. Classify minimal $\mathrm{SL}(2;\mathbb{R})$-Lie foliations whose leaves are hyperbolic plane.

By a theorem of Carrière [4], for a G-Lie foliation on a compact manifold, G is solvable if and only if each leaf admits a Følner sequence. Thus Lie foliations in Question 1.2 are of the lowest dimension among Lie foliations with hyperbolic leaves.

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QUESTION 1.3. Classify 3-dimensional minimal Lie foliations which admits leafwise geometrization in the sense of Thurston.

One may conjecture that any 3-dimensional minimal Lie foliations may admits leafwise geometrization. Thus Question 1.3 may be considered as a step for the classification of 3-dimensional minimal Lie foliations.

Main results. This talk is based on work in progress. The main result is the following:

Theorem 1.4. *Let $(M, \mathcal{F})$ be a compact manifold with a minimal G-Lie foliation. Assume that M admits a Riemannian metric such that every leaf of $\mathcal{F}$ is isometric to a symmetric space $X = \prod X_i$, where X_i is an irreducible Riemannian symmetric space of noncompact type of dimension greater than two. Then $(M, \mathcal{F})$ is homogeneous.*

This result gives a complete answer for Question 1.3 in the case where the leaves are $\mathbf{H}^3$. We describe the proof in detail in Section 3. The key step of the proof is to show that the geodesic boundary of hyperbolic leaves admits a $\pi_1 M$-invariant conformal structure thanks to ergodicity of the $\pi_1 M$-action or the leafwise geodesic flow. This phenomenon can be regarded as a certain family version of strong Mostow rigidity for locally symmetric spaces [19]. Our proof is not sufficient to solve Question 1.2 in the same reason why Mostow strong rigidity fails to hold for Riemann surfaces.

We deduce two consequences of Theorem 1.4. We need the following result, which will be proved in Section 4.

Proposition 1.5. *If a homogeneous Lie foliation $(K \backslash H \times G/\Gamma, \mathcal{F})$ in Example 1.1 satisfies the assumption of Theorem 1.4, then G is semisimple and the projection of Γ to any connected normal subgroup of $H \times G$ is dense.*

A lattice Γ of a connected Lie group G is called *superrigid* if, for any real algebraic group H containing no connected simple compact normal subgroups, any homomorphism $\Gamma \to H$ with Zariski dense image virtually extends to a continuous homomorphism $G \to H$. Combining Theorem 1.4, Proposition 1.5 with an extension of Margulis' superrigidity theorem due to Starkov [23, Theorem 4.6], we get the following.

Corollary 1.6. *Under the assumption of Theorem 1.4, $\pi_1 M$ is isomorphic to a superrigid cocompact lattice in $H \times G$.*

Combining with Theorem 1.4 and Proposition 1.5 with Margulis' arithmeticity theorem [12, Theorem A in p. 298], we get the following consequence, which implies a generalization of a theorem of Zimmer [24, Theorem A-3], which says that the holonomy group of Lie foliation whose leaves are isometric to a product of symmetric space of noncompact type of rank greater than one is arithmetic.

Corollary 1.7. *In addition to the assumption of Theorem 1.4, we assume that X is of rank greater than one. Then $\pi_1 M$ is isomorphic to an S-arithmetic subgroup of $H \times G$.*

The advantage of arithmeticity is that arithmetic subgroups can be listed up in a sense. Thus Lie foliations in Corollary 1.7 are classified in a sense.

2. Questions

The following is related to Question 1.2.

Question 2.1. Does there exist a non-homogeneous minimal Lie foliation on a closed manifold whose leaves are isometric to hyperbolic planes?

The following is a question concerning the possibility of generalizations of a theorem of Matsumoto-Tsuchiya [14] on homogeneity of solvable Lie foliations.

Question 2.2. Find a good condition which implies the rigidity of minimal G-Lie foliations when G is solvable.

Tits buildings are arithmetic analog of symmetric spaces which have similar rigidity theoretic properties.

Question 2.3. Construct minimal Lie foliations whose leaves are quasi-isometric to Tits buildings of rank greater than one. Do they have rigidity?

Question 2.4. The leaves of the example [9, Section 6] of a minimal $\mathrm{SL}(2;\mathbb{R})$-Lie foliation are quasi-isometric to a Tits building of rank one. Does it have rigidity?

3. Outline of the proof of Theorem 1.4

Step I. The leafwise boundary of foliations. First we explain the proof of Theorem 1.4 in the case where $X = \mathbf{H}^n_{\mathbb{R}}$ $(n \geq 3)$.

Let $(M, \mathcal{F})$ be a minimal G-Lie foliation on a compact manifold. Assume that M admits a Riemannian metric such that each leaf of $\widetilde{\mathcal{F}}$ is isometric to X. Let $(\widetilde{M}, \widetilde{\mathcal{F}})$ be the universal cover of $(M, \mathcal{F})$. Let $\mathcal{G}(\widetilde{\mathcal{F}}) = \{\gamma \mid \gamma$ is a geodesic in a leaf of $\widetilde{\mathcal{F}}\}$. We define the leafwise geodesic boundary $\partial\widetilde{\mathcal{F}}$ of $(\widetilde{M}, \widetilde{\mathcal{F}})$ by

$$\partial\widetilde{\mathcal{F}} = \mathcal{G}(\widetilde{\mathcal{F}})/\sim\,,$$

where $\gamma \sim \gamma'$ if and only if γ and γ' are contained in a leaf of $\widetilde{\mathcal{F}}$ and asymptotic to each other. By the structure theory of Lie foliations (see [17, Section 4.2]), we have an X-bundle dev : $\widetilde{M} \to G$ whose fibers are the leaves of $\widetilde{\mathcal{F}}$. We also have a homomorphism hol : $\pi_1 M \to G$, which makes dev a $\pi_1 M$-equivariant X-bundle. Then $\partial\widetilde{\mathcal{F}}$ is the total space of a $\pi_1 M$-equivariant ∂X-bundle $\partial\,\mathrm{dev} : \partial\widetilde{\mathcal{F}} \to G$, where ∂X is the geodesic boundary of X.

Step II. Ergodicity of the $\pi_1 M$-action on the leafwise boundary. Let $H = \operatorname{Isom} X = \mathrm{PSO}(n,1)$ and K the isotropy group of a point on X so that $X = K\backslash H$. Since each leaf of $\mathcal{F}$ is isometric to X, we have a canonical K-principal bundle $N \to M$ over M with an isometric H-action. Here we have $\partial\widetilde{\mathcal{F}} = \widetilde{N}/P$ for a parabolic subgroup P. Thus $\partial\widetilde{\mathcal{F}}$ has a Lebesgue measure. The following is the key step in the case where $X = \mathbf{H}^n_{\mathbb{R}}$.

Proposition 3.1. *The $\pi_1 M$-action on $\partial\widetilde{\mathcal{F}}$ constructed in Step I is ergodic with respect to the Lebesgue measure.*

In the sequel, we consider Lebesgue measures on smooth manifolds. We will use the following results.

Lemma 3.2 (A modification of [18, Proposition 4])**.** *Let Γ_1 and Γ_2 be two groups. Let Z be a smooth manifold with a $(\Gamma_1 \times \Gamma_2)$-action such that Z/Γ_1 and Z/Γ_2 are smooth manifolds. Then the Γ_1-action on Z/Γ_2 is ergodic if and only if the Γ_2-action on Z/Γ_1 is ergodic.*

Theorem 3.3 (A part of [18, Theorem 1])**.** *Let H be a semisimple Lie group with no compact connected subgroup. Let P be a subgroup of H. Then the following are equivalent:*

1. *The image of P under the projection from H to each connected simple normal subgroup of H is noncompact.*
2. *For any unitary H-representation π in the Hilbert space V and any vector $v \in V$, if $\pi(P)v = v$, then $\pi(H)v = v$.*

Proof of Proposition 3.1. By Lemma 3.2, the $\pi_1 M$-action on $\partial\widetilde{\mathcal{F}}$ is ergodic if and only if the P-action on N is ergodic. By Theorem 3.3 for

Hilbert space $L^2(N)$, any P-invariant L^2-function on N is H-invariant. Thus the latter condition is equivalent to the ergodicity of the H-action on N. Lemma 3.2 implies that the H-action on N is ergodic if and only if the Γ-action on $\widetilde{N}/H = G$ is ergodic. Since Γ is a dense subgroup of G, the Γ-action on G is ergodic (see the proof of [18, Proposition 4]). $\square$

Step III. Construction of a homomorphism $\pi_1 M \to H$. We construct a trivialization of $\partial\widetilde{\mathcal{F}}$ as a ∂X-bundle over G based on the construction [9, Section 3]. For $g \in G$, denote the leaf of $\widetilde{\mathcal{F}}$ which is the fiber of dev over g by $L(g)$. Take a $\pi_1 M$-invariant metric on $(\widetilde{M}, \widetilde{\mathcal{F}})$. Any left invariant vector field ξ on G can be horizontally lifted to $\widetilde{M}$ along $\mathrm{dev} : \widetilde{M} \to G$ so that the lift $\widetilde{\xi}$ is tangent to $(T\widetilde{\mathcal{F}})^{\perp}$. Since dev is $\pi_1 M$-equivariant and M is compact, the flow on $\widetilde{M}$ generated by $\widetilde{\xi}$ is bi-Lipschitz. For each $g \in G$, take the left invariant vector field ξ on G such that $\exp \xi = g$. By the flow on $\widetilde{M}$ generated by $\widetilde{\xi}$, we have a map $\Phi(g) : \widetilde{M} \to \widetilde{M}$ whose restriction to $L(h)$ is bi-Lipschitz for any $h \in G$. Here $\Phi(g)$ induces a map $\partial\Phi(g) : \partial\widetilde{\mathcal{F}} \to \partial\widetilde{\mathcal{F}}$ whose restriction to $\partial L(h)$ is a quasi-conformal homeomorphism (see [19, Section 21]). Clearly we have $\partial\Phi(g_1) \circ \partial\Phi(g_2) = \partial\Phi(g_1 g_2)$. Then we get a trivialization $\partial\widetilde{\mathcal{F}} \cong \partial L(e) \times G$. Let e_G be the unit element of G. we obtain a $\pi_1 M$-action on $\partial L(e_G)$ given by

$$\begin{array}{ccc} \pi_1 M \times \partial L(e_G) & \longrightarrow & \partial L(e_G) \\ (c, [\gamma]) & \longmapsto & \partial\Phi(\mathrm{hol}(c)^{-1})([c \cdot \gamma]) \ , \end{array} \tag{3.4}$$

where $\cdot$ denotes the $\pi_1 M$-action on the space $\mathcal{G}(\widetilde{\mathcal{F}})$ of geodesics.

Since a quasi-conformal homeomorphism is absolutely continuous, the trivialization preserves the Lebesgue measure class. Thus, by Proposition 3.1, we have ergodicity of (3.4). Here we apply the following.

Proposition 3.5 ([19, Section 22])**.** *Let $n \geq 2$ and $q : S^n \to S^n$ be a quasi-conformal homeomorphism. If q is equivariant with respect to an ergodic group action, then q is conformal.*

Then we conclude that the $\pi_1 M$-action (3.4) on $\partial L(e_G)$ is conformal. We get a homomorphism $\pi_1 M \to \mathrm{Conf}(\partial X) \cong \mathrm{Isom}\, X = H$.

Step IV. Construction of a homogeneous Lie foliation $(M_0, \mathcal{F}_0)$. Let $\rho : \pi_1 M \to H$ be the homomorphism constructed in Step III. Consider the direct product $\rho \times \mathrm{hol} : \pi_1 M \longrightarrow H \times G$. Let $\Gamma = (\rho \times \mathrm{hol})(\pi_1 M)$.

We show that Γ is discrete in $H \times G$. Assume that Γ is not discrete. Then there exists a sequence $\{c_k\}$ in $\pi_1 M$ such that $\rho(c_k) \to e_H$ and $\mathrm{hol}(c_k) \to e_G$. Let $\psi_k : L(e_G) \to L(e_G)$ be the isometry which induces a conformal transformation $\rho(c_k)$ on $\partial L(e_G)$. Take a point x in $L(e_G)$ and consider a sequence $\{a_k\}$ in $L(e_G)$ defined by

$$a_k = \psi_k^{-1}\big(\Phi(\mathrm{hol}(c_k)^{-1})(c_k \cdot x)\big) ,$$

where $\Phi(\mathrm{hol}(c_k)^{-1}) : L(\mathrm{hol}(c_k)) \to L(e_G)$ is the bi-Lipschitz map constructed in Step III and $\cdot$ denotes the $\pi_1 M$-action on $\widetilde{M}$. By construction, the map

$$\begin{array}{rccc} \chi_k : & L(e_G) & \longrightarrow & L(e_G) \\ & y & \longmapsto & \psi_k^{-1}\big(\Phi(\mathrm{hol}(c_k)^{-1})(c_k \cdot y)\big) \end{array}$$

is a bi-Lipschitz map which induces the identity on $\partial L(e_G)$. Since $\{\mathrm{hol}(c_k)\}$ converges to e_G, there exists a positive number C such that, for any k, $\Phi(\mathrm{hol}(c_k)^{-1})|_{L(\mathrm{hol}(c_k))}$ is bi-Lipschitz with Lipschitz constant C. Then, χ_k is a bi-Lipschitz with Lipschitz constant C for any k. By the Morse lemma (see, for example, [2, 8.4.20]), there exists $r > 0$ such that χ_k maps any geodesic τ in $L(e_G)$ into an r-neighborhood of τ. This implies that $d(y, \chi_k(y)) < r$, where d is the distance on $L(e_G)$. Then $\{\chi_k(x)\}$ admits a converging subsequence. By construction, this implies that $\{c_k \cdot x\}$ admits a converging subsequence. This contradicts with the properly discontinuity of the $\pi_1 M$-action on $\widetilde{M}$. Thus Γ is discrete in $H \times G$.

We show that Γ is cocompact in $H \times G$. We denote the real cohomological dimension of manifolds and groups by rcd. First we compute $\mathrm{rcd}\,\Gamma$. By applying [5, Lemme 2.4] to $\mathrm{dev} : \widetilde{M} \to G$, we have

$$\mathrm{rcd}\,\widetilde{M} \leq \mathrm{rcd}\,\widetilde{L} + \mathrm{rcd}\,G ,$$

$$\mathrm{rcd}\,M \leq \mathrm{rcd}\,\widetilde{M} + \mathrm{rcd}\,\Gamma ,$$

where $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}$. Since $\widetilde{L}$ is contractible, $\mathrm{rcd}\,\widetilde{L}$ is zero. Since M is compact, we have $\mathrm{rcd}\,M = \dim M$. Thus we get

$$\mathrm{rcd}\,G + \mathrm{rcd}\,\Gamma \geq \dim M .$$

Let K_G be a maximal compact subgroup of G. Let $X_G = K_G \backslash G$. Recall that K is a maximal compact subgroup of H such that $X = K \backslash H$. Since $\mathrm{rcd}\,G = \dim K$, $\dim G = \dim X_G + \dim K$ and $\dim M = \dim G + \dim X$. We get

$$\mathrm{rcd}\,\Gamma \geq \dim X_G + \dim X .$$

On the other hand, since a finite index subgroup of Γ acts freely on $X \times X_G$ which is contractible, we get

$$\mathrm{rcd}\,\Gamma \leq \dim X_G + \dim X .$$

Thus we get $\mathrm{rcd}(\Gamma) = \dim X_G + \dim X$. This implies that $H^n\big((X \times X_G)/\Gamma; \mathbb{R}\big)$ is nontrivial, where $n = \dim(X \times X_G)/\Gamma$. Thus Γ is cocompact in $H \times G$.

Then $M_0 = K\backslash H \times G/\Gamma$ is a closed manifold. Here M_0 admits a G-Lie foliation $\mathcal{F}_0$ which is induced from the product foliation $K\backslash H \times G = \sqcup_{g \in G} K\backslash H \times \{g\}$ and whose leaves are isometric to $X = K\backslash H$.

Step V. Construction of a diffeomorphism. Here we will show that $(M, \mathcal{F})$ is diffeomorphic to $(M_0, \mathcal{F}_0)$. Since $(M, \mathcal{F})$ and $(M_0, \mathcal{F}_0)$ are classifying spaces of G-Lie foliations with the same holonomy group as explained in the last paragraph, there exist smooth maps $f : M \to M_0$ and $f_0 : M_0 \to M$ such that $f^*\mathcal{F}_0 = \mathcal{F}$, $f_0^*\mathcal{F} = \mathcal{F}_0$, $f_0 \circ f \simeq \mathrm{id}_M$ and $f \circ f_0 \simeq \mathrm{id}_{M_0}$. Let $\widetilde{f}$ and $\widetilde{f_0}$ be lifts of f and f_0 to the universal covers. Since M and M_0 are compact, by using $\widetilde{f_0}$, we can show that $\widetilde{f}$ is a quasi-isometry on each leaf. Thus $\widetilde{f}$ induces a $\pi_1 M$-equivariant homeomorphism $\partial\widetilde{f} : \partial\widetilde{\mathcal{F}} \to \partial\widetilde{\mathcal{F}_0}$ which is quasi-conformal on the geodesic boundary of each leaf. The $\pi_1 M$-equivalence of $\partial\widetilde{f}$ and Proposition 3.5 imply that $\partial\widetilde{f}$ is conformal on the geodesic boundary of each leaf. Since $H = \mathrm{Isom}\, X$, for each $g \in G$, there is a unique way to extend $\partial\widetilde{f}|_{\partial L(g)}$ to an isometry on L_g. It is easy to see that, by this extension, we get a well-defined $\pi_1 M$-equivariant diffeomorphism $\widetilde{f_1} : \widetilde{M} \to \widetilde{M_0}$. Thus the proof is concluded.

The case where X is an irreducible symmetric space of rank one. Now X is one of the following: $\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$ and $\mathbf{H}^2_{\mathbb{O}}$. In the case where $X = \mathbf{H}^n_{\mathbb{C}}$ $(n \geq 2)$, Theorem 1.4 is proved in a way similar to the real hyperbolic case by replacing $\mathbf{H}^n_{\mathbb{R}}$ with $\mathbf{H}^n_{\mathbb{C}}$ and by using quasi-conformal mappings over $\mathbb{C}$ (see [19, Section 21]).

If $X = \mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$, then Theorem 1.4 is proved in a way simpler than the above two cases thanks to the following result of Pansu.

Theorem 3.6 ([20]). *For any quasi-isometry φ on $\mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$, there exists an isometry φ_1 such that $\varphi \circ \varphi_1^{-1}$ is bounded.*

By this theorem, we can skip Step II. In Step III, we get a homomorphism $\pi_1 M \to H$ without Step II. In the last step, we do not need to show that ∂f is conformal. The rest of the proof is the same.

The case where X is an irreducible symmetric space of rank $r \geq 2$. We refer to [19] for facts used in this paragraph. A *flat* in X is a totally geodesic flat submanifold of dimension r. Let ∂X be the Furstenberg maximal boundary of X, which is defined as a set of asymptotic classes of flats in X. Here ∂X has a structure of a spherical Tits building whose automorphism group $\mathrm{Aut}(\partial X)$ is isomorphic to H. Theorem 1.4 can be proved in this case by replacing the geodesic boundary of hyperbolic spaces to Tits building ∂X. The following is well known.

Proposition 3.7 (see [19, Section 15]). *Any quasi-isometry on X induces an automorphism of Tits building ∂X.*

We define the leafwise boundary $\partial\widetilde{F}$ like in Step I but by replacing geodesics with flats. We skip Step II. By Proposition 3.7, we get a homomorphism $\pi_1 M \to H$ in Step III without Step II. To show the discreteness Γ in $H \times G$ in Step IV, we need to use the following result instead of Morse lemma:

Theorem 3.8 (A special case of [11, Theorem 1.1.3]). *Let Z be an irreducible symmetric space of noncompact type of rank greater than one. Then, for any bi-Lipschitz self-map with Lipschitz constant C on Z, there exists a homothety on Z at distance less than S, where S is a function of C.*

In the last step, we do not need to show that ∂f is conformal. The rest of the proof of Theorem 1.4 is the same as the case where $X = \mathbf{H}^n_{\mathbb{R}}$.

The general case. By a theorem of Kapovich-Kleiner-Leeb [10], for a quasi-isometry ϕ on $\prod_{i=1}^{\ell} X_i$, there exists a quasi-isometry ϕ_i for each i such that $p_i \circ \phi$ is equal to $\phi \circ p_i$ up to a bounded error. If X_i is $\mathbf{H}^n_{\mathbb{H}}$, $\mathbf{H}^2_{\mathbb{O}}$ or an irreducible symmetric space of rank greater than one for any i, then we finish the proof by applying the above argument to each component.

Assume that $X_i = \mathbf{H}^n_{\mathbb{R}}$ $(n \geq 3)$ or $\mathbf{H}^n_{\mathbb{C}}$ $(n \geq 2)$ for some i. Then, in Step II, we need to show that the $\pi_1 M$-action on the geodesic boundary ∂X_i is ergodic. We consider a subfoliation $\mathcal{F}_i$ of $\mathcal{F}$ which is defined by the X_i-factor in each leaf of $\mathcal{F}$. Since the X_i-factor is determined by the holonomy of the given smooth metric, $\mathcal{F}_i$ is a smooth foliation. Let $H_i = \mathrm{X}_i$ and take a subgroup K_i so that $X_i = H_i/K_i$. Since each leaf of $\mathcal{F}_i$ is isometric to X_i, we have a canonical principal K_i-bundle $W_i \to M$. We can lift the foliation $\mathcal{F}'_i$ horizontally to get an $(H' \times G)$-Lie foliation on W_i, where $H' = H/H_i$. By the structure theorem of Lie foliations [17, Theorem 4.2], the closure of a leaf is a submanifold M_i of M. Here $(M_i, \mathcal{F}'_i|_{M_i})$ is a minimal Lie foliation whose leaves are isometric to X_i. We apply the above Step II for $(M_i, \mathcal{F}'_i|_{M_i})$ to show the ergodicity of the $\pi_1 M_i$-action on the geodesic boundary of a leaf of $\mathcal{F}_i$. This implies that the $\pi_1 M$-action on the geodesic boundary of a leaf of $\mathcal{F}_i$ is ergodic. Applying this argument for each i such that X_i is of rank one, we can get a homomorphism $\pi_1 M \to \operatorname{Isom} \prod X_i = H$ in Step III. The rest of the proof is the same.

4. Proof of Proposition 1.5.

Let $G = L \ltimes R$ be the Levi decomposition of G, where L is semisimple with trivial center and R is solvable and normal in G. By the assumption that the leaves of $\mathcal{F}$ are simply-connected, the H-action on $H \times G/\Gamma$ is free.

Then, by [24, Lemma 5.2], $\Gamma \cap R$ is discrete. Since $R \cap \Gamma$ is discrete and normal in $H \times G$, $R \cap \Gamma$ is central in $H \times G$. Thus, by taking quotient of G and Γ by $R \cap \Gamma$, the proof of Proposition 1.5 can be reduced to the case where $R \cap \Gamma$ is trivial. Let $L = SK$ be the decomposition of L such that $\mathrm{Lie}(S)$ is the sum of noncompact semisimple Lie algebras and $\mathrm{Lie}(K)$ is the sum of compact semisimple Lie algebras. Since Γ is a cocompact lattice of $H \times G$, by a consequence of Auslander's theorem [22, Theorem E.10], $R \cap \Gamma$ is a cocompact lattice of KR. Thus R is compact, hence the identity component R_0 is abelian.

We will show that the projection of Γ to any connected normal simple subgroup of $H \times G$ is dense. Let $p : H \times G \to H \times G/KR$ be the projection. Since KR is compact, $p(\Gamma)$ is a lattice of $H \times G/KR$. Then, since $H \times G/KR$ is a semisimple group without connected compact subgroup, by a well known result (see [21, Theorem 5.22]), $p(\Gamma)$ has a finite index subgroup T such that $T = \prod_{i=1}^{m} T_i$, where T_i is an irreducible lattice of a product of some connected normal simple subgroup of $H \times G/KR$. Since the leaves of $\mathcal{F}$ is simply-connected, the restriction of the projection $H \times G \to G$ to Γ is injective. Hence we get $m = 1$ and S is an irreducible lattice of $H \times G/KR$, which implies that so is $p(\Gamma)$ (see [21, Corollary 5.21]). Then the projection of Γ to any normal simple subgroup of $H \times G$ is dense.

To show that G is semisimple, it suffices to show that R is finite. Since R_0 is abelian, the kernel of $R \to G/[G, G]$ is finite. On the other hand, since Γ is a lattice of $H \times G$ and the projection of Γ to any normal simple subgroup of $H \times G$ is dense, a vanishing theorem of Starkov [23] implies that $\Gamma/[\Gamma, \Gamma] = 0$. Since Γ is dense in G, it follows that $G/[G, G] = 0$. Hence R is finite.

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Quasi-invariant measures for non-discrete groups on S^1

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Whereas the discussion is primarily conducted for finitely generated subgroups of $\mathrm{Diff}^{\omega}(S^1)$, almost all results can be adapted to similar subgroups of smooth diffeomorphism of the circle. In terms of higher dimensional manifolds, however, our results will only admit convenient generalizations if the corresponding group of diffeomorphisms are supposed to contain a Morse-Smale element (see [Re-2]).

Consider then a finitely generated subgroup G of $\mathrm{Diff}^{\omega}(S^1)$. The group G is said to be *non-discrete* if it contains a sequence of elements $\{h_i\}$ converging to the identity in the (say) C^{∞}-topology (and such that $h_i \neq \mathrm{id}$ for every $i \in \mathbb{N}$). Concerning the existence of non-discrete groups as before, the following result due to Ghys may be quoted.

Theorem (Ghys [Gh]). *Consider the group* $\mathrm{Diff}^{\omega}(S^1)$ *equipped with the analytic topology. Then there is a neighborhood* $\mathcal{U}$ *of the identity such that every non-solvable group* $G \subset \mathrm{Diff}^{\omega}(S^1)$ *generated by a finite set* $S \subset \mathcal{U}$ *is non-discrete.*

Our purpose will be to study some subtle aspects of the ergodic theory of non-discrete groups as above. In particular, we would like to investigate the structure of quasi-invariant measures (always supposed to be non-atomic) with special interest in the case of stationary measures. Stationary measures are defined as follows. Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a finitely generated group equipped with a probability measure ν which is non-degenerate in the sense that its support generates G as semi-group. A probability measure μ on S^1 is said to be *stationary* for G with respect to ν if the equation

$$\mu(\mathcal{B}) = \sum_{g \in G} \nu(g)\mu(g^{-1}(\mathcal{B})) \; . \tag{1}$$

holds for every Borel set $\mathcal{B} \subset S^1$. A simple adaptation of Krylov-Bogoloubov theorem suffices to ensure that stationary measures always exist. Also, assuming that G has no invariant measure, it easily follows that every stationary measure is quasi-invariant and gives no mass to points. Moreover, this measure is often unique as proved by Deroin, Kleptsyn and Navas:

Theorem (Deroin-Kleptsyn-Navas [DKN]). *Suppose that G is a group of diffeomorphisms of S^1 leaving no probability measure on S^1 invariant. If G is equipped with a non-degenerate probability measure ν, then the resulting stationary measure μ is unique.*

Thus, whereas every theorem valid for quasi-invariant measures will automatically hold for the stationary measure, many easy constructions of singular quasi-invariant measures do not apply to stationary measures. Similarly it is easy to produce several examples of singular measures that are quasi-invariant by non-discrete groups as above but most of the corresponding constructions yield measures that are certainly not stationary. Yet, singular stationary measures for groups as above do exist and the difficulty of providing a criterion to ensure that stationary measures must be regular is illustrated by the following theorem due to Kaimanovich and Le Prince [K-LP]: every Zariski-dense finitely generated subgroup of $\mathrm{PSL}\,(2,\mathbb{R})$ can be equipped with a non-degenerate measure ν giving rise to a singular stationary measure μ on S^1.

Besides stationary measures, Patterson-Sullivan measures are among the best known examples of quasi-invariant measures (in the case for Fuchsian or Kleinian groups). A very distinguished feature of Patterson-Sullivan measure is its *d-quasiconformal character*. Given $d \in \mathbb{R}_+^*$, recall that a probability measure μ on S^1 is said to be d-quasiconformal for G if there exists a constant C such that, for every Borel set $\mathcal{B} \subset S^1$ and every $g \in G$, we have

$$\frac{1}{C}|g'(x)|^d \leq \frac{d\mu}{dg_*\mu}(x) \leq C|g'(x)|^d\,.$$

In particular, d-quasiconformal measures are closely related to the d-dimensional Hausdorff measure. Also we can wonder whether d-quasiconformal measures on the circle exist beyond the class of Fuchsian groups, whether or not we are dealing with “discrete groups”. In fact, this problem can be viewed as a far reaching extension of Patterson-Sullivan theory. Concerning the case of non-discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$, we have the following theorem.

Theorem (Uniqueness of Lebesgue, [Re-1]). *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a finitely generated non-solvable and non-discrete group. Assume also that G has no finite orbit and that μ is a d-quasiconformal measure for G. Then μ is absolutely continuous and $d = 1$.*

Since we mentioned that some statements for the circle generalize to higher dimensions in the presence of a Morse-Smale dynamics, here it is a good point to give an example of these generalizations. To keep statements as simple as possible, consider non-discrete groups of analytic diffeomorphism of the sphere S^2. Since these diffeomorphisms need not be conformal, the notion of d-quasiconformal measures can be replaced by the following definition:

Definition. Let μ be a probability measure on S^2 and consider a group $G \subset \mathrm{Diff}^{\omega}(S^2)$. Given $d \in \mathbb{R}_+^*$, the measure μ will be called a d-quasi-volume for G if there is a constant C such that for every point $x \in S^1$ and every element $g \in G$, the Radon-Nikodym derivative $d\mu/dg_*\mu$ satisfies the estimate

$$\frac{1}{C}\|\mathrm{Jac}\,[Dg](x)\|^d \leq \frac{d\mu}{dg_*\mu}(x) \leq C\|\mathrm{Jac}\,[Dg](x)\|^d\,,$$

where $\mathrm{Jac}\,[Dg](x)$ stands for the Jacobian determinant of Dg at the point x.

The more general statements in [Re-2] imply the following:

Theorem ([Re-2]). *Suppose that $G \subset \mathrm{Diff}^{\omega}(S^2)$ is non-discrete and contains a Morse-Smale element. Suppose also that G leaves no proper analytic subset of S^2 invariant. Then every d-quasi-volume μ for G is absolutely continuous (in particular $d = 2$).*

Going back to the circle, it was observed that d-quasiconformal measures behave similarly to the d-dimensional Hausdorff measure. Since the examples in Kaimanovich-Le Prince [K-LP] possess Hausdorff dimension comprised between 0 and 1, it is natural to wonder whether these stationary measures are comparable to Hausdorff measures of same dimension. To help to make sense of these possible comparisons, the notion of *Lusin sequences* can be used. A *Lusin sequence* for a probability measure μ consists of a sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ such that $\mu(K_n) \to 1$. The advantage of using Lusin sequences is to work with compact sets as opposed to general Borel sets while recovering the standard definitions of Hausdorff measures/dimensions and so on. Denoting by μ_d the d-dimensional Hausdorff measure, we have:

Theorem ([Re-1]). *Let $G \subset \mathrm{Diff}^{\omega}(S^1)$ be a finitely generated non-solvable and non-discrete group. Suppose that μ is an ergodic (non-atomic) singular quasi-invariant measure for G whose Hausdorff dimension d belongs to $(0, 1]$. Denoting by μ_d the d-dimensional Hausdorff measure, the following alternative holds:*

- *Either there is a Lusin sequence $\{K_n\}$ for μ such that $\mu_d(K_n) = 0$ for every n or*
- *Every Lusin sequence $\{K_n\}$ verifies $\mu_d(K_n) \to \infty$ when $n \to \infty$.*

In particular, the Borel set $K = \bigcup_{n=1}^{\infty} K_n$ is such that $\mu(K) = 1$ and $\mu_d(K)$ is either zero or infinite.

If time permits, we shall conclude with a more detailed discussion of stationary measures along with some regularity criteria for them.

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Generic pseudogroups on $(\mathbb{C}, 0)$ and the topology of leaves

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This is an extended abstract for the material in the papers [MRR] and [RR] jointly with J.-F. Mattei and J. Rebelo.

In the study of some well-known problems about singular holomorphic foliations, we usually experience difficulties concerning to greater or lesser extent the topology of their leaves. Yet, most of these problems are essentially concerned with pseudogroups generated by certain local holomorphic diffeomorphisms defined on a neighborhood of $0 \in \mathbb{C}$. In this sense, results about pseudogroups of $\mathrm{Diff}\,(\mathbb{C}, 0)$ generated by a finite number of local holomorphic diffeomorphisms are crucial for the understanding of certain singular foliations defined about the origin of $\mathbb{C}^2$. Also, as it will be seen below, for most of these problems it is necessary to consider classes of pseudogroups with a distinguished generating set all of whose elements have fixed conjugacy class in $\mathrm{Diff}\,(\mathbb{C}, 0)$.

In the above mentioned works, some well-known questions about singular holomorphic foliations on $(\mathbb{C}^2, 0)$ are answered. These questions have first arisen as an outgrowth of the problem of classifying germs of plane analytic curves (Zariski problem). The key to answer them will be the introduction of a theory of pseudogroups obtained out of "generic" elements in $\mathrm{Diff}\,(\mathbb{C}, 0)$ having fixed conjugacy class. We shall explain these problems before presenting our main results.

Recall that a local singular holomorphic foliation on a neighborhood of $(0, 0) \in \mathbb{C}^2$ is nothing but the foliation induced by the local orbits of a holomorphic vector field having isolated singularities and defined on the mentioned neighborhood. In particular singular points of a foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ are always isolated and, besides, two holomorphic vector fields representing $\mathcal{F}$ differ by an invertible multiplicative holomorphic function. Assume that the origin is a singular point for a given foliation $\mathcal{F}$ and let X be a representative of $\mathcal{F}$. The eigenvalues of $\mathcal{F}$ at the origin correspond to the eigenvalues of the linear part of X at the same point. It is well-known that every foliation on $(\mathbb{C}^2, 0)$ can be transformed by a finite sequence of blow-up maps into a new foliation $\widetilde{\mathcal{F}}$ possessing singularities that are "simple", i.e. $\widetilde{\mathcal{F}}$ has at least one eigenvalue different from zero at each of its singular points. This sequence of blow-up maps leading to $\widetilde{\mathcal{F}}$ is called the *resolution of* $\mathcal{F}$.

The study of singularities of foliations and of their deformations, paralleling Zariski problem, led to the introduction of the *Krull topology* in the space of these foliations. In this topology, a sequence of foliations $\mathcal{F}_i$ is said to converge to $\mathcal{F}$ if there are representatives X_i for $\mathcal{F}_i$ and X for $\mathcal{F}$ such that X_i is tangent to X, at the origin, to arbitrarily high orders (modulo choosing i large enough). It should be noted that, given a foliation $\mathcal{F}$, its resolution depends only on a finite jet of the Taylor series of X at the singular point. Therefore, if $\mathcal{F}'$ is close to $\mathcal{F}$ in the Krull topology, then these foliations admit exactly the same resolution. Furthermore the position of the singularities of the resolved foliations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ coincide and so do their corresponding eigenvalues.

A prototypical problem in this direction that will also help us to clarify the contents of the above discussion is provided by the nilpotent foliations associated to Arnold singularities A^{2n+1}. These are local foliations $\mathcal{F}$ defined by a (germ of) vector field X having nilpotent linear part, i.e. $X = y\partial/\partial x + \cdots$, and a unique separatrix S that happens to be a curve analytically equivalent to $\{y^2 - x^{2n+1} = 0\}$. Let us discuss the simplest case $n = 1$ in detail (the general case is very similar).

Consider a nilpotent foliation $\mathcal{F}$ associated to Arnold singularity A^3, i.e. a nilpotent foliation admitting a unique separatrix that happens to be a curve analytically equivalent to $\{y^2 - x^3 = 0\}$. For this type of foliation, the desingularization of the separatrix coincides with the resolution of the foliation itself. More precisely, the map associated to the desingularization of the separatrix $E_S : M \to \mathbb{C}^2$ reduces also the foliation $\mathcal{F}$ (see Figure 1 for the corresponding resolution).

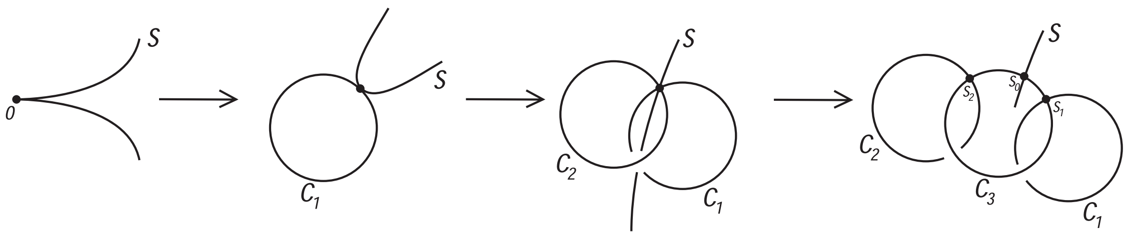

Figure 1

The corresponding exceptional divisor $\mathcal{D} = E_S^{-1}(0)$ consists of the union of 3 rational curves as indicated in Figure 1. The singular points of $\widetilde{\mathcal{F}}$ are the intersection points of consecutive components in the tree along with a point s_0 that corresponds to the intersection of the transformed of the separatrix with $E_S^{-1}(0)$. This intersection takes place in the component C_3 as indicated in Figure 1. All these singular points possess two eigenvalues different from zero. The corresponding eigenvalues can precisely be determined by using the self-intersection of the various components of the exceptional divisor.

For example, the eigenvalues of $\widetilde{\mathcal{F}}$ at s_1 are $1, -3$ whereas the eigenvalues of $\widetilde{\mathcal{F}}$ at s_2 are $1, -2$.

It should be noted that the regular leaf $C_1 \setminus \{s_1\}$ is isomorphic to $\mathbb{C}$ and thus simply connected. This implies that the local holonomy map associated to a path contained in $C_1 \setminus \{s_1\}$ and winding around s_1 coincides with the identity. This assertion combined with the fact that $\widetilde{\mathcal{F}}$ has eigenvalues $1, -3$, guarantees that the germ of $\widetilde{\mathcal{F}}$ at s_1 is linearizable. Thus the local holonomy map f associated to a small loop about s_1 and contained in C_3 must be of finite order equal to 3, i.e. it is conjugate to a rotation of order 3. A similar discussion applies to the component C_2 and leads to the conclusion that the local holonomy map g associated to a small loop around s_2 and contained in C_3 has order equal to 2, i.e. it is conjugate to a rotation of order 2. Since $C_3 \setminus \{s_0, s_1, s_2\}$ is a regular leaf of $\widetilde{\mathcal{F}}$, we conclude that the (image of the) holonomy representation of the fundamental group of $C_3 \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}\,(\mathbb{C}, 0)$ is nothing but the group generated by f, g. The reader will easily convince himself/herself that the dynamics of this holonomy group encodes all the information about the corresponding foliation.

It should be noted that the conclusion above depends only on the configuration of the reduction tree which, in turn, is determined by a finite jet of the Taylor series of X at the singular point. Hence, if the coefficients of Taylor series of the vector field X are perturbed starting from a sufficiently high order, the new resulting vector field X' will still give rise to a foliation whose singularity is reduced by the same blow-up map associated to the divisor of Figure 1. In particular, the holonomy representation of the fundamental group of $C_3 \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}\,(\mathbb{C}, 0)$, obtained from this new foliation, is still generated by two elements of $\mathrm{Diff}\,(\mathbb{C}, 0)$ having finite orders respectively equal to 2 and to 3. Since every local diffeomorphism of finite order as above is conjugate to the corresponding rotation, it follows in particular that their conjugacy classes in $\mathrm{Diff}\,(\mathbb{C}, 0)$ are fixed.

From what precedes, it follows that whenever $\mathcal{F}$ is a foliation as above and $\mathcal{F}'$ is close to $\mathcal{F}$ in the Krull topology, then $\mathcal{F}'$ is also a nilpotent foliation of type A^3. It is then natural to wonder what type of dynamical behavior can be expected from these foliations, or more precisely, from a "typical" foliation in this family. Inasmuch the space of foliations was endowed with the Krull topology, which fails to have the Baire property, questions about "dense sets of foliations" can still be asked. The following is an example of long-standing problem in the area:

QUESTION. Does there exist a nilpotent foliation $\mathcal{F}$ in A^3 whose leaves are simply connected (apart maybe from a countable set)? Is the set of these foliations dense in the Krull topology, i.e. given a nilpotent foliation $\mathcal{F}$ in A^3, does there exist a sequence of foliations $\mathcal{F}_i$ converging to $\mathcal{F}$ in the Krull topology and such that every $\mathcal{F}_i$ has simply connected leaves (with possible exception of a countable set of leaves)?

Our methods are powerful enough to affirmatively settle both questions above. A crucial point is the understanding of groups generated by f, g at level of pseudogroup and not only at germ level. In fact, the local dynamics of the holonomy pseudogroup arising from the leaf $C_3 \setminus \{s_0, s_1, s_2\}$ on a *fixed* neighborhood of $0 \in \mathbb{C}$ must be studied.

In the case of nilpotent foliations in the class A^3, it was seen that pseudogroups given by generating sets with elements possessing fixed conjugacy classes play a central role in the description of the corresponding foliations. This phenomenon is not peculiar to the mentioned family of foliations and, indeed, appears quite often. To have a better insight in the nature of the mentioned phenomenon, suppose we are given a foliation $\mathcal{F}$ and consider $\mathcal{F}'$ very close to $\mathcal{F}$ in the Krull topology. In particular, the resolutions $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ of $\mathcal{F}$, $\mathcal{F}'$ turn out to coincide. The positions of the singular points of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ in the common exceptional divisor coincide as well and so do their corresponding eigenvalues. Suppose now that $\widetilde{\mathcal{F}}$ has only hyperbolic singularities i.e. the singularities of $\widetilde{\mathcal{F}}$ have two eigenvalues different from zero and such that their quotient lies in $\mathbb{C} \setminus \mathbb{R}$. The same holds for $\widetilde{\mathcal{F}}'$ since corresponding singularities of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ have the same eigenvalues. By Poincaré theorem, both singularities are then conjugate to the corresponding linear model and, thus, they are conjugate to each other. Thus the corresponding local holonomy maps arising from a small loop encircling the singularity in question are themselves conjugate by a local diffeomorphism. In other words, the pseudogroups generated by these holonomy maps for $\widetilde{\mathcal{F}}$ and for $\widetilde{\mathcal{F}}'$ naturally have generating sets whose elements have the same conjugacy classes. The latter are, indeed, fixed since it corresponds to the class of a hyperbolic element of $\mathrm{Diff}\,(\mathbb{C}, 0)$ with fixed multiplier.

Having explained the need for considering pseudogroups with generating sets all of whose elements possess a fixed conjugacy class in $\mathrm{Diff}\,(\mathbb{C}, 0)$, we can now proceed to state our main results. Let us begin with the results concerning pseudogroups generated by a finite number of elements in $\mathrm{Diff}\,(\mathbb{C}, 0)$ which will later allow us to answer the above stated questions on nilpotent foliations. For this, let us equip $\mathrm{Diff}\,(\mathbb{C}, 0)$ with the so-called analytic topology, that was first considered by Takens in the context of real diffeomorphisms of an analytic manifold and further discussed in the case of $\mathrm{Diff}\,(\mathbb{C}, 0)$ in [MRR]. Unlike the Krull topology, the analytic topology has the Baire property. Now, consider a k-tuple of local holomorphic diffeomorphisms $f_1, \ldots, f_k$ fixing $0 \in \mathbb{C}$. The first theorem states that the local diffeomorphisms f_i can be perturbed inside their conjugacy classes so as to generate a pseudogroup isomorphic to the free product of the corresponding cyclic groups. Indeed, the perturbation can be made inside a

G_δ-dense subset of $(\mathrm{Diff}\,(\mathbb{C},0))^k$. Also, it can be proved that the mentioned perturbation can be made inside the class of diffeomorphisms tangent to the identity to every a priori fixed order (which for technical reasons is also necessary to solve the corresponding questions on foliations). More precisely, letting $\mathrm{Diff}_\alpha(\mathbb{C},0)$ stand for the normal subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$ consisting of elements tangent to the identity to order α, we have the following:

Theorem A ([MRR]). *Fixed $\alpha \in \mathbb{N}$, let $f_1,\dots,f_k$ be given elements in* $\mathrm{Diff}\,(\mathbb{C},0)$ *and consider the corresponding cyclic groups $G_1,\dots,G_k$. Then, there exists a G_δ-dense set $\mathcal{V} \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^k$ such that, whenever $(h_1,\dots,h_k) \in \mathcal{V}$, the following holds:*

(1) *The group generated by $h_1^{-1}\circ f_1\circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k$ induces a group in $\mathrm{Diff}\,(\mathbb{C},0)$ that is isomorphic to the free product $G_1 * \cdots * G_k$.*

(2) *Let $f_1,\dots,f_k$ and $h_1,\dots,h_k$ be identified to local diffeomorphisms defined about $0 \in \mathbb{C}$. Suppose that none of the local diffeomorphisms $f_1,\dots,f_k$ has a Cremer point at $0 \in \mathbb{C}$. Denote by Γ^h the pseudogroup defined on a neighborhood V of $0 \in \mathbb{C}$ by the mappings $h_1^{-1}\circ f_1\circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k$, where $(h_1,\dots,h_k) \in \mathcal{V}$. Then V can be chosen so that, for every non-empty reduced word $W(a_1,\dots,a_k)$, the element of Γ^h associated to $W(h_1^{-1}\circ f_1\circ h_1,\dots,h_k^{-1}\circ f_k\circ h_k)$ does not coincide with the identity on any connected component of its domain of definition.*

Item (1) of the previous result concern groups at the germ level, while item (2) concerns pseudogroups. Note that the assumption that none of the fixed diffeomorphisms $f_1,\dots,f_k$ has a Cremer point at $0 \in \mathbb{C}$ is not necessary for the first conclusion of Theorem A. This assumption is, however, indispensable for the second item due to certain examples of dynamics near Cremer points that were constructed by Perez-Marco.

Item (2) ensures the existence of a point p possessing an infinite orbit of hyperbolic fixed points for the pseudogroup Γ^h. In other words, p has an infinite orbit under Γ^h and, for every point q lying in the orbit of p, there is an element $g \in \Gamma^h$ for which q is a hyperbolic fixed point (i.e. $\|g'(q)\| \neq 0$). In fact, the existence of this type of point p associated to a pseudogroup whose germ at $0 \in \mathbb{C}$ is not solvable has been known for a while (see [Lo] and their references). However the question on whether or not these pseudogroups exhibit more than one single orbit of hyperbolic "fixed points", at least in the case of "typical" pseudogroups, has remained open. In [RR], we provide "generic" answers for this question and for the question on the nature of the stabilizer of points $p \neq 0$. This is as follows:

Theorem B ([RR]). *Suppose we are given f, g in α and denote by D an open disc about $0 \in \mathbb{C}$ where f, g and their inverses are defined. Assume that* *none of the local diffeomorphisms f, g has a Cremer point at $0 \in \mathbb{C}$. Then, there is a G_δ-dense set $\mathcal{U} \subset \mathrm{Diff}_\alpha(\mathbb{C}, 0) \times \mathrm{Diff}_\alpha(\mathbb{C}, 0)$ such that, whenever (h_1, h_2) lies in $\mathcal{U}$, the pseudogroup Γ_{h_1,h_2} generated by $\tilde{f} = h_1^{-1} \circ f \circ h_1$, $\tilde{g} = h_2^{-1} \circ g \circ h_2$ on D satisfies the following:*

(1) *The stabilizer of every point $p \in D$ is either trivial or cyclic.*
(2) *There is a sequence of points $\{Q_i\}$, $Q_i \neq 0$ for every $i \in \mathbb{N}^*$, converging to $0 \in \mathbb{C}$ and such that every Q_n is a hyperbolic fixed point of some element $W_i(\tilde{f}, \tilde{g}) \in \Gamma_{h_1,h_2}$. Furthermore the orbits under Γ_{h_1,h_2} of Q_{n_1}, Q_{n_2} are disjoint provided that $n_1 \neq n_2$.*

Let us now show how the previous theorems can be translated in terms of nilpotent foliations in the class A^{2n+1}. The above conducted discussion can be expanded to show the existence of an injection from the set of nilpotent foliations associated to Arnold singularities A^{2n+1} in the space of subgroups of Diff $(\mathbb{C}, 0)$ generated by two diffeomorphisms such that one of them has order 2 and the other has order $2n + 1$. Denote by Γ the pseudogroup generated by f, g on a neighbourhood V of $0 \in \mathbb{C}$. A necessary condition for a foliation as above to have simply connected leaves (up to a countable set of them), is that every element on Γ cannot coincide with the identity on any connected component of its domain of definition. Owing to Theorem A, the diffeomorphisms f, g can be perturbed into $\tilde{f} = h_1^{-1} \circ f \circ h_1$ and $\tilde{g} = h_2^{-1} \circ g \circ h_2$ so as to satisfy this condition. It remains the problem of realizing these diffeomorphisms as the generators of the holonomy of another nilpotent foliation associated to the Arnold singularity A^{2n+1}. In this direction, we proved that the existence of an actual correspondence between the space of these foliations and the space of subgroups of Diff $(\mathbb{C}, 0)$ generated by two holomorphic diffeomorphims conjugate to the rotations of order 2 and order $2n + 1$ (cf. [MRR]).

To formulate our statement in terms of "Krull denseness", as in the original questions, let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a holomorphic vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}, in particular $\mathcal{F}$ possesses one unique separatrix. Now by putting together the construction in [MRR] with Theorems A and B above, we obtain:

Theorem C ([MRR, RR]). *Let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}. Then, for every $N \in \mathbb{N}$, there exists a vector field $X' \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ defining a germ of foliation $\mathcal{F}'$ and satisfying the following conditions:*

(a) $J_0^N X' = J_0^N X$.
(b) *$\mathcal{F}$ and $\mathcal{F}'$ have S as a common separatrix.*

(c) *there exists a fundamental system of open neighborhoods* $\{U_j\}_{j\in\mathbb{N}}$ *of* S*, inside a closed ball* $\bar{B}(0,R)$*, such that the following holds for every* $j\in\mathbb{N}$*:*

(c1) *The leaves of the restriction of* $\mathcal{F}'$ *to* $U_j\setminus S$, $\mathcal{F}'|_{(U_j\setminus S)}$ *are simply connected except for a countable number of them.*

(c2) *The countable set constituted by non-simply connected leaves is, indeed, infinite.*

(c3) *Every leaf of* $\mathcal{F}'|_{(U_j\setminus S)}$ *is either simply connected or homeomorphic to a cylinder.*

The item (c1) in Theorem C appears already in [MRR] whereas items (c2) and (c3) require Theorem B proved in [RR]. The realization of pseudogroups as in the statement of Theorems A and B as holonomy pseudogroups of nilpotent foliations was carried out in [MRR] and relies heavily on the techniques of [MS].

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On the space of left-orderings of solvable groups

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1. Introduction

A left-orderable group is a group Γ which admits a total ordering $\preceq$ invariant under multiplications, that is, $f \prec g \Rightarrow hf \prec hg$ for all $f, g, h \in \Gamma$. Equivalently Γ is left-orderable if we can find $P \subset \Gamma$ satisfying

i) $P\,P \subseteq P$, so P is a semigroup.

ii) $\Gamma = P \sqcup P^{-1} \sqcup \{id\}$, where the unions are disjoint.

The set P is usually called the positive cone of an ordering $\preceq$, since the equivalence between the two above definitions is given by $P_{\preceq} = \{f \in \Gamma \mid f \succ id\}$.

Given a left-orderable group Γ, we shall denote by $\mathcal{LO}(\Gamma)$ its associated space of left-orderings, which consists of all possible left-orderings on Γ. A natural topology can be put in $\mathcal{LO}(\Gamma)$ by considering the inclusion $P \mapsto \chi_P \in \{0,1\}^{\Gamma}$, where χ_P denotes the characteristic function over P, and the topology on $\{0,1\}^{\Gamma}$ is the product topology. In this way, we have that two left-orderings are close if they coincide on a large finite set. Moreover, one can check that the inclusion $\mathcal{LO}(\Gamma) \to \{0,1\}^{\Gamma}$ is closed, hence proving

Theorem 1.1 (Sikora [12])**.** *With the above topology, $\mathcal{LO}(\Gamma)$ is compact and totally disconnected. Moreover, if Γ is countable, then $\mathcal{LO}(\Gamma)$ is metrizable.*

It is interesting to observe that if Γ is countable and $\mathcal{LO}(\Gamma)$ has no isolated left-orderings, then $\mathcal{LO}(\Gamma)$ is homeomorphic to the Cantor set. The problem of relating the topology of $\mathcal{LO}(\Gamma)$ with the algebraic structure of Γ has been of increasing interest since the discovery by Dubrovina and Dubrovin that the space of left-orderings of the braid groups is infinite and yet contains isolated points [2]. Recently, more examples of groups showing these two behaviors have appeared in the literature [1, 4, 5, 8]. Although all this groups contain free subgroups, it is known that non-trivial free products of groups have no isolated left-orderings [10]. In the same spirit, it is a result of Navas [7], that for finitely generated groups with subexponential growth (*e.g.* nilpotent groups), the associated space of left-orderings is either finite or homeomorphic to the Cantor set.

2. Main results

In this talk I will try to convince you of the following result.

Theorem A(Rivas-Tessera [11]): *The space of left-orderings of a countable virtually solvable group is either finite or homeomorphic to a Cantor set.*

There are at least three main ingredients, the first one being the notion of convex subgroup of an ordered group (see for instance [6]).

Definition 2.1. A subset C of a left-ordered group $(\Gamma, \preceq)$ is *convex* if the relation $c_1 \prec f \prec c_2$, for c_1 and c_2 in C, implies that $f \in C$.

For us, the main utility of this notion is the following

Proposition 2.2. *Let $\preceq$ be a left-ordering on Γ and let H be a convex subgroup. Then there is a continuous injection $\mathcal{LO}(H) \to \mathcal{LO}(\Gamma)$, having $\preceq$ in its image. Moreover, if in addition H is normal, then there is a continuous injection $\mathcal{LO}(H) \times \mathcal{LO}(G/H) \to \mathcal{LO}(G)$ having $\preceq$ in its image.*

Therefore, to prove Theorem A, given a left-ordering $\preceq$ it is enough to find subgroup H that is convex for $\preceq$ and such that $\mathcal{LO}(H)$ has no isolated left-orderings, or such that H is normal and $\mathcal{LO}(\Gamma/H)$ has no isolated left-orderings.

The second main ingredient is the following nice characterization of left-orderability (see [3])

Proposition 2.3. *For a countable group Γ, the following assertions are equivalent*

- Γ *is left-orderable.*
- Γ *acts faithfully by order preserving homeomorphisms of the real line.*

This puts at our disposal the strong machinery of group actions on the real line. For instance, of mayor importance for us will be the following theorem.

Theorem 2.4 (Plante [9]). *Every finitely generated nilpotent group of $Homeo_+(\mathbb{R})$, acting without global fixed point, preserves a measure on the real line, which is finite on compact sets and has no atoms (a Radon measure for short).*

Finally, the last main ingredient is the notion of Conradian orderings. Recall that a left-ordering $\preceq$ is called Conradian, if in addition it satisfies that $f \succ id$, $g \succ id \Rightarrow fg^2 \succ id$. What it is so important about Conradian orderings is their nice dynamical counterpart discovered by Navas in [7].

Theorem 2.5 (Navas [7]). *Let $\preceq$ be a Conradian ordering on a group Γ. Then, the action on the real line associated to $\preceq$ is an action without crossings.*

The easiest definition of a crossing is the following picture.

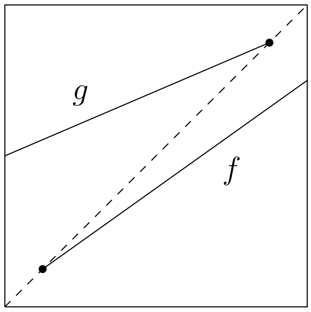

Figure 1: The graphs of the crossed homeomorphisms f and g.

Equivalently, a group $\Gamma \subset Homeo_+(\mathbb{R})$ is said to acts without crossings, if whenever $f \in \Gamma$ fixes a open interval I_f, but has no fixed point in it, then for any $g \in \Gamma$ we have that

$$g(I_f) \cap I_f = \begin{cases} I_f, \text{ or} \\ \emptyset. \end{cases}$$

This three main ingredient will be put to work together in order to show Theorem A. We shall put some emphasis in the case where Γ is a polycyclic group (that is when Γ is finitely generated solvable, and its successive quotient in the derived series are cyclic), which is the simpler non-trivial incarnation of Theorem A.

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Hopf Conjecture holds for k-basic, analytic Finsler metrics on two tori

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1. Introduction

The theory of metric structures in the torus all of whose geodesics are global minimizers was totally understood in the Riemannian case after the solution of the so-called Hopf conjecture: every Riemannian metric in the torus without conjugate points is flat. This statement was proved by Hopf [12] in the 1940's and by Burago-Ivanov [2] in the early 1990's. However, if we widen our scope to the family of Finsler metrics the theory still poses many interesting, unsolved problems.

Definition 1. Let M be a n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In canonical coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM; y \neq 0\}$ be the complement of the zero section. A C^k $(k \geq 2)$ *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:
(i) F is C^k on TM_0;
(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y) \;\forall\; \lambda > 0$$

(iii) The Hessian matrix of $F^2 = F \cdot F$

$$g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Given a Tonelli Hamiltonian in a compact manifold (i.e., a Hamiltonian that is convex and superlinear in the vertical fibers of the cotangent bundle) the Hamiltonian flow in a sufficiently high enery level can be reparametrized

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to become the geodesic flow of a Finsler metric, so Finsler theory is as general as Hamiltonian theory. Finsler manifolds have curvature tensors which generalize Riemannian curvature tensors, in particular the so-called flag curvature $K(p, v)$ that extends the notion of Riemannian sectional curvature (see [1] for instance for the basic theory of Finsler manifolds). Since the Finsler metric is defined in the tangent bundle of the manifold, the flag curvature depends in general on the vertical variable v. Finsler surfaces where $K(p, v) = K(p)$ are called **k-basic**. Well known examples of non-Riemannian, k-basic Finsler surfaces are given by Randers metrics: those obtained by adding a Riemannian norm and a one form.

Finsler manifolds have geodesics, solutions of the Euler-Lagrange equation defined by the Finsler function F. We say that a complete Finsler manifold has no conjugate points if every geodesic is a global minimizer of the Lagrangian action associated to the Finsler metric (i.e., the Finsler length). Since Busemann examples [3] of non-flat Finsler metrics in the two torus without conjugate points it is known that the Hopf conjecture is false in the Finsler realm. Nevertheless, Finsler metrics in the torus without conjugate points enjoy many properties in common with flat metrics. One of them is their connection with weakly integrable systems in the sense of [11]: there exists a continuous, Lagrangian, invariant foliation by tori of the unit tangent bundle ([6]). The existence of a Lagrangian, C^k invariant foliation of the unit tangent bundle is called in [11] C^k integrability of the geodesic flow of the Finsler metric. Moreover, in all known examples of smooth Finsler metrics without conjugate points ([3], [15] for instance) such foliation is smooth. In the Riemannian case the smoothness of the foliation follows from the rigidity of the metric: since the metric is flat the Riemannian metric is Euclidean. The smoothness of the foliation is not part of the proof of the Hopf conjecture but one of its consequences.

So two questions arise naturally from the above discussion. Do C^0 integrable Finsler geodesic flows on tori are C^k for some $k \geq 1$? Does the C^k integrability of such geodesic flows for $k \geq 1$ play any role in the proof of rigidity results? The first question has been already considered in [6], where it is proved that Lipschitz integrability of the geodesic flow of a Finsler metric on the torus without conjugate points implies C^1 integrability. However, a full answer to the question is still open.

2. Main results

Our main results provide substantial information concerning the above problems. The main contribution of our work is the solution of the Hopf conjecture in the analytic, two dimensional case.

Theorem 1. *An analytic k-basic Finsler metric without conjugate points in the two torus is flat.*

This result is the combination of two results. First of all, the main theorem in [11],

Theorem 2. *$C^{1,L}$ integrable geodesic flows of k-basic Finsler metrics on two tori are flat.*

$C^{1,L}$ means C^1 with Lipschitz first derivatives. The second result [8] deals with the smoothness of the so-called Busemann foliation of Finsler tori without conjugate points.

Theorem 3. *Let (T^2, F) be an analytic, k-basic Finsler metric without conjugate points in the two torus T^2. Then the geodesic flow is analytically integrable, namely, there exists an analytic foliation by invariant tori of the unit tangent bundle of the metric which are graphs of the canonical projection.*

The last Theorem is the first result, as far as we know, to show that a Finsler, non-Riemannian metric in the two torus without conjugate points is smoothly integrable without using geometric rigidity. It is remarkable that in the literature about the link between smoothness of invariant foliations of Hamiltonian flows and geometric rigidity, the most common assumption is hyperbolicity (see for instance [14], [5], [7], [9] with results for surfaces of higher genus. So most of the ideas applied to such manifolds do not hold on tori.

Outline of Proof. Let us give a sketch of the proof of the above results. The proof of Theorem 2 involves the calculation of the Godbillon-Vey number of the Busemann foliation of a Finsler geodesic flow in the two torus provided that the foliation is $C^{1,L}$. This result in itself is very interesting and gives a sort of generalized Gauss-Bonnet formula for Finsler geodesic flows on tori without conjugate points which are smoothly integrable:

Proposition 1. *Let (T^2, F) be a C^∞ Finsler metric without conjugate points whose geodesic flow preserves a codimension 1, $C^{1,L}$ foliation $\mathcal{F}$ of the unit tangent bundle. Then (T^2, F) has no conjugate points and there exists a Riccati operator u associated to the foliation. Moreover, the Godbillon-Vey number of $\mathcal{F}$ is*

$$gv(\mathcal{F}) = \int \eta \wedge d\eta = \int [3(Vu)^2 + u^2]\omega_1 \wedge \omega_2 \wedge \omega_3$$
$$+ \int [4uVJ - 2uXVI - IVK]\omega_1 \wedge \omega_2 \wedge \omega_3,$$

where the integration is taken over T_1M, and

(1) *X is the geodesic vector field, V is a unit vertical field and the triple X, Y, V is a Cartan frame for the unit tangent bundle with $\omega_1, \omega_2, \omega_3$ as their dual Cartan 1-forms.*
(2) *K is the flag curvature, I is the Cartan tensor, and J is the Landsberg tensor.*

This is essentially Proposition 2.1 in [11]. We would like to remark that when the Finsler metric is Riemannian, we get $J = I = 0$ and the Godbillon-Vey formula reduces to Mitsumatsu's formula in [14]. Then we show (Theorem 4.2 in [11]),

Proposition 2. *Let (T^2, F) be a C^∞ Finsler metric without conjugate points whose geodesic flow preserves a codimension 1 C^1 foliation $\mathcal{F}$ in the unit tangent bundle. Then $\mathcal{F}$ is the Busemann foliation and moreover, if $\mathcal{F}$ is $C^{1,L}$ its Godbillon-Vey number is zero.*

Finally, we use Riemann-Finsler geometry to show that when the Finsler metric is k-basic the Landsberg tensor J and the Cartan tensor I are related by the formula

$$J = uI$$

at every point in the unit tangent bundle. Replacing this identity in the Godbillon-Vey formula we get that the Riccati operator must vanish everywhere, and so the flag curvature as well.

The proof of Theorem 3 [8] relies on the application of Riemann-Finsler geometry to link the singularities of the Busemann foliation (as a foliation) with the zeroes of the Cartan tensor in the case of k-basic Finsler metrics. So first of all we show (Proposition 2.2 and Lemma 3.2 in [8])

Proposition 3. *Let (T^2, F) be an analytic k-basic Finsler metric without conjugate points. Then*

(1) *Each leaf of the Busemann foliation is analytic.*
(2) *The Riccati operator associated to Busemann leaves is given by $u = J/I = X(I)/I$ whenever $I \neq 0$.*
(3) *The Busemann foliation is analytic in the set where $I \neq 0$.*

So the study of the analyticity of the Busemann foliation is reduced to show that the function $u = X(I)/I$ has removable singularities in the unit tangent bundle. This is proved in Lemma 3.4 in [8] where we show that the function u has a real analytic extension. $\square$

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Riemannian manifolds not quasi-isometric to leaves in codimension one foliations

Paul A. SCHWEITZER, S.J.

The question of when an open (i.e. noncompact) connected manifold can be realized up to diffeomorphism as a leaf in a foliation of a compact differentiable manifold was first posed by Sondow in 1975 for surfaces in 3-manifolds, and was solved positively for all open surfaces by Cantwell and Conlon [3] in 1987. In the opposite direction, in 1985 Ghys [4] and (independently) Inaba, Nishimori, Takamura and Tsuchiya [5] constructed open 3-manifolds—infinite connected sums of lens spaces—that cannot be leaves in a foliation of a compact 4-manifold. Attie and Hurder [2] in 1996 gave an uncountable family of smooth simply connected 6-dimensional manifolds that are not diffeomorphic to leaves in a compact 7-manifold. It is still an open problem whether every smooth open manifold of dimension greater than 2 is diffeomorphic to a leaf of a foliation of codimension two or greater.

For the related question of when an open Riemannian manifold can be realized up to quasi-isometry as a leaf in a foliation of a compact manifold, Attie and Hurder [2] also produced an uncountable family of quasi-isometry types of Riemannian metrics on the 6-manifold $S^3 \times S^2 \times \mathbb{R}$, each with bounded geometry, which are not quasi-isometric to leaves in any codimension one foliation of a compact 7-manifold. (Note that leaves of a foliation on a compact Riemannian manifold must have bounded geometry, such as injectivity radius and curvatures.) Their results extend to codimension one foliations of dimensions greater than 6, but they asked (Question 2 in [2]) whether there exist examples in the lower dimensions $3, 4$, and 5. We answer this question in the following theorem.

Theorem 1. *Every connected non-compact smooth p-manifold L of dimension $p \geq 2$ possesses C^∞ complete Riemannian metrics g with bounded geometry that are not quasi-isometric to any leaf of a codimension one $C^{2,0}$ foliation on any compact differentiable $(p+1)$-manifold.*

Furthermore g can be chosen such that no end is quasi-isometric to an end of a leaf of such a foliation, and also to have any growth type compatible with bounded geometry. Hence there are uncountably many quasi-isometry classes of such metrics g on every such manifold L.

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Consequently no bounded local geometric invariants of an open Riemannian p-manifold with $p \geq 2$ can be obstructions to its being quasi-isometric to a leaf.

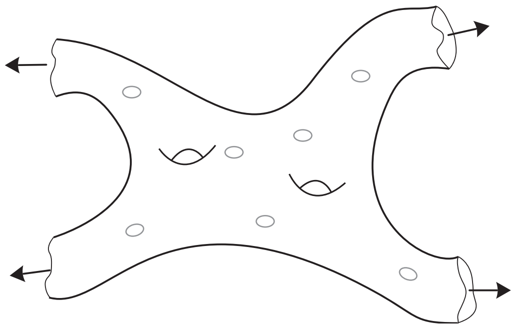

Figure 1: The manifold L with the original metric.

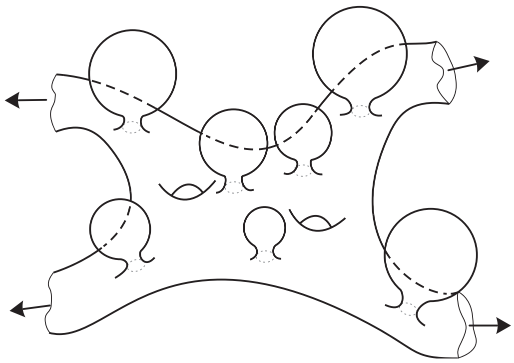

Figure 2: The manifold L with "balloons".

Our construction of Riemannian metrics on open manifolds modifies an arbitrary given metric by replacing open disks of a fixed small radius δ by large balloons, which are the complements of δ-disks in spheres of arbitrarily large radius. This can be done so that the curvature and injectivity radius remain globally bounded. By inserting these balloons in disks that converge rapidly to the ends of the manifold, the original growth rate can be preserved. We show that open manifolds with arbitrarily large balloons cannot be codimension one leaves in a compact manifold.

For surfaces the theorem was proven in [7] using a certain 'bounded homotopy property'. For manifolds of dimension $p \geq 3$, we introduce an analogous 'bounded homology property' which must be satisfied for all leaves of $C^{2,0}$ codimension one foliations of compact $(p+1)$-manifolds [8]. The essential idea is that an embedded vanishing cycle of limited size that bounds on its p-dimensional leaf must bound a region C of the leaf that in a certain sense is 'small'. From consideration of the leaves in a Reeb component it is obvious that the p-volume of the bounded region may be arbitrarily large. Hence we need a different notion of size. We consider non-negative Morse functions on the region C that vanish on its boundary (the vanishing cycle), and we require that there be a Morse function whose level sets have bounded $(p-1)$-volume. The minimum over all such Morse functions of the maximum $(p-1)$-volume of the level sets is called the *Morse volume* of the bounded set C. A manifold L possesses the *bounded homology property* if, for every constant $K > 0$ there is an $\epsilon >$ such that for every connected embedded cycle that bounds on L and has volume less than ϵ, the manifold it bounds must have Morse volume less than K.

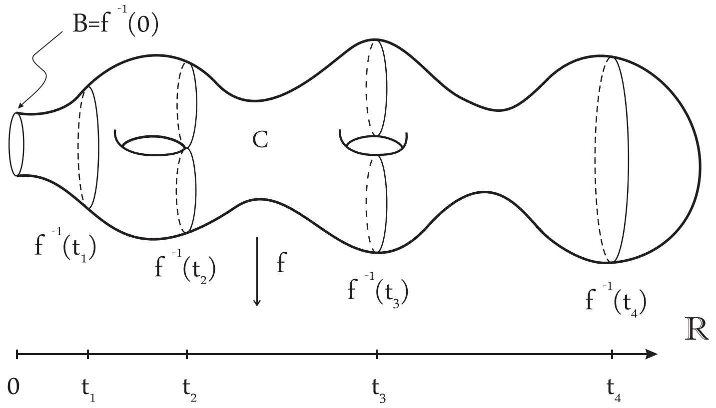

Figure 3: Morse volume of a compact manifold C with boundary B.

It is easy to see that the leaves in a Reeb component have uniformly bounded Morse volume.

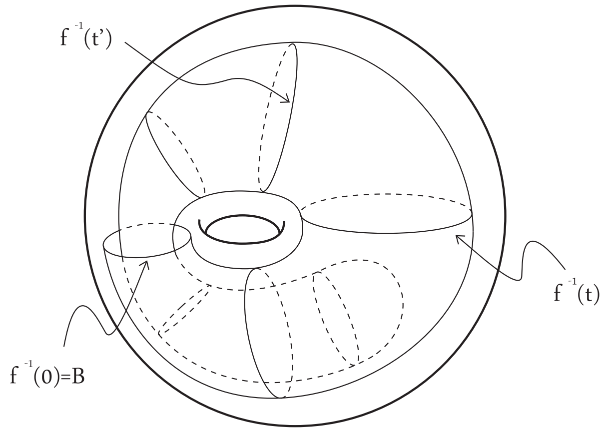

Figure 4: The Morse volume of a set C in a leaf of a Reeb component.

We prove that leaves in a codimension one foliation of a compact $(p+1)$-manifold $(p \geq 3)$ must have the bounded homology property. It is clear that a Riemannian manifold in which arbitrary large balloons have been inserted does not have the bounded homology property, thus showing the theorem. At a certain point in the proof of the theorem we need a weak generalization of Novikov's celebrated theorem on the existence of Reeb components: Every connected $(p-1)$-dimensional vanishing cycle *embedded* on a p-dimensional leaf in a compact foliated $(p+1$-manifold must lie on the boundary of a (generalized) Reeb component with connected boundary; a generalized Reeb component with connected boundary is defined to be a compact foliated $(p+1)$-manifold with connected non-empty boundary whose interior foliates over the circle with the leaves as fibers. We give a proof of this weak generalization.

A diffeomorphism $f : L \to L'$ between two Riemannian manifolds L and L' is defined to be a **quasi-isometry** if there exist constants $C, D > 0$ such that the distance functions d and d' on L and L' satisfy

$$C^{-1}d'(f(x), f(y)) - D \leq d(x, y) \leq Cd'(f(x), f(y)) + D$$

for all points $x, y \in L$. For example, any diffeomorphism between compact smooth Riemannian manifolds is a quasi-isometry. The presence of the constant $D > 0$ in this definition requires some technical details in the definition of the bounded homology property so that it will be an invariant of quasi-isometry.

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Several problems on groups of diffeomorphisms

Takashi TSUBOI

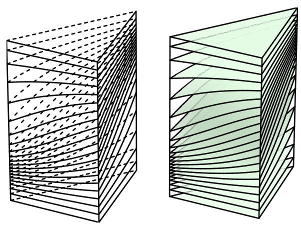
1. Introduction

This is a discussion on several problems related to the study of groups of diffeomorphisms which the author worked on for a while with some hope to find new phenomena.

For a compact manifold M, let $\mathrm{Diff}^r(M)$ ($r = 0$, $1 \leqq r \leqq \infty$, or $r = \omega$) denote the group of C^r diffeomorphisms of M. $\mathrm{Diff}^r(M)$ is equipped with the C^r topology and let $\mathrm{Diff}^r(M)_0$ denote the identity component of it. The family of diffeomorphisms generated by a time dependent vector field is called an isotopy. A diffeomorphism near the identity is contained in an isotopy. $\mathrm{Diff}^r(M)$ has a manifold structure modelled on the space of C^r vector fields. It is worth noticing that the composition $(g_1, g_2) \longrightarrow g_1 \circ g_2$ in $\mathrm{Diff}^r(M)$ $(1 \leqq r < \infty)$ is C^∞ with respect to g_1 but not continuous with respect to g_2.

2. Foliated products

A smooth singular simplex $\sigma : \Delta^m \longrightarrow \mathrm{Diff}^r(M)$ corresponds to the multi dimensional isotopy which is the foliation of $\Delta^m \times M$ transverse to the fibers of the projection $\Delta^m \times M \longrightarrow \Delta^m$ whose leaf passing through (t, x) is $\{\sigma(s)\sigma(t)^{-1}(x) \mid s \in \Delta\}$. These multi isotopies naturally match up along the boundary and form the universal foliated M-product over the classifying space $B\overline{\mathrm{Diff}}^r(M)$.



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Let $B\overline{\Gamma}_n^r$ be the classifying space for Haefliger's Γ_n^r structures with trivialized normal bundles. Since $B\overline{\Gamma}_n^r$ classifies C^r foliations with trivialized normal bundles, for an n-dimensional parallelized manifold M^n, we obtain the map $B\overline{\mathrm{Diff}}^r(M^n) \times M^n \longrightarrow B\overline{\Gamma}_n^r$, and hence the map $B\overline{\mathrm{Diff}}^r(M^n) \longrightarrow \mathrm{Map}(M, B\overline{\Gamma}_n^r)$. The deep result by Mather-Thurston says that the last map induces an isomorphism in integral homology.

Theorem 2.1 (Mather-Thurston). *For* $1 \leqq r \leqq \infty$*,*

$$H_*(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_*(\mathrm{Map}(M^n, B\overline{\Gamma}_n^r); \boldsymbol{Z}).$$

In particular, $H_*(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) \cong H_*(\Omega^n B\overline{\Gamma}_n^r; \boldsymbol{Z})$ *for the group* $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ *of* C^r *diffeomorphisms of* $\boldsymbol{R}^n$ *with compact support.*

On the other hand, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ $(1 \leqq r \leqq \infty, r \neq n+1)$ has been shown by Herman-Mather-Thurston. Note that $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) \cong H_1(B\widetilde{\mathrm{Diff}}^r(M^n)_0^\delta; \boldsymbol{Z})$, where $\widetilde{\mathrm{Diff}}^r(M^n)_0$ is the universal covering group and $^\delta$ means that the group is equipped with the discrete topology when we take its classifying space. In general, the abelianization of a group G is isomorphic to $H_1(BG^\delta; \boldsymbol{Z})$ and a group is said to be perfect if its abelianization is trivial. Moreover, by the fragmentation technique, $H_1(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{Z}) = 0$ is equivalent to $H_1(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0$, and if $\widetilde{\mathrm{Diff}}_c^r(\boldsymbol{R}^n)_0$ is perfect, then $\widetilde{\mathrm{Diff}}^r(M^n)_0$ and $\mathrm{Diff}^r(M^n)_0$ are perfect.

Theorem 2.2 (Herman-Mather-Thurston). $\mathrm{Diff}_c^r(M^n)_0$ $(1 \leqq r \leqq \infty$*,* $r \neq n+1)$ *is a perfect group. It is a simple group if* M^n *is connected.*

It is known that for $r > 2 - 1/(n+1)$, there is a characteristic cohomology class called the Godbillon-Vey class in $H^{n+1}(B\overline{\mathrm{Diff}}^r(M^n); \boldsymbol{R})$. $B\overline{\mathrm{Diff}}^r(M^n)$ is conjectured to be n-acyclic. For the higher dimensional homology, it is only known [ASPM (1985), Annals (1989)] that

$$\begin{array}{ll} H_2(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{if } 1 \leqq r < [n/2], \\ H_m(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{if } 1 \leqq r < [(n+1)/m] - 1 \text{ and} \\ H_m(B\overline{\mathrm{Diff}}_c^1(\boldsymbol{R}^n); \boldsymbol{Z}) = 0 & \text{for } m \geqq 1. \end{array}$$

The main technical reason of the above regularity conditions can be seen in the infinite iteration construction using $(\boldsymbol{Z}_+ * \boldsymbol{Z}_+)^n$ action on $\boldsymbol{R}^n$. As is well-known, by the homothety of ratio A, the C^r-norm of a foliated $\boldsymbol{R}^n$-product is multiplied by A^{1-r}. For the easiest case of divisible abelian m-cycle c represented by time 1 maps of commuting vector fields, we divide it into 2^m pieces $[n/m]$ times and we use $\boldsymbol{Z}_+^{2^n}$ action generated by homotheties of ratio $A = 1/(2+\varepsilon)$, then the infinite iteration construction converges in the C^r topology if $2^{-[n/m]}/(2+\varepsilon)^{1-r} < 1$, that is, if $r - [n/m] - 1 < 0$. To treat general cycles we loose a little more regularity.

For the connectivity of $B\overline{\Gamma}^r_n$, it seems that it increases when r tends to 1. It is true that in $\mathrm{Diff}_c^{1+\alpha}(\boldsymbol{R}^n)$, we can construct a $\boldsymbol{Z}^k$ action which permutes open sets, where k tends to infinity as α tends to 0 [JMSJ (1995)], and we think that we can use it to construct infinite iterations of chains. The bound of the rank of such action has been studied by Andrés Navas which gave rise to a new direction of study of group of diffeomorphisms.

For seeking more regular construction, it is necessary to know that abelian cycles are null homologous.

Problem 2.3. For the action $\varphi : \boldsymbol{R}^m \longrightarrow \mathrm{Diff}^r(M^n)$, show that $B(\boldsymbol{R}^m)^\delta \longrightarrow B\overline{\mathrm{Diff}}^r(M^n)$ induces the trivial homomorphism in integral homology.

Remark 2.4. It is true for $\mathrm{Diff}_c^\infty(\boldsymbol{R})$ [Fourier (1981), Fete of Topology (1988)]. It is probably true for $m = 1$ and $\mathrm{Diff}_c^\infty(\boldsymbol{R}^n)$. The first interesting case is $\boldsymbol{R}^2 \longrightarrow \mathrm{Diff}_c^\infty(\boldsymbol{R}^2)$.

To treat non abelian cycles, we notice that the theorem of Mather-Thurston implies that any class of $H_2(B\overline{\mathrm{Diff}}^r(M^n);\boldsymbol{Z})$ $(r \neq n+1)$ can be represented by a foliated M^n product over the surface Σ_2 of genus 2.

For the smooth codimension 1 foliations, there is the interesting problem of determining the kernel of the Godbillon-Vey class.

Problem 2.5. Determine the kernel of $GV : H_2(B\overline{\mathrm{Diff}}_c^r(\boldsymbol{R});\boldsymbol{Z}) \longrightarrow \boldsymbol{R}$.

Remark 2.6. There is a group G which contains both $\mathrm{Diff}_c^r(\boldsymbol{R})$ $(r > 1 + 1/2)$ and the group $PL_c(\boldsymbol{R})$ of piecewise linear homeomorphisms of $\boldsymbol{R}$ with compact support, with a metric such that GV cocycle is continuous [Fourier (1992)]. We know that for a G-foliated $\boldsymbol{R}$-product $\mathcal{F}$ over a surface, $GV(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is homologous to a G-foliated $\boldsymbol{R}$-product $\mathcal{H}_0$ over a surface Σ which is the limit of G-foliated $\boldsymbol{R}$-products $\mathcal{H}_k$ over the surface Σ representing 0 in $H_2(G;\boldsymbol{Z})$ [Proc. Japan Acad (1992)]. $\mathcal{H}_k$ are in fact transversely piecewise linear foliations and the topology of $BPL_c(\boldsymbol{R})^\delta$ has been known by the work of Peter Greenberg. It will be nice if we can take $\mathcal{H}_k$ to be C^1 piecewise $PSL(2;\boldsymbol{R})$ foliated S^1-products. The group of C^1 piecewise $PSL(2;\boldsymbol{R})$ diffeomorphisms of S^1 contains the Thompson simple group (consisting of C^1 piecewise $PSL(2;\boldsymbol{Z})$ diffeomorphisms) which gives other interests to study this group.

3. $B\overline{\Gamma}_1^\omega$

Many years ago, Haefliger showed that $B\overline{\Gamma}_1^\omega$ is a $K(\pi,1)$ space. If one understands the definition of the $\overline{\Gamma}_1^\omega$ structures, though π is a huge group, it is easy to show that $H_1(B\overline{\Gamma}_1^\omega;\boldsymbol{Z}) = 0$.

Problem 3.1. Prove or disprove that $H_2(B\overline{\Gamma}_1^\omega;\boldsymbol{Z}) = 0$.

Remark 3.2. The homology group $H_2(B\overline{\Gamma}_1^\omega;\boldsymbol{Z})$ is generated by cycles represented by surfaces Σ_2 of genus 2 with C^ω singular foliations with 2 saddles. Since $H_1(B\overline{\Gamma}_1^\omega;\boldsymbol{Z}) = 0$ is a $K(\pi,1)$, a homology class represented by the map from S^2 is trivial. A homology class represented by the map from T^2 is homologous to a union of suspensions of C^ω diffeomorphisms of S^1, and these are trivial because $\mathrm{Diff}^\omega(S^1)_0$ is perfect by a result of Arnold.

As for the perfectness of the group $\mathrm{Diff}^\omega(M^n)_0$ of real-analytic diffeomorphisms of M^n, Herman showed that $\mathrm{Diff}^\omega(T^n)_0$ is simple almost 40 years ago. Rather recently, we could show that if M^n admits a nice circle action then $\mathrm{Diff}^\omega(M^n)_0$ is perfect [Ann. ENS (2009)]. These are applications of Arnold's work on the small denominators. With this method, it should be at least generalized to the manifolds with circle actions. There are torus bundles which admits a flow whose orbit closures are fibers. It might be possible to apply the argument of [Ann. ENS (2009)].

4. Uniform perfectness

For a perfect group G, every element g can be written as a product of commutators. The least number of commutators to write g is called the commutator length of g and written as $cl(g)$. A group G is uniformly perfect if cl is a bounded function. The least bound $cw(G)$ is called the commutator width. After the result by Burago-Ivanov-Polterovich [ASPM (2008)], we showed that for a compact n-dimensional manifold M^n which admits a handle decomposition without handles of the middle index $n/2$, $cw(\mathrm{Diff}^r(M^n)_0) \leqq 3$ if n is even, $cw(\mathrm{Diff}^r(M^n)_0) \leqq 4$ if n is odd $(r \neq n+1)$. For a compact $2m$-dimensional manifold M^{2m} $(2m \geqq 6)$, $cw(\mathrm{Diff}^r(M^{2m})_0) < \infty$ $(r \neq 2m+1)$ [CMH (2012)].

Problem 4.1. Estimate $cw(\mathrm{Diff}^r(T^2)_0)$, $cw(\mathrm{Diff}^r(\mathbf{C}P^2)_0)$, $cw(\mathrm{Diff}^r(S^2 \times S^2)_0)$, ...

For the group of homeomorphisms, we managed to prove that for the spheres S^n and the Menger compact space μ^n, $cw(\mathrm{Homeo}(S^n)_0) = 1$ and

$cw(\mathrm{Homeo}(\mu^n)) = 1$ [Proc. AMS (2013)]. It is probably true that for the Menger-type compact space μ^n_k, $cw(\mathrm{Homeo}(\mu^n_k)_+) = 1$, where $_+$ means a certain condition concerning the orientation. The idea of proof comes from the fact that the typical homeomorphism of such a space is the one with one source and one sink and that the conjugacy class of such a homeomorphism should be unique.

PROBLEM 4.2. Find other examples of groups of commutator width 1.

In 1980, Fathi showed that for the group $\mathrm{Homeo}_\mu(M^n)_0$ of homeomorphisms preserving a good measure μ of M^n ($n \geqq 3$), the kernel of the flux homomorphism $\mathrm{Homeo}_\mu(M^n)_0 \longrightarrow H^{n-1}(M^n; \boldsymbol{R})$ is perfect. It seems that he proved that the kernel is uniformly perfect (at least he proved it for the spheres). For the group $\mathrm{Diff}_{\mathrm{vol}}(M^n)_0$ of volume preserving diffeomorphisms, Thurston showed that the kernel of the flux homomorphism is perfect.

PROBLEM 4.3. Prove or disprove that $\mathrm{Diff}_{\mathrm{vol}}(S^n)_0$ ($n \geqq 3$) is uniformly perfect.

Burago-Ivanov-Polterovich gave the notion of norms on the group and studied its properties. $\nu : G \longrightarrow \boldsymbol{R}_{\geqq 0}$ is a (conjugate invariant) norm if it satisfies (i) $\nu(1) = 0$; (ii) $\nu(f) = \nu(f^{-1})$; (iii) $\nu(fg) \leqq \nu(f) + \nu(g)$; (iv) $\nu(f) = \nu(gfg^{-1})$ and (v) $\nu(f) > 0$ for $f \neq 1$. For a symmetric subset $K \in G$ normally generating G, any $f \in G$ can be written as a product of conjugates of elements of K and the function giving the minimum number $q_K(f)$ of the conjugates is a norm. Then $cl(f) = q_K(f)$ for K being the set of single commutators.

For the groups of diffeomorphism with the fragmentation property, the perfectness implies the simplicity. For a simple group G, the norm $q_{\{g,g^{-1}\}} : G \longrightarrow \boldsymbol{Z}_{\geqq 0}$ is defined for $g \in G$. If $\{q_{\{g,g^{-1}\}}\}_{g \in G \setminus \{1\}}$ is bounded then G is said to be uniformly simple. In other words, for any $f \in G$ and $g \in G \setminus \{1\}$, f is written as a product of a bounded number of conjugates of g or g^{-1}. We have a distance function d on the set $\{C_{\{g,g^{-1}\}}\}_{g \neq 1}$ of symmetrized nontrivial conjugate classes:

$$d(C_{\{f,f^{-1}\}}, C_{\{g,g^{-1}\}}) = \log \max\{q_{\{f,f^{-1}\}}(g), q_{\{g,g^{-1}\}}(f)\}$$

For simple groups which are not uniformly simple, for example, $\mathrm{Diff}_{\mathrm{vol},c}(\boldsymbol{R}^n)_0$ ($n \geqq 3$), A_∞, etc, it is interesting to study the metric d. For the infinite alternative group A_∞, Kodama and Matsuda told me that d is quasi-isometric to the half line.

A real valued function ϕ on a group G is a homogeneous quasimorphism if $(g_1, g_2) \mapsto \phi(g_2) - \phi(g_1 g_2) + \phi(g_1)$ is bounded and $\phi(g^n) = n\phi(g)$ for $n \in \boldsymbol{Z}$. Put

$$D(\phi) = \sup\{|\phi(g_2) - \phi(g_1 g_2) + \phi(g_1)| \mid (g_1, g_2) \in G \times G\}.$$

Then Bavard's duality says that

$$\mathrm{scl}(g) = \frac{1}{2} \sup_{\phi \in Q(G)/H^1(G;\boldsymbol{R})} \frac{\phi(g)}{D(\phi)},$$

where $\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$ (stable commutator length) and $Q(G)$ is the real vector space of homogeneous quasimorphisms on G. Of course, for groups with infinite commutator width, we need to study their stable commutator length function. If the commutator width of a group G is infinite, G is not uniformly simple, hence the distance function d is unbounded. We might have more information on the distance d by looking at relative quasimorphisms. Let $Q(G, K)$ be the real vector space of homogeneous quasimorphisms on G which vanishes on K. If there is a nontrivial element $\phi \in Q(G, K)$ (for example, if $\dim Q(G)$ is larger than the number of K), then $\phi(f) \leqq (q_K(f) - 1)D(\phi)$ and q_K is not bounded. Since Entov-Polterovich, Gambaudo-Ghys, Ishida, and others have shown that $Q(\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2))$ is infinite dimensional and hence the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2) \longrightarrow \boldsymbol{R}$ is not uniformly simple.

Problem 4.4. For the kernel of the Calabi homomorphism $\mathrm{Diff}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2) \longrightarrow \boldsymbol{R}$, show that $\{C_{\{g,g^{-1}\}}\}_{g\neq 1}$ with metric d is not quasi-isometric to the half line.

As for the group $\mathrm{Homeo}_{\mathrm{vol}}(D^2, \mathrm{rel}\partial D^2)$, despite attemps by many people, its simplicity is still an open problem. The following problem seems to be the first step to show it.

Problem 4.5. Using area preserving homeomorphisms with the Calabi invariant being infinity, show that an area preserving diffeomorphism with nontrivial Calabi invariant is a product of commutators.

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Hedlund's theorem for compact laminations by hyperbolic surfaces

Alberto VERJOVSKY

If $\mathcal{L}$ is a compact minimal lamination by surfaces of negative curvature, we give a sufficient condition for the horocycle flow on its unit tangent bundle to be minimal. The geodesic and horocycle flows over compact hyperbolic surfaces have been studied in great detail since the pioneering work in the 1930's by E. Hopf and G. Hedlund. Such flows are particular instances of flows on homogeneous spaces induced by one-parameter subgroups, namely, if G is a Lie group, K a closed subgroup and N a one-parameter subgroup of G, then N acts on the homogeneous space $K\backslash G$ by right multiplication on left cosets. One very important case is when $G = SL(n, \mathbb{R})$, $K = SL(n, \mathbb{Z})$ and N is an unipotent one parameter subgroup of $SL(n, \mathbb{R})$, i.e., all elements of N consists of matrices having all eigenvalues equal to one. In this case $SL(n, \mathbb{Z})\backslash SL(n, \mathbb{R})$ is the space of unimodular lattices. By a theorem by Marina Ratner , which gives a positive answer to the Raghunathan conjecture, the closure of the orbit under the unipotent flow of a point $x \in SL(n, \mathbb{Z})\backslash SL(n, \mathbb{R})$ is the orbit of x under the action of a closed subgroup $H(x)$. This particular case already has very important applications to number theory, for instance, it was used by G. Margulis and Dani and Margulis to give a positive answer to the Oppenheim conjecture. When $n = 2$ and Γ is a discrete subgroup of $SL(2, \mathbb{R})$ such that $M := \Gamma\backslash SL(2, \mathbb{R})$ is of finite Haar volume, and N is any unipotent one-parameter subgroup acting on M, Hedlund proved that any orbit of the flow is either a periodic orbit or dense. When Γ is cocompact the flow induced by N has every orbit dense, so it is a minimal flow. The horocycle flow on a compact hyperbolic surface is a homogeneous flow of the previous type and most of the important dynamic, geometric and ergodic features are already present in this 3-dimensional case.

On the other hand, the study of Riemann surface laminations has recently played an important role in holomorphic dynamics polygonal tilings of the Euclidean or hyperbolic plane , moduli spaces of Riemann surfaces, etc. It is natural then to consider compact laminations by surfaces with a Riemannian metric of negative curvature and consider the positive and negative horocycle flows on the unit tangent bundle of the lamination. In this paper we give a condition that guarantees that both these flows are minimal if the lamination is minimal.

joint work with Matilde Martínez

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Uniqueness of the contact structure approximating a foliation

Thomas VOGEL

1. Introduction

We study the relationship between foliations by surfaces and contact structures on oriented 3-manifolds. Let us recall that a positive contact structure ξ is a smooth plane field locally defined by a 1-form α such that $\alpha \wedge d\alpha > 0$. In the following we assume that all plane fields are cooriented (and hence oriented) and all contact structures are positive. The first result indicating that there are connections between foliations and contact structures on 3-manifolds is the following theorem from [3].

Theorem 1.1 (Eliashberg-Thurston)**.** *Let $\mathcal{F}$ be a C^2-foliation on a compact 3-manifold such that $\mathcal{F}$ is not diffeomorphic to a foliation by spheres on $S^2 \times S^1$. Then every C^0-neighbourhood of $\mathcal{F}$ in the space of plane fields contains a positive contact structure.*

Example 1.2. The foliation of $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ given by the 2-tori $\{z = \text{const}\}$ is approximated by the contact structures

$$\xi_{k,\varepsilon} = \ker\left(\alpha_{k,\varepsilon} = dz + \varepsilon\left(\cos(2\pi kz)dx - \sin(2\pi kz)dy\right)\right)$$

as $0 \neq \varepsilon \to 0$ provided that k is a positive integer. According to Gray's theorem, contact structures which are homotopic through contact structures are isotopic. This ensures that $\xi_{k,\varepsilon}$ is independent from ε, so we omit the ε from the notation. However, it is well known that the contact structures ξ_k and ξ_l are isotopic if and only if $k = l$. Therefore one cannot expect that there is a neighbourhood of $\mathcal{F}$ such that all positive contact structures in that neighbourhood are pairwise isotopic.

In this talk we present a complete characterization of those foliations which have a C^0-neighbourhood in the space of plane fields such that all positive contact structures in that neighbourhood are pairwise isotopic. Our result can be applied to show that the space of taut foliations on certain 3-manifolds is not connected. This is of interest in view of the work of H. Eynard [4] and this question was investigated further by J. Bowden [1].

2. Main results

It turns out that the presence of torus leaves as in Example 1.2 is the main source of non-isotopic contact structures in arbitrarily small neighbourhoods of a foliation.

Theorem 2.1 (Vogel). *Let $\mathcal{F}$ be a C^2-foliation on a closed 3-manifold such that*

(i) there is no torus leaf,

(ii) not every leaf is a plane, and

(iii) not every leaf is a cylinder.

Then there is a C^0-neighbourhood of $\mathcal{F}$ in the space of plane fields such that all positive contact structures in that neighbourhhood are pairwise isotopic.

This theorem remains true for confoliations (i.e. smooth plane fields defined by a 1-form α such that $\alpha \wedge d\alpha \geq 0$) instead of foliations. Let us also note that the main use of the C^2-assumption is through Sacksteder's theorem which guarantees the existence of curves with attractive holonomy in exceptional minimal sets. Both the existence result of Eliashberg-Thurston and our uniqueness result remain valid for stable/unstable foliations of Anosov flows on 3-manifolds although these foliations are not C^2-smooth in general.

Let recall that according to theorems of H. Rosenberg and G. Hector, C^2-foliations of the type described in (ii) respectively (iii) occur only on T^3 respectively on parabolic T^2-bundles over S^1. Thus if M is not a torus fibration over S^1, then (i) is the only restriction on the foliation in order to ensure that the contact structures approximating the foliation are unique up to isotopy.

Remark 2.2. It can be shown (by explicit construction) that every neighbourhood of a foliation as in (i),(ii),(iii) of the above theorem contains infinitely many pairwise non-isotopic contact structures.

The uniqueness theorem can be extended to the case when torus leaves are present provided that the torus leaves have attractive holonomy (this condition can be weakened a little bit, however it cannot be omitted completely). Then every two contact structures in a sufficiently small C^0-neighbourhood of $\mathcal{F}$ become isotopic after a stabilization operation is applied to both them.

The proof of Theorem 2.1 is rather intricate. The overall structure is similar to the structure of the proof of Theorem 1.1 but the order of the steps is reversed. For the purposes of this exposition we assume that $\mathcal{F}$ has only one minimal set, namely a closed leaf Σ of genus $g \geq 2$. The two main steps of the proof are then as follows:

1. Fix a pair of tubular neighbourhoods $V_{out}(\Sigma) \supset V_{in}(\Sigma)$ of Σ. Given two contact structures ξ_0, ξ_2 sufficiently close to $\mathcal{F}$ show that there is a contact structure ξ_1 on M such that ξ_0 is isotopic to ξ_1 and $\xi_1 = \xi_2$ on the complement of $V_{in}(\Sigma)$. This step uses an adaptation of the methods used in [2] by V. Colin.
2. Show that the restriction of ξ_1, ξ_2 to $V_{out}(\Sigma) \setminus V_{in}(\Sigma)$ completely determines ξ_1 and ξ_2 on $V_{out}(\Sigma)$ up to isotopy relative to the boundary provided that ξ_2 is sufficiently close to $\mathcal{F}$. For this we appeal to classification results of K. Honda, W. Kazez and G. Matić [6] and we use the technique developed in [5] by E. Giroux.

The above strategy works if a finite list of assumptions on the distance of the contact planes from $\mathcal{F}$ is satisfied. We thus obtain the required neighbourhood of $\mathcal{F}$ in the space of plane fields. Above we have constructed a homotopy through contact structures which is turned into an isotopy by Gray's theorem.

3. Applications and a question

Theorem 1.1 has the following applications: Every construction of an interesting foliation on a 2-manifold can be viewed as construction of a potentially interesting contact structure. Conversely, Theorem 2.1 allows us to associate every invariant of a contact structure to a foliation which satisfies the hypothesis of Theorem 2.1. It is rather easy to show that this invariant does not change when the foliation $\mathcal{F}$ is deformed through a continuous path of foliations satisfying the hypotheses. Therefore, Theorem 2.1 can be used to show that the space of taut foliations is not connected on some manifolds.

For this recall that on the one hand foliations without torus leaves are always taut. On the other hand if a foliation has no Reeb components, then all torus leaves are incompressible. Hence contact invariants can be applied effectively to the study of connectivity properties of spaces of taut foliations on atoroidal manifolds. This should be compared with theorems of H. Eynard which imply that two taut foliations are homotopic through foliations (which may have Reeb components) provided that the two foliations are homotopic through plane fields.

Question 3.1. Theorem 2.1 can be viewed as a statement about the relationship between the topology space of contact structures and the topology of the C^0-closure of the space of contact structures. What else can be said?

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On a peculiar conformally defined class of surfaces and foliations

Paweł G. WALCZAK

In [LW1], the authors proposed to study the extrinsic conformal geometry of foliations on 3-manifolds of constant sectional curvatures. By *extrinsic conformal geometry*, we mean such geometric properties which (1) can be expressed in terms of the second fundamental form (of the leaves) and (2) are invariant under Möbius transformations.

The simplest property of this sort is *umbilicity*: a surface (in particular, a leaf) L is umbilical whenever the two principal curvatures k_1 and k_2 of L are equal. In [LW1], the authors proved that umbilical foliations (that is, foliations by umbilical leaves) on compact manifolds of non-zero constant curvature do not exist.

Since then, we (working, in different configurations, together with Adam Bartoszek, Gilbert Hector and Rémi Langevin), obtained a number of negative results concerning existence of foliations by leaves enjoying several geometric properties. For example, we proved that (nonsingular!) foliations by Dupin cyclides [LW1] and surfaces with constant local conformal invariants [BW] (see [CSW] for the definitions) do not exist on compact manifolds of constant non-zero curvature and that foliations by canal surfaces (that is envelopes of one-parameter families of spheres) do not exist on closed hyperbolic manifolds [HLW].

Positive results of this sort have been obtained as well: on the sphere S^3, nonsingular foliations by canal surfaces (and by *special canal surfaces* defined and studied in [BLW]) and singular foliations by Dupin cyclides do exist and have been classified ([LW2] and [LS]).

In this talk, we will define and discuss a new (?) class $\mathcal{S}$ of surfaces: those built of pieces of canals and pieces of spheres. Roughly speaking, a surface L belongs to $\mathcal{S}$ whenever one of its principal conformal curvatures (again, see [CSW] for a definition) vanishes at all the non-umbilical points. We will show that, from the topological point of view, any "reasonable" surface can be represented by a surface of our class $\mathcal{S}$ and that there exist many foliations of 3-manifolds by the leaves of this class. In particular, we shall show that

(1) all the surfaces listed as *generic leaves* in either [CC] or [Gh] can be represented by elements of this class,

(2) several closed 3-manifolds (like T^3, S^3 and some others) admit

foliations by surfaces of class $\mathcal{S}$.

It seems that our class $\mathcal{S}$ should be interesting not only from the point of view of pure mathematics (geometry) but also for computer aided geometric design (CAGD) [1].

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On Lagrangian submanifolds in the Euclidean spaces

Naohiko KASUYA and Toru YOSHIYASU

1. Introduction

In this paper, we study the problem of realizing an n-manifold M^n as a Lagrangian submanifold in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with a not fixed symplectic structure.

For the standard symplectic structure, there are several conditions on Lagrangian submanifolds.

Theorem 1.1 (Gromov [5]). *Let L^n be a closed Lagrangian submanifold of the $2n$-dimensional Euclidean space with the standard symplectic structure, $(\mathbb{R}^{2n}, \omega_0)$. Then*

$$[\omega_0] \neq 0 \in H^2(\mathbb{R}^{2n}, L; \mathbb{R}),$$

and therefore $H^1(L; \mathbb{R}) \neq 0$.

Theorem 1.2 (Fukaya [3]). *Let $(\mathbb{R}^6, \omega_0)$ be the 6-dimensional Euclidean space with the standard symplectic structure and L be an oriented connected closed prime 3-manifold. Then L can be embedded in $(\mathbb{R}^6, \omega_0)$ as a Lagrangian submanifold if and only if L is diffeomorphic to $S^1 \times \Sigma_g$, where Σ_g is an oriented closed 2-dimensional manifold of genus $g \geq 0$.*

By Theorem 1.1 and Theorem 1.2, the topology of a Lagrangian submanifold of $\mathbb{R}^{2n}$ with the standard symplectic structure is strongly restricted. On the other hand, we will see that almost of all the closed parallelizable manifolds can be Lagrangian submanifolds of the Euclidean spaces with not fixed symplectic structures.

2. Main Result

The main result is the following.

Theorem 2.1. *Let M^n be a closed parallelizable n-manifold. If $n \neq 7$, or if $n = 7$ and the Kervaire semi-characteristic $\chi_{\frac{1}{2}}(M^7)$ is zero, then for any embedding of M^n in $\mathbb{R}^{2n}$, there exists a symplectic structure on $\mathbb{R}^{2n}$ such that the embedding is Lagrangian.*

Remark 2.2. For $n = 2$, the only closed parallelizable 2-manifold is the 2-torus and for $n = 3$, any closed orientable 3-manifold is parallelizable. There are infinitely many isotopy classes of embeddings of the 2-torus in the 4-dimensional Euclidean space. For $n \geq 3$, there is a surjection from the set of isotopy classes of embeddings of M^n in the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ to the homology group $H_1(M^n; \mathbb{Z})$ if n is odd, and to $H_1(M^n; \mathbb{Z}_2)$ if n is even [9], [10].

3. Preliminary

To obtain a Lagrangian embedding of an n-manifold in $\mathbb{R}^{2n}$, we embed its cotangent bundle in $\mathbb{R}^{2n}$ and extend its canonical symplectic structure to $\mathbb{R}^{2n}$.

Proposition 3.1. *Let M^n be a closed parallelizable n-manifold embedded in $\mathbb{R}^{2n}$. Then its normal bundle is trivial.*

Proof. It is an immediate consequence of Kervaire's theorem that for any stably parallelizable manifold K^d embedded in $\mathbb{R}^{2d}$, the normal bundle is trivial [7]. $\square$

Therefore, for a closed parallelizable n-manifold M^n, any embedding of M^n in $\mathbb{R}^{2n}$ extends to an embedding of T^*M^n in $\mathbb{R}^{2n}$. To extend the canonical symplectic structure on T^*M^n, we review Gromov's h-principle for symplectic structures on an open manifold and the space of non-degenerate 2-forms on $\mathbb{R}^{2n}$.

Theorem 3.2 (Gromov [4]). *Let N^{2n} be a triangulated open $2n$-manifold and ω be a non-degenerate 2-form on N^{2n}. Then there is a symplectic form $\tilde{\omega}$ on N^{2n}. Moreover, if ω is closed on a neighborhood of a subset M of a core C of N^{2n}, then we can choose $\tilde{\omega}$ which coincides with ω on a neighborhood of M.*

By Theorem 3.2, to extend the canonical symplectic structure, it is sufficient to extend the canonical symplectic structure as a non-degenerate 2-form. We prepare some propositions to apply Theorem 3.2.

Proposition 3.3 (See the section 4.3 of [2]). *Let N be a triangulated open manifold. Then there exists a subpolyhedron $C \subset N$ such that* $\dim C < \dim N$ *and N can be compressed by an isotopy $\varphi_t \colon N \to N$, $t \in [0, 1]$, into any neighborhood of C.*

We call C a *core* of the open manifold N.

Proposition 3.4. *There is a diffeomorphism from the space of linear symplectic structures on* $\mathbb{R}^{2n}$ *to the quotient space* $\mathrm{GL}(2n;\mathbb{R})/\mathrm{Sp}(2n)$*. Moreover, the connected component* $\mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n)$ *corresponds to the space of linear symplectic structures on* $\mathbb{R}^{2n}$ *which give the positive orientation on* $\mathbb{R}^{2n}$ *where*

$$\mathrm{GL}_+(2n;\mathbb{R}) = \{A \in \mathrm{GL}(2n;\mathbb{R}) \mid \det A > 0\}.$$

Proof. For a linear symplectic structure Ω on $\mathbb{R}^{2n}$, we can take a symplectic basis $\langle u_1, v_1, \ldots, u_n, v_n\rangle$ which is determined up to linear transformations by the symplectic group $\mathrm{Sp}(2n)$. That is, the map

$$\Omega \mapsto [A] \in \mathrm{GL}(2n;\mathbb{R})/\mathrm{Sp}(2n) \ \big(A = (u_1 \ \cdots u_n \ v_1 \ \cdots \ v_n)\big)$$

is well defined. Its inverse is given by

$$[A] \mapsto {}^tA^{-1}\Omega_0 A^{-1} \ \left(\Omega_0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \in \mathrm{GL}(2n;\mathbb{R})\right).$$

□

Then we can identify a positive non-degenerate 2-form on $\mathbb{R}^{2n}$ with a smooth map

$$\mathbb{R}^{2n} \to \mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n).$$

We note that the map represents a symplectic basis of the non-degenerate 2-form at each point of $\mathbb{R}^{2n}$.

Proposition 3.5 (See the section 2.2 of [8])**.** *The map*

$$\mathrm{GL}_+(2n;\mathbb{R})/\mathrm{Sp}(2n) \to \mathrm{SO}(2n)/\mathrm{U}(n), \ [A] \mapsto [B],$$

where $({}^tA^{-1}\Omega_0 A^{-1}\,{}^tA^{-1}\,{}^t\Omega_0 A^{-1})^{-\frac{1}{2}}({}^tA^{-1}\Omega_0 A^{-1}) = B\Omega_0 B^{-1}$*, is a homotopy equivalence.*

By Proposition 3.4 and 3.5, we can identify the canonical symplectic structure ω on T^*M^n with the continuous map

$$\omega\colon T^*M^n \to \mathrm{SO}(2n)/\mathrm{U}(n).$$

Actually, the possibility of extending the canonical symplectic structure ω as a non-degenerate 2-form depends only on the homotopy type of ω.

Remark 3.6. For an n-manifold M^n, the existence of a Lagrangian embedding of M^n in $\mathbb{R}^{2n}$ with a not fixed symplectic structure is equivalent to the existence of a totally real embedding of M^n in $\mathbb{C}^n$. Indeed, we can check it by applying Theorem 3.2 and Gromov's h-principle for totally real embeddings [6]. Audin gave a necessary and sufficient condition for the existence of a totally real embedding of M^n in $\mathbb{C}^n$ in [1]. In particular, the existence part of Theorem 2.1 is a part of Audin's theorem if $n \neq 7$.

4. Outline of the Proof of Theorem 2.1

Proof. We prove only the case where $n = 3$. It suffices to show that the map $\omega\colon T^*M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$ is null-homotopic. Let us take a triangulation of M^3, $M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3$ be the skeletons, $f\colon M^3 \to \mathbb{R}^6$ be the embedding. First, we denote the Gauss map of f by g_0. Since M^3 is parallelizable, the Gauss map of f takes the value in the Stiefel manifold $V_{6,3}$,

$$g_0\colon M^3 \to V_{6,3}.$$

The map g_0 is null-homotopic on $M^{(2)}$ because $V_{6,3}$ is 2-connected. Thus there exists a homotopy $g_t^{(2)}\colon M^{(2)} \to V_{6,3}$, $t \in [0,1]$, with $g_0^{(2)} = g_0 \mid_{M^{(2)}}$ and $g_1^{(2)}$ is a constant map. By the covering homotopy property of the fibration $\mathrm{SO}(6) \to V_{6,3}$, we can take the lift $G_t^{(2)}$ of $g_t^{(2)}$,

$$G_t^{(2)}\colon M^{(2)} \to \mathrm{SO}(6).$$

Since the fiber of the fibration $\mathrm{SO}(6) \to V_{6,3}$ is $\mathrm{SO}(3)$ and the homotopy group $\pi_2\big(\mathrm{SO}(3)\big) = 0$, $G_0^{(2)}$ extends to the map $G_0\colon M^3 \to \mathrm{SO}(6)$ which formed by an orthonormal tangent 3-frame field and an orthonormal normal 3-frame field of M^3. On the other hand, $G_1^{(2)}$ extends to a constant map $G_1\colon M^3 \to \mathrm{SO}(6)$. Next, we composes these map with the projection $\pi\colon \mathrm{SO}(6) \to \mathrm{SO}(6)/\mathrm{U}(3)$ which we denote $\bar{G}_0 = \pi \circ G_0$, $\bar{G}_t^{(2)} = \pi \circ G_t^{(2)}$, and $\bar{G}_1 = \pi \circ G_1$. We note that the map $\bar{G}_0 = \omega\colon T^*M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$ and the map $\bar{G}_1$ is a constant map. Lastly, since the homotopy group $\pi_3\big(\mathrm{SO}(6)/\mathrm{U}(3)\big) = 0$, $\bar{G}_t^{(2)}$ extends to the map $\bar{G}_t\colon M^3 \to \mathrm{SO}(6)/\mathrm{U}(3)$. Therefore, ω is null-homotopic.

The remaining cases are similar by using the Kervaire semi-characteristic. $\square$

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Entropy-like invariants for groups, pseudogroups and foliations

Andrzej BIŚ

1. Introduction

In classical dynamical systems one of the most fundamental invariants of a continuous map $f : X \to X$ is its topological entropy $h_{top}(f)$ which measure the complexity of the system. When the entropy is positive, it reflects some chaotic behavior of the map f. In foliation theory, any so called nice covering $\mathcal{U}$ of a compact foliated manifold (M, F) determines a finitely generated holonomy pseudogroup $(H_{\mathcal{U}}, H_{\mathcal{U}1})$ generated by a finite generating set $H_{\mathcal{U}1}$. Also, there exists a corresponding notion of a topological entropy for a group action or pseudogroup action on a compact metric space. For any foliated manifold (M, F), the action of a holonomy pseudogroup on a complete transversal contains complete information about the dynamics of (M, F). It does not depend on the choice of the transversal up to an equivalence of pseudogroups. Therefore, a foliated manifold can be considered as a generalized dynamical system.

In classical theory of dynamical systems a continuous map $f : X \to X$ determines an f-invariant measure μ and one can define a measure entropy $h_\mu(f)$ with respect to μ. The important relation between topological entropy and measure entropy of a map $f : X \to X$ is established by the Variational Principle, which asserts that

$$h_{top}(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\}$$

i.e. topological entropy equals to the supremum $h_\mu(f)$, where μ ranges over the set $M(X, f)$ of all f-invariant Borel probability measures on X.

In classical dynamical systems there are relations between the topological entropy of a continuous map $f : X \to X$ and Hausdorff dimension. More than thirty years ago Bowen [4] provided a definifion of topological entropy of a map which resembles the definition of Hausdorff dimension. A dimensional type approach to topological entropy of a single continuous map one can find for example in [1], [10] or [7]. A cyclic group or semigroup $< f >$ generated by a single map $f : X \to X$ has linear growth. Therefore it is difficult to adopt ideas and techniques presented for groups of linear growth to finitely generated groups or pseudogroups which growth is rarely linear. The goal of the talk is to present interrelations between dimension theory and the theory of generalized dynamical systems.

2. Topological entropy of a pseudogroup and local measure entropy

In [6] Ghys, Langevin and Walczak defined the topological entropy of a finitely generated pseudogroup and introduced a notion of a geometric entropy of a foliation. The problem of defining good measure theoretical entropy for foliated manifolds which would provide an analogue of the variational principle for geometric entropy of foliations is still open. In general, there are many examples of foliations that do not admit any non-trivial invariant measure. Even in a case when an invariant measure exists, it is not clear how to define its measure-theoretic entropy.

We generalize the notion of local measure entropy introduced by Brin and Katok [5] for a single map $f : X \to X$ to a finitely generated pseudogroup (G, G_1) acting on a metric space X. We define a local upper measure entropy $h^G_\mu(x)$ and a local lower measure entropy $h_{\mu,G}(x)$ of (G, G_1) at a point $x \in X$ with respect to a Borel probability measure μ on X.

The main result of [2] is an analogue of the partial Variational Principle for pseudogroups which reads as follows:

Theorem 2.1. *Let (G, G_1) be a finitely generated group of homeomorphisms of a compact closed and oriented manifold M. Let E is a Borel subset of M, $s > 0$ and μ_{vol} the natural volume measure on M.*

If the local measure entropy $h^G_{\mu_{vol}}(x) \leq s$, for all $x \in E$, then the topological entropy $h_{top}((G, G_1), E) \leq s$.

Theorem 2.2. *Let (G, G_1) be a finitely generated pseudogroup on a compact metric space X. Let E is a Borel subset of X and $s > 0$. Denote by μ a Borel probability measure on X.*

If the local measure entropy $h_{\mu,G}(x) \geq s$, for all $x \in E$, and $\mu(E) > 0$ then the topological entropy $h_{top}((G, G_1), E) \leq s$.

Next, we introduce a special class class of measures, called $G-$ homogeneous measures, on X.

Theorem 2.3. *If a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d) admits a $G-$homogeneous measure then the local measure entropy $h^G_\mu(x)$ is constant and it does not depend on the point $x \in X$. Moreover:*

For a finitely generated pseudogroup (G, G_1) acting on a compact metric space X and admitting a $G-$ homogeneous measure μ on X we have

$$h_{top}(G, G_1) = h^G_\mu,$$

where h^G_μ is the common value of local measure entropies $h^G_\mu(x)$.

3. New entropy-like invariants

In the talk we present and apply the theory of Carathéodory structures (or C-structures), studied by Pesin ([8], [9]) and Pesin and Pitskel ([10]), which are the powerful generalization of the classical construction of Hausdorff measure. Pesin introduced a C-structure axiomatically by describing its elements and relation between them. A Carathéodory structure τ defined on a metric space X determines the Carathéodory dimension $dim_{C,\tau}(Z)$ of a subset $Z \subset X$. Another procedure leads to definition of two other basic characteristics of dimensional type: the lower and upper capacity of a set $Z \subset X$.

The main results of [3] are as follows.

Theorem 3.1. *For a finitely generated pseudogroup (G, G_1) there exists a C- structure with upper capacity that coincides with the topological entropy of (G, G_1).*

We denote by E a class of continuous and decreasing functions $f : [0, \infty) \to [0, \infty)$ with $\lim_{x\to\infty} f(x) = 0$. Now, we fix a pseudogroup (H, H_1) acting on a metric space X. Any function $f \in E$ and the pseudogroup (H, H_1) determine a class of C-structures $\Gamma(f)_\delta = \{(F_\delta, \xi, \eta, \psi) : \delta > 0\}$ and the limit C-structure $\Gamma(f)$ on X. The upper capacity of a set $Z \subset X$, with respect the limit C-structure $\Gamma(f)$, is denoted here by $\overline{CP(f)}_Z$.

We apply the Theorem 3.1 to get some estimations of the geometric entropy $h_{geom}(F, g)$ of a compact foliated manifold (M, F), which describes the global dynamics of (M, F). It is known that a compact foliated manifold (M, F) with fixed so called nice covering U determines a finitely generated holonomy pseudogroup $(H(U), H_1(U))$ acting on the transversal T_U. Here, the finite generating set $H_1(U)$ consists of elementary holonomy maps corresponding to overlapping charts of U.

Theorem 3.2. *Given a finitely generated pseudogroup (H, H_1) acting on a compact metric space X. Assume that for $f, g \in E$ and for any $x \in [0, \infty)$ the inequalities $f(x) \le e^{-x} \le g(x)$ hold. Then, for any subset $Z \subset X$ we get*

$$\overline{CP(f)}_Z \le h_{top}((H, H_1), Z) \le \overline{CP(g)}_Z.$$

As a corollary we get two classes of dimensional type estimations of the geometric entropy of foliations.

Theorem 3.3. *Assume that for $f_1, f_2 \in E$ and for any $x \in [0, \infty)$ the inequalities $f_1(x) \le e^{-x} \le f_2(x)$ hold. For any nice covering U of a compact foliated manifold(M, F) endowed with a Riemannian structure g, denote by* *diam(U) the maximum of the diameters of the plaques of U measured with respect to the Riemannian structures induced on the leaves. Then*

$$h_{geom}^{lower}(F, f_1) \leq h_{geom}(F, g) \leq h_{geom}^{upper}(F, f_2),$$

where

$$h_{geom}^{lower}(F, f_1) = \sup\{\frac{1}{diam(U)}\overline{CP(f_1)(H(U), H_1(U))}_{T_U} : U\text{-}nice\ cover\ of\ M\},$$

$$h_{geom}^{upper}(F, f_2) = \sup\{\frac{1}{diam(U)}\overline{CP(f_2)(H(U), H_1(U))}_{T_U} : U\text{-}nice\ cover\ of\ M\}.$$

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Index theory for basic Dirac operators on Riemannian foliations

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Let $(M, \mathcal{F})$ be a smooth, closed manifold endowed with a Riemannian foliation. Let $D_b^E : \Gamma_b(M, E^+) \to \Gamma_b(M, E^-)$ be a basic, transversally elliptic differential operator acting on the basic sections of a foliated complex vector bundle E over M, of rank N. The basic index $\operatorname{ind}_b\left(D_b^E\right)$ is known to be a well-defined integer, and it has been an open problem since the 1980s to write this integer in terms of the geometric and topological data. Our main theorem expresses $\operatorname{ind}_b\left(D_b^E\right)$ as a sum of integrals over the different strata of the Riemannian foliation, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is

$$\operatorname{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0}\diagup\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^{r} \beta(M_j)\ ,$$

$$\beta(M_j) = \frac{1}{2}\sum_{\tau} \frac{1}{n_\tau \operatorname{rank} W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}^{\tau}(x)\ \widetilde{|dx|}\ .$$

Here, the integrands $A_{0,b}(x)$ and $A_{j,b}^{\tau}(x)$ are the familiar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds, $D_j^{S+,r}$ is a first order differential operator on a round sphere, explicitly computable from local information provided by the operator and the foliation, while $\eta(D_j^{S+,r})$ and $h(D_j^{S+,r})$ denote its eta-invariant and kernel, respectively. Even in the case when the operator D is elliptic, such a result was not known previously. We emphasize that every part of the formula is local in the data, even $\eta(D_j^{S+,r})$ is calculated directly from the principal transverse symbol of the operator D_b^E at any point of a singular stratum. The de Rham operator provides an important example illustrating the computability of the formula, yielding the basic Gauss-Bonnet Theorem.

The Theorem is proved by first writing $\operatorname{ind}_b\left(D_b^E\right)$ as the invariant index of a G-equivariant, transversally elliptic operator $\mathcal{D}$ over a G-manifold $\widehat{W}$

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associated to the foliation, where G is a compact Lie group of isometries. Precisely, we lift the given foliation $\mathcal{F}$ to a foliation $\widehat{\mathcal{F}}$ on the principal frame bundle, $\widehat{M}$, associated to $Q \otimes E$, with structure group $G := O(q) \times U(N)$, where Q is the normal bundle to $\mathcal{F}$ of rank q. Then $\mathcal{F}$ is transversally parallelizable. Hence we deduce from Molino's structure theory that the leaf closures of $\widehat{\mathcal{F}}$ are the fibers of the *basic fibration* $\widehat{\pi} : \widehat{M} \to \widehat{W}$. Since G acts isometrically on $\widehat{W}$, we have reduced the problem to the computation of a G-equivariant index. Using our equivariant index theorem, we obtain an expression for this index in terms of the geometry and topology of $\widehat{W}$ and then rewrite this formula in terms of the original data on the foliation.

We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations in a rather special case. Using a different technique, the authors prove a formula for the index of a basic Dirac operator that is distinct from our formula, assuming that all the infinitesimal holonomy groups of the foliation are connected tori and that Molino's commuting sheaf is abelian and has trivial holonomy. Our result requires only mild topological assumptions on the transverse structure of the strata of the Riemannian foliation. In particular, the Gauss-Bonnet Theorem for Riemannian foliations is a corollary and requires no assumptions on the structure of the Riemannian foliation.

We add some remarks on the proof of the Theorem. Using the notions of basic sections, holonomy-equivariant vector bundles, basic Clifford bundles, and basic Dirac-type operators, we describe the Fredholm properties of these basic operators, and we show how to construct the G-manifold $\widehat{W}$ of leaf closures and the G-equivariant operator $\mathcal{D}$, using a slight generalization of Molino theory. We also use our construction to obtain asymptotic expansions and eigenvalue asymptotics of transversally elliptic operators on Riemannian foliations, which are of independent interest. We also construct bundles associated to represention of the isotropy subgroups of the G-action; these bundles are used in the main theorem. In the course of the proof, we describe the construction of the *desingularization* of a Whitney stratified space, i. e. a method of cutting out tubular neighborhoods of the singular strata and doubling the remainder to produce a Whitney stratified space with fewer strata. We also deform the operator and the metric and determine the effect of this desingularization and deformation operation on the basic index. Finally, we prove a generalization of this theorem to representation-valued basic indices.

The theorem is illustrated with a collection of examples. These include foliations by suspension, a Transverse Signature, and the Basic Gauss-Bonnet Theorem.

One known application of our theorem is Kawasaki's Orbifold Index Theorem. It is known that every orbifold is the leaf space of a Riemannian foliation, where the leaves are orbits of an orthogonal group action such that all isotropy subgroups have the same dimension. In particular, the contributions from the eta invariants in our Transverse Signature Theorem agree exactly with the contributions from the singular orbifold strata when the orbifold is four-dimensional.

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Contact manifolds with symplectomorphic symplectizations

Sylvain COURTE

1. Introduction

Contact geometry and symplectic geometry are very much related. Given a contact manifold (M, ξ), we can associate a symplectic manifold $(\mathrm{S}_\xi M, \omega_\xi)$, called its *symplectization*. Topologically, the symplectization of M is just the product $\mathbb{R} \times M$. There is an $\mathbb{R}$-action on $\mathrm{S}_\xi M$ which allows to reinterpret contact geometry as $\mathbb{R}$-equivariant symplectic geometry without any loss of information. On one hand, many contact invariants are constructed from symplectizations using holomorphic curves techniques. It is therefore tempting to think that contact manifolds with symplectomorphic symplectizations are contactomorphic. On the other hand, in smooth topology it is well-known that there exist manifolds M and M' that are not diffeomorphic but for which $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic (see [2]). Using flexibility results of Eliashberg and Cieliebak [4], we can realize these examples in a symplectic setting to construct non-diffeomorphic contact manifolds with symplectomorphic symplectizations.

Definition 1.1. Let $(M, \xi = \ker \alpha)$ be a contact manifold. The symplectic manifold $(\mathrm{S}_\xi M, \omega_\xi) = (\mathbb{R} \times M, d(e^t\alpha))$ is called the *symplectization* of (M, ξ). It is endowed with an $\mathbb{R}$-action given by translation in the $\mathbb{R}$ factor.

Proposition 1.2. *Any $\mathbb{R}$-equivariant symplectomorphism $\mathrm{S}_\xi M \to \mathrm{S}_{\xi'} M'$ induces a contactomorphism $(M, \xi) \to (M', \xi')$.*

Now if we relax the hypothesis that the symplectomorphism is $\mathbb{R}$-equivariant in the proposition above, does it still follow that M and M' are contactomorphic?

2. Main results

Theorem 2.1. *[1] For any closed contact manifold (M, ξ) of dimension at least 5 and any closed manifold M' such that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic, there is a contact structure ξ' on M' such that $S_\xi M$ and $S'_\xi M'$ are symplectomorphic.*

Outline of proof. The diffeomorphism $\Psi : \mathbb{R} \times M \to \mathbb{R} \times M'$ produces two h-cobordisms (W, M, M') and (W', M', M) such that the compositions in both senses are trivial :

$$W \cup W' \simeq [0,1] \times M \text{ and } W' \cup W \simeq [0,1] \times M'$$

(for example, W is obtained as the region in $\mathbb{R} \times M$ between $\{0\} \times M$ and $\Psi^{-1}(\{c\} \times M')$ for a sufficiently large positive number c).

Using Eliashberg and Cieliebak's results from [4] we can endow W and W' with *flexible* symplectic structures that induce the contact structure ξ on M and a new contact structure ξ' on M' and we still have, now *symplectically*:

$$W \cup W' \simeq [0,1] \times M \text{ and } W' \cup W \simeq [0,1] \times M'.$$

We apply the Mazur trick (see [2]) and consider the infinite composition V:

$$\cdots (W \cup W') \cup (W \cup W') \cdots = \cdots (W' \cup W) \cup (W' \cup W) \cdots$$

We get from the left hand side that V symplectomorphic to $S_\xi M$ and from the right hand side that V is symplectomorphic to $S_{\xi'} M'$. $\square$

For example, let us consider $M = L(7,1) \times S^2$ endowed with the canonical contact structure ξ coming from the unit tangent bundle of $L(7,1)$. It was proved by Milnor (see [3]) that M is not diffeomorphic to $M' = L(7,2) \times S^2$ but they are h-cobordant. It follows from the s-cobordism theorem and the Mazur trick as in the proof above that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic (see [2]). Hence theorem 2.1 provides a contact structure ξ' on M' such that $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are symplectomorphic.

We now discuss an application of this result to the symplectic topology of Stein manifolds. Stein manifolds (of finite type) admit contact manifolds at infinity, given by level sets above any critical value of positive proper plurisubharmonic functions. However we may wonder if this contact manifold depends only on the Stein manifold or may change when we pick a different proper plurisubharmonic function. Again using results from [4] to go from Weinstein to Stein, we can apply the method of Theorem 2.1 to provide different contact boundaries for a given Stein manifold.

Corollary 2.2. *[1] Let V be a Stein manifold of finite type. Let (M, ξ) be the contact manifold at infinity given by a plurisubharmonic function ϕ. Then for any closed manifold M' such that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic, there is a plurisubharmonic function ψ on V with contact manifold at infinity diffeomorphic to M'.*

3. Questions

Does there exist contact structures ξ and ξ' on a given manifold M such that ξ and ξ' are not conjugated by a diffeomorphism of M but $S_\xi M$ and $S'_\xi M$ are symplectomorphic? Any contact invariant which is functorial with respect to symplectic cobordisms (such as contact homology) could not distinguish between ξ and ξ'.

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Transverse Ricci flow as a tool for classification of Riemannian flows

Andrzej CZARNECKI

1. Introduction

A Riemannian flow is a 1-dimensional Riemannian foliation. It provides just enough structure (an invariant metric on the normal bundle) with just enough flexibility (the flow need not to be isometric) to be both treatable and interesting for geometers. For similar reasons, it is also of interest to people working in dynamical systems.

In 1984 Yves Carrière gave a classification of Riemannian flows on 3-manifolds, [1]. This stemed from his previous work on the topic, but with more emphasis on Molino's structural approach. Indeed, not long after Molino and Almeida classified Riemannian flows on 4-manifolds, [2]. The two theorems can be summarized as folows

Theorem 1.1 (Carrière). *A 3-manifold with a Riemannian flow*

- *is either foliated-diffeomorphic to a dense linear flow on a torus, or*
- *a suspension of a prescribed type, or*
- *foliated-diffeomorphic to a prescribed flow on a torus, or*
- *foliated-diffeomorphic to a prescribed flow on a lens space, or*
- *a Seifert fibration,*

and those instances are distinguished by presence or absence of dense or closed leaves and their holonomy. Only the second dot cannot be endowed with a metric that makes the flow isometric.

Theorem 1.2 (Molino, Almeida). *A 4-manifold with a Riemannian flow*

- *is either foliated-diffeomorphic to a dense linear flow on a torus, or*
- *a suspension of a prescribed type, or*
- *foliated-diffeomorphic to a prescribed flow on a (twisted) double of $\mathbb{T}^2 \times \mathbb{D}^2$, or*
- *has a 2-dimensional orbifold with boundary as the space of leaves, or*

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- *is a Seifert fibration,*

and, again, those instances are distinguished by occuring closures of leaves and their holonomy. This time both the second and the fourth dots provide examples of non-isometric flows.

It seems there was no substantial progress in classifying Riemannian flows in higher dimensions and this is not very surprising – this problem essentially generalizes classification of manifolds. We emphasize that both proofs rely on Molino's theory and structural approach presented in [3].

We would like to recover those theorems using different methods, namely geometric flows. In 1986 Min-Oo and Ruh restated Hamilton's result on Ricci flow in terms of flow of Cartan connections, [4]. Recently, with Lovrić, in [5], they were able to apply this technique to a flow of connections on transverse bundle of a foliation, essentially obtaining the Ricci flow on the transverse manifold – with usual consequences:

Theorem 1.3 (Lovrić, Min-Oo, Ruh)**.** *Suppose a codimension 3 Riemannian foliation with positive definite Ricci curvature. Then the metric can be deformed to a metric of constant sectional curvature.*

The metric here is understood only on the transverse bundle. We would like to better comprehend and to apply this transverse Ricci flow.

2. Main results

The present work is concerned with recovering Theorems 1.1 and 1.2 with methods developed in [5].

Proposition 2.1. *Holonomy and stratification that classify codimension 2 and 3 Riemannian flows can be deduced from curvature properties of the deformed metric.*

It is plausible that transverse Ricci flow in those codimensions should prove as useful as it's usual, non-transverse counterpart. Of course, topological considerations threaten to be much more involved (we have metric information only in a normal, non-integrable direction), but on the other hand, we know – a posteriori – that spaces in question are quite simple.

Another step is a geometric proof of the following

Theorem 2.2. *Compact, connected, orientable, irreducible, atoroidal 3-orbifold is geometric.*

This is one possible statement of Thurston's Geometrization Conjecture for orbifolds. Recall that an orbifold is a topological space locally homeomorphic to an euclidean space divided by a finite group action. Original (quite involved) proofs are due to Cooper, Hodgson, Kerckhoff (cf. [6]) and Boileau, Leeb, Porti (cf. [7]) and carefully reduce the problem to Geometrization Conjecture for manifolds. Recently, Kleiner and Lott provided in [8] a direct proof, describing Ricci flow on orbifolds. It still seems a worthwile task to prove Theorem 2.2 using the folowing desingularization procedure (cf. [9])

Theorem 2.3. *Every orbifold can be realized as a space of leaves of a Riemannian foliation. Geometry of the orbifold is the transverse geometry of that foliation.*

and using [5]. Note that this realization may produce a foliation of high dimension, although the codimension is preserved.

Proposition 2.4. *Transverse flow of Cartan connections of [5] and Ricci flow on orbifolds of [8] coincide.*

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Classsification of maximal codimension totally geodesic foliations of the complex hyperbolic space

Maciej CZARNECKI

1. Totally geodesic foliations of $\mathbb{H}^n$

Totally geodesic foliations of the real hyperbolic space $\mathbb{H}^n$ in codimension 1 are well understood. The first classification given by Ferus in [5] concentrates on geometry of orthogonal transversal. Browne observed that it is enough to study vector fields along geodesics (cf. [2]). Lastly, Lee and Yi classified totally geodesic codimension 1 foliations of $\mathbb{H}^n$ through closed curves on S^{n-1} which represent the ideal boundary of leaves. For short explanation compare [4] and [1].

2. Complex hyperbolic space and complex de Sitter space

The complex hyperbolic space $\mathbb{C}H^n$ is one of the easiest examples of the Hadamard manifold with nonconstant sectional curvature. Even here there is no (real) codimesion 1 totally geodesic submanifolds; in fact only totally geodesic submanifolds are totatlly complex or totally real (cf. [6]).

Define *complex de Sitter space* $\mathbb{C}\Lambda^n$ as the (complex) projectivization of positive vectors with respect to the Hermitian form in $\mathbb{C}^{n+1}$ given by

$$\langle Z, W\rangle = -Z_0\overline{W_0} + Z_1\overline{W_1} + \ldots + Z_n\overline{W_n}.$$

Recall that $\mathbb{C}H^n$ is simply projectivization of negative vectors in $\mathbb{C}^{n+1}$.

Every totally geodesic codimension 2 submanifold of $\mathbb{C}H^n$ is the projectivization of complex hyperplane which is complex-time-like. Thus it is represented by a positive vector i.e. belonging to $\mathbb{C}\Lambda^n$.

3. Classsification of totally geodesic codimension 2 foliations of $\mathbb{C}H^n$

In [4] Czarnecki and Walczak stated the problem of geometric classification of foliations of $\mathbb{C}H^n$ with leaves isometric to $\mathbb{C}H^{n-1}$, i.e. of the real codimension 2.

This problem could be studied similarly to the real case when the conformal geometry is applied. Using methods developed in [7] Czarnecki and Langevin (see [3]) gave local and global conformal condition for curves in de Sitter space Λ^{n+2} to represent a totally geodesic codimension 1 folations of $\mathbb{H}^n$.

Totally geodesic codimension 2 foliations are curves in $\mathbb{C}\Lambda^n$ such that its tangent vector is of complex-time-like. Therefore, totally geodesic maximal codimension foliations of $\mathbb{C}H^n$ are those which are orthogonal to a complex curve of holomorphic curvature bounded by 1. Such a curve is an Hadamard 2–dimensional submanifold of bounded negative curvature.

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Birkhoff sections for geodesic flows of hyperbolic surfaces

Norikazu HASHIGUCHI

1. Birkhoff section

Definition 1.1. A Birkhoff section for a flow φ_t defined on a closed 3-manifold is an embedded surface satisfying that its interior is transverse to φ_t and that its boundaries are consist of closed orbits of φ_t.

Example 1.2. 1. Let T^2 be a flat torus. Now we construct a Birkhoff section for the geodesic flow g_t of T^2 in the unit tangent vector bundle T_1T^2. We take closed geodesics C_1, C_2, C_3, C_4 of T^2 (see Figure 1). The complement of these closed geodesics is 4 rectangles. We choose two rectangles R_1 and R_2 which are not adjacent. Next we consider a family $\mathcal{C}_i$ $(i = 1, 2)$ of convex simple closed curves which fills the interior of R_i with one singularity deleted. Let S be the closure of the union of unit tangent vectors of all curves of $\mathcal{C}_1$ and $\mathcal{C}_2$. Then, S is a torus with 8 discs deleted and the boundaries of S are close oriented geodesics corresponding to C_1, C_2, C_3, C_4.

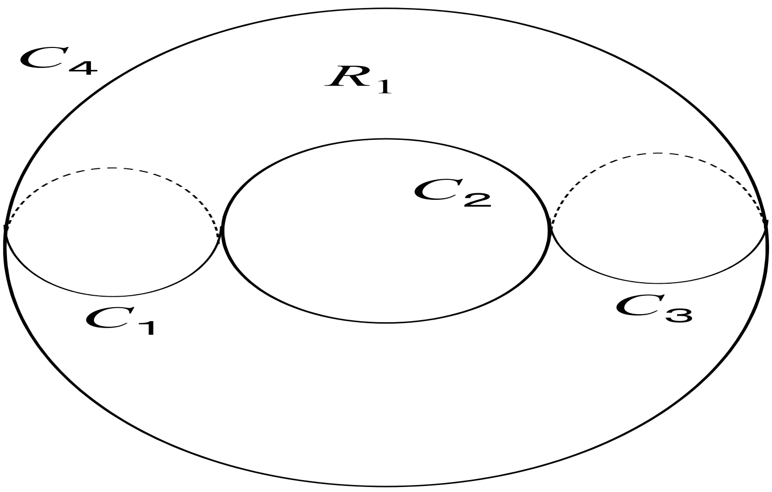

Figure 1: Geodesics and rectangles of T^2

S is a Birkhoff section for g_t. The first return map of g_t associated with S is topologically semiconjugate to the toral automorphism induced by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

2. In the hyperbolic case, we construct a genus one Birkhoff section of the geodesic flow by the same method of the above case.
   Let Σ_g $(g \geq 2)$ be a genus g orientable closed surface with a hyperbolic metric. The geodesic flow of Σ_g has genus one Birkhoff sections [1, 2, 3]. The first return maps associated with these sections are topologically semiconjugate to hyperbolic toral automorphisms. These toral automorphisms are induced by

$$A_g = \begin{pmatrix} 2g^2-1 & 2g(g-1) \\ 2g(g+1) & 2g^2-1 \end{pmatrix}$$

   and

$$B_g = \begin{pmatrix} 4g^2-2g-1 & 2g^2-2g \\ 8g^2-2 & 4g^2-2g-1 \end{pmatrix}$$

   [1, 4].

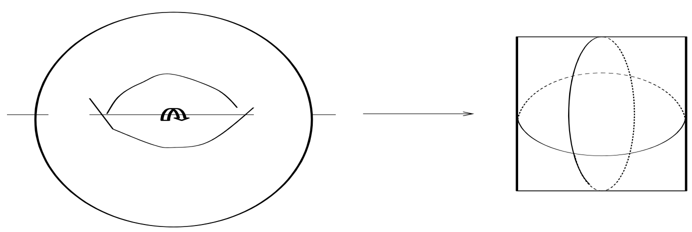

Figure 2: Branched covering $\gamma : T^2 \to P$

3. Let P be a flat pillowcase i.e. a 2-dimensional sphere with 4 singular points. P is also considered as a quotient space $\mathbb{R}^2/\Gamma$ where Γ is the group of isometries of $\mathbb{R}^2$ generated by π-rotations centered at $(0, \pm\frac{1}{2})$ and $(\pm\frac{1}{2}, 0)$. We consider a branched covering $\gamma : T^2 \to P$

(see Figure 2). The differential $T_1\gamma$ of γ preserves geodesic flows and $S' = T_1\gamma(S)$ is also a genus one Birkhoff section for the geodesic flow f_t of P. The double covering $T_1\gamma|_S : S \to S'$ is induced by the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Hence, the first return map of f_t associated with S' is topologically semiconjugate to a toral automorphism induced by $DA_1D^{-1} = \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}$.

2. Main Results

In [1], Brunella showed the method to construct genus one Birkhoff sections. We apply this method to geodesic flows of 2-spheres with singularities.

For any three positive integers p, q, r satisfying that the hyperbolic condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, let $S(p, q, r)$ be a 2-sphere with three singular points whose cone angles are $\frac{2\pi}{p}, \frac{2\pi}{q}, \frac{2\pi}{r}$. If we consider the hyperbolic metric on $S(p, q, r)$, then the geodesic flow F_t of $S(p, q, r)$ is an Anosov flow on a triangular Seifert fibred space.

Using Scott's result about closed geodesics of F_t [5], we have the next theorem.

Theorem 2.1. *If (p, q, r) is not $(2, 3, u)$ $(u \geq 7)$ nor $(2, 4, u)$ $(u \geq 5)$ up to permutation of p, q, r, then the geodesic flow F_t of $S(p, q, r)$ has a genus one Birkhoff section and F_t is topologically constructed by doing Dehn surgeries along two closed orbits of the suspension of the hyperbolic toral automorphism induced by a matrix $A_{p,q,r} \in SL(2; \mathbf{Z})$.*

In some special cases, we can calculate $A_{p,q,r}$ by the same way of the above flat pillowcase case. There exist branched coverings $\Sigma_g \to S(2g + 2, 2g + 2, g + 1)$ and $\Sigma_g \to S(2g + 1, 2g + 1, 2g + 1)$. Since these branched covering preserve the geodesic flows, they are used to calculate $A_{2g+2,2g+2,g+1}$ and $A_{2g+1,2g+1,2g+1}$.

Theorem 2.2.

$$A_{2g+2,2g+2,g+1} = \begin{pmatrix} 2g^2 - 1 & g(g^2 - 1) \\ 4g & 2g^2 - 1 \end{pmatrix}$$

$$A_{2g+1,2g+1,2g+1} = \begin{pmatrix} 4g^2 - 2g - 1 & 2g(g-1)(2g+1) \\ 2(2g-1) & 4g^2 - 2g - 1 \end{pmatrix}$$

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Lie foliations transversely modeled on nilpotent Lie algebras

Naoki KATO

1. Introduction

Let M be an n-dimensional closed orientable smooth manifold and let $\mathcal{F}$ be a codimension q transversely orientable smooth foliation of M. Let $\mathfrak{g}$ be a q-dimensional Lie algebra over $\mathbb{R}$.

Definition 1.1. The foliation $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation if there exists a non-singular Maurer-Cartan form $\omega \in A^1(M, \mathfrak{g})$ such that $T\mathcal{F} = \mathrm{Ker}(\omega)$.

P. Molino [4] proved that the following structure theorem.

Theorem 1.2 (Molino)**.**

1. *There exists a locally trivial fibration $\pi\colon M \to W$ such that each fiber is the closure of a leaf of $\mathcal{F}$.*
2. *There exists a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is uniquely determined by $\mathcal{F}$ such that, for each fiber F of the fibration π, the induced foliation $\mathcal{F}|_F$ is a Lie $\mathfrak{h}$-foliation.*

The Lie algebra $\mathfrak{h}$ is called the structure Lie algebra of $\mathcal{F}$.

By Theorem 1.2, to each Lie foliation $\mathcal{F}$, there are associated two Lie algebras, the model Lie algebra $\mathfrak{g}$ and the structure Lie algebra $\mathfrak{h}$. Hence, we have a natural question to determine the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which can be realized as a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M with structure Lie algebra $\mathfrak{h}$.

Definition 1.3. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. $(\mathfrak{g}, \mathfrak{h})$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of M such that the structure Lie algebra of $\mathcal{F}$ is $\mathfrak{h}$.

If $\mathcal{F}$ is a flow, then the structure Lie algebra $\mathfrak{h}$ is abelian and thus it is isomorphic to $\mathbb{R}^m$ for some m.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra and m be an integer. $(\mathfrak{g}, m)$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-flow $\mathcal{F}$ of M such that the structure Lie algebra of $\mathcal{F}$ is $\mathbb{R}^m$, that is the dimension of the structure Lie algebra is equal to m.

E. Gallego, B. Herrera, M. Llabrés and A. Reventós completely solved this problem in the case where the dimension of the Lie algebras $\mathfrak{g}$ is three (cf. [2], [3]).

We study the realizing problems of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, m)$ in the case where $\mathfrak{g}$ is nilpotent Lie algebras of general dimensions.

2. Main results

Theorem 2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra which has a rational structure. Then $(\mathfrak{g}, m)$ is realizable if and only if $m \leq \dim \mathfrak{c}(\mathfrak{g})$, where $\mathfrak{c}(\mathfrak{g})$ is the center of $\mathfrak{g}$.*

Theorem 2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is realizable if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the quotient Lie algebra $\mathfrak{h}\backslash\mathfrak{g}$ has a rational structure.*

Corollary 2.3. *For any nilpotent Lie algebra $\mathfrak{g}$, there exists a minimal Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M.*

Since nilpotent Lie algebras has a non-trivial center, by Theorem 2.1, any nilpotent Lie algebra $\mathfrak{g}$ with a rational structure can be realized as a Lie $\mathfrak{g}$-flow. On the other hand, there exists a nilpotent Lie algebra $\mathfrak{g}$ with no rational structures which can not be realized as a Lie $\mathfrak{g}$-flow.

Example 2.4 (Chao). Let $c_{ij}^k, 1 \leq i,j \leq m, 1, \leq k \leq n$ be real numbers such that $c_{ij}^k = -c_{ji}^k$. Assume that c_{ij}^k are algebraically independent over $\mathbb{Q}$. Let $\mathfrak{g}$ be the Lie algebra defined by a basis

$$\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$$

with the products

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k Y_k$$

for $i, j = 1, \ldots, m$ and all other products being zero. Then $\mathfrak{g}$ is nilpotent a Lie algebra and $[\mathfrak{g}, \mathfrak{g}] = \langle Y_1, \ldots, Y_n \rangle_{\mathbb{R}}$. This Lie algebra $\mathfrak{g}$ has no rational structure if $(n/2)(m^2 - m) > m^2 + n^2$.

Proposition 2.5. *Let $\mathfrak{g}$ be the Lie algebra constructed above. If $(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2$, then $\mathfrak{g}$ can not be realized as a Lie $\mathfrak{g}$-flow.*

However there exists a nilpotent Lie algebra with no rational structures which can be realized as a Lie flow.

Proposition 2.6. *There exists a nilpotent Lie algebra* $\mathfrak{g}$ *which has no rational structures such that* $\mathfrak{g}$ *can be realized as a Lie* $\mathfrak{g}$*-flow.*

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Superheavy subsets and noncontractible Hamiltonian circle actions

Morimichi KAWASAKI

1. Introduction

Let (M, ω) be a symplectic manifold. In this paper a diffeomorphism f of M is called *a symplectomorphism* if f preserves the symplectic form ω.

Our result is as follows:

Theorem 1.1. *Let $(\mathbb{T}^2, \omega_{\mathbb{T}^2}) = (\mathbb{R}^2/\mathbb{Z}^2, \omega_{\mathbb{T}^2})$ be the 2-torus with the coordinates (p, q) and the symplectic form $dp \wedge dq$. The union $M \cup L$ of the meridian curve M and the longitude curve L is a "$[\mathbb{T}^2]$-superheavy" subset of $(\mathbb{T}^2, \omega_{\mathbb{T}^2})$.*

As a corollary of Proposition 1.1, we have the following result:

Corollary 1.2. *Let $(\mathbb{C}P^n, \omega_{FS})$ be the complex projective space with the Fubini-Study form ω_{FS} and C be the Clifford torus $\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n; |z_0| = \cdots = |z_n|\}$ of $\mathbb{C}P^n$. Then there exists no symplectomorphism f of $(\mathbb{C}P^n \times \mathbb{T}^2, \omega_{FS} \oplus \omega_{\mathbb{T}^2})$ such that $C \times (M \cup L) \cap f(C \times (M \cup L)) = \emptyset$.*

2. Preliminaries

2.1. Definitions

For a function $F\colon M \to \mathbb{R}$ with compact support, we define *the Hamiltonian vector field* $\operatorname{sgrad} F$ associated with F by

$$\omega(\operatorname{sgrad} F, V) = -dF(V) \text{ for any } V \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M.

For a function $F\colon M \times [0,1] \to \mathbb{R}$ and $t \in [0,1]$, we define $F_t\colon M \to \mathbb{R}$ by $F_t(x) = F(x,t)$. We denote by $\{f_t\}$ the isotopy which satisfies $f_0 = \mathrm{id}$ and $\frac{d}{dt} f_t(x) = (\operatorname{sgrad} F_t)_{f_t(x)}$. We call this *the Hamiltonian path generated by the Hamiltonian function F*. The time-1 map f_1 of $\{f_t\}$ is called *the Hamiltonian diffeomorphism generated by the Hamiltonian function F*. A diffeomorphism f is called *a Hamiltonian diffeomorphism* if there exists a

Hamiltonian function with compact support generating f. A Hamiltonian diffeomorphism is a symplectomorphism.

For a symplectic manifold (M, ω), we denote by $\mathrm{Symp}(M, \omega)$, $\mathrm{Ham}(M, \omega)$ and $\widetilde{\mathrm{Ham}}(M, \omega)$, the group of symplectomorphisms, the group of Hamiltonian diffeomorphisms of (M, ω) and its universal cover, respectively. We denote by $\mathrm{Symp}_0(M, \omega)$ the identity component of $\mathrm{Symp}(M, \omega)$. Note that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}_0(M, \omega)$.

Definition 2.1. For functions F and G and a symplectic manifold (M, ω), *the Poisson bracket* $\{F, G\} \in C^\infty(M)$ is defined by

$$\{F, G\} = \omega(\mathrm{sgrad}\, G, \mathrm{sgrad}\, F).$$

Definition 2.2 ([1]). Let (M, ω) be a symplectic manifold.

A subset U of M is called *displaceable* if there exists a Hamiltonian diffeomorphism $f \in \mathrm{Ham}(M, \omega)$ such that $f(U) \cap \bar{U} = \emptyset$.

A subset U of M is called *strongly displaceable* if there exist a symplectomorphism $f \in \mathrm{Symp}(M, \omega)$ such that $f(U) \cap \bar{U} = \emptyset$.

We consider the cotangent bundle $T^*\mathbb{S}^1 = \mathbb{R} \times \mathbb{S}^1$ of the circle $\mathbb{S}^1$ with the coordinates (r, θ) and the symplectic form $dr \wedge d\theta$. A subset U of M is called *stably displaceable* if $U \times \{r = 0\}$ is displaceable in $M \times T^*\mathbb{S}^1$ equipped with the split symplectic form $\bar{\omega} = \omega \oplus (dr \wedge d\theta)$.

If U is displaceable, then U is stably displaceable. Since $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$, if U is displaceable, then U is strongly displaceable.

2.2. Spectral invariants

For a closed connected symplectic manifold (M, ω), define

$$\Gamma = \frac{\pi_2(M)}{\mathrm{Ker}(c_1) \cap \mathrm{Ker}([\omega])},$$

where c_1 is the first Chern class of TM with an almost complex structure compatible with ω. The Novikov ring of the closed symplectic manifold (M, ω) is defined as follows:

$$\Lambda = \left\{ \sum_{A \in \Gamma} a_A A; a_A \in \mathbb{Q}, \#\{A; a_A \neq 0, \int_M \omega < R\} < \infty \text{ for any real number } R \right\}$$

The quantum homology $QH_*(M, \omega)$ is a Λ-module isomorphic to $H_*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$ and $QH_*(M, \omega)$ has a ring structure with the multiplication called *quantum product* [3]. To each element $a \in QH_*(M, \omega)$, a functional $c(a, \cdot) \colon C^\infty(M \times [0, 1]) \to \mathbb{R}$ is defined in terms of Hamiltonian Floer theory. The functional

$c(a,\cdot)$ is called *spectral invariant* ([3]). To describe the properties of a spectral invariant, we define the spectrum of a Hamiltonian function as follows:

Definition 2.3 ([3]). Let $H \in C^\infty(M\times[0,1])$ be a Hamiltonian function on a closed symplectic manifold M. *Spectrum* $\mathrm{Spec}(H)$ of H is defined as follows:

$$\mathrm{Spec}(H) = \left\{ \int_0^1 H(h_t(x),t)dt + \int_{\mathbb{D}^2} u^*\omega \right\} \subset \mathbb{R},$$

where $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by H and $x \in M$ is a fixed point of h_1 whose orbit defined by $\gamma^x(t) = h_t(x)$ $(t \in [0,1])$ is a contractible loop and $u\colon \mathbb{D}^2 \to M$ is a disc in M such that $u|_{\partial\mathbb{D}^2} = \gamma^x$.

We define the *non-degeneracy* of Hamiltonian functions as follows:

Definition 2.4. $H \in C^\infty(M\times[0,1])$ is called *non-degenerate* if the graph of the Hamiltonian diffeomorphism h generated by H is transverse to the diagonal in $M\times M$.

The followings are well-known properties of spectral invariants ([3], [4]).

Non-degenerate spectrality $c(a,H) \in \mathrm{Spec}(H)$ for every non-degenerate $H \in C^\infty(M\times[0,1])$.

Hamiltonian shift property $c(a, H+\lambda(t)) = c(a,H) + \int_0^1 \lambda(t)dt$.

Monotonicity property If $H_1 \leq H_2$, then $c(a,H_1) \leq c(a,H_2)$.

Lipschitz property The map $H \mapsto c(a,H)$ is Lipschitz on $C^\infty(M\times[0,1])$ with respect to the C^0-norm.

Symplectic invariance $c(a, f^*H) = c(a,H)$ for any $f \in \mathrm{Symp}_0(M,\omega)$ and any $H \in C^\infty(M\times[0,1])$.

Homotopy invariance $c(a,H_1) = c(a,H_2)$ for any normalized H_1 and H_2 generating the same $h \in \widetilde{\mathrm{Ham}}(M)$. Thus one can define $c(a,\cdot)\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ by $c(a,h) = c(a,H)$, where H is a normalized Hamiltonian function generating h.

Triangle inequality $c(a*b, fg) \leq c(a,f) + c(b,g)$ for elements f and $g \in \widetilde{\mathrm{Ham}}(M,\omega)$, where $*$ denotes the quantum product.

2.3. Heaviness and superheaviness

M. Entov and L. Polterovich ([1]) defined the *heaviness* and the *superheaviness* of closed subsets in closed symplectic manifolds and gave stably non-displaceable subsets and strongly non-displaceable subsets.

For an idempotent a of the quantum homology $QH_*(M,\omega)$, define the functional $\zeta_a\colon C^\infty(M)\to\mathbb{R}$ by

$$\zeta_a(H)=\lim_{l\to\infty}\frac{c(a,lH)}{l},$$

where $c(a,H)$ is the spectral invariant ([3], see Section 2.2).

Definition 2.5 ([1]). Let (M,ω) be a $2n$-dimensional closed symplectic manifold. Take an idempotent a of the quantum homology $QH_*(M,\omega)$.

A closed subset X of M is called *ζ_a-heavy* (or *a-heavy*) if

$$\zeta_a(H)\geq\inf_X H \text{ for any } H\in C^\infty(M),$$

and is called *ζ_a-superheavy* (or *a-superheavy*) if

$$\zeta_a(H)\leq\sup_X H \text{ for any } H\in C^\infty(M).$$

A closed subset X of M is called heavy (respectively, superheavy) if X is ζ_a-heavy (respectively, ζ_a-superheavy) for some idempotent a of $QH_*(M,\omega)$.

For a oriented closed manifold M, we denote its fundamental class by $[M]$.

Theorem 2.6 (A part of Theorem 1.4 of [1]). *For a non-trivial idempotent a of $QH_*(M,\omega)$, the followings hold.*

(1) Every ζ_a-superheavy subset is ζ_a-heavy.

(2) Every ζ_a-heavy subset is stably non-displaceable.

(3) Every $[M]$-superheavy subset is strongly non-displaceable.

□

Example 2.7. (1) Let $(\mathbb{T}^2,\omega_{\mathbb{T}^2})=(\mathbb{R}^2/\mathbb{Z}^2,\omega_{\mathbb{T}^2})$ be the 2-torus with the coordinates (p,q) and the symplectic form $dp\wedge dq$. Then the meridian curve $M=\{(p,q)\in\mathbb{T}^2; q=0\}$ and the longitude curve $L=\{(p,q)\in\mathbb{T}^2; p=0\}$ are $[\mathbb{T}^2]$-heavy subsets of $(\mathbb{T}^2,\omega_{\mathbb{T}^2})$, hence they are stably non-displaceable ([1] Example 1.18).

(2) Let $(\mathbb{C}P^n,\omega_{FS})$ be the complex projective space with the Fubini-Study form. The Clifford torus $C=\{[z_0:\cdots:z_n]\in\mathbb{C}P^n; |z_0|=\cdots=|z_n|\}\subset\mathbb{C}P^n$ is a $[\mathbb{C}P^n]$-superheavy subset of $(\mathbb{C}P^n,\omega_{FS})$, hence they are strongly non-displaceable ([1] Theorem 1.8).

Definition 2.8. Let (M,ω) be a $2n$-dimensional closed symplectic manifold. Take an idempotent a of the quantum homology $QH_*(M,\omega)$. An open subset U of M is said to be ζ_a-null if for $G \in C^\infty(U)$,

$$\zeta_a(G) = 0.$$

An open subset U of M is said to be strongly ζ_a-null if for $F \in C^\infty(M)$ and $G \in C^\infty(U)$ such that $\{F, G\} = 0$,

$$\zeta_a(F + G) = \zeta_a(F).$$

A subset X of M is said to be (strongly) ζ_a-null if there exists a (strongly) ζ_a-null open neighborhood U of X.

3. Main proposition

Definition 3.1. A closed symplectic manifold (M,ω) is called *rational* if $\omega(\pi_2(M))$ is a discrete subgroup of $\mathbb{R}$.

The main result is the following proposition. We use this proposition to prove Theorem 1.1 by using the argument of stems.

Proposition 3.2. *Let (M,ω) be a rational closed symplectic manifold. Let α be a nontrivial free homotopy class of free loops on M; $\alpha \in [\mathbb{S}^1, M]$, $\alpha \neq 0$. Let U be an open subset of M. Assume that there exists a Hamiltonian function $H \in C^\infty(M \times [0,1])$ which satisfies the followings:*

(1) $h_1|_U = \mathrm{id}_U$,

(2) for any $x \in U$, the free loop $\gamma^x \colon \mathbb{S}^1 \to M$ defined by $\gamma^x(t) = h_t(x)$ belongs to α, and

(3) $\alpha \notin i_*([\mathbb{S}^1, U])$.

Here $i \colon U \to M$ is the inclusion map and $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by H. Then U is strongly ζ_a-null for any idempotent a of $QH_*(M,\omega)$.

□

The proof of Theorem 3.2 is based on the idea of K. Irie in the proof of Theorem 2.4 of [2].

4. Proof of Theorem 1.1

M. Entov and L. Polterovich defined stems to give examples of superheavy subsets. We define ζ_a-stems which generalizes a little the notion of stems and there exhibits ζ_a-superheaviness.

We generalize the argument of Entov and Polterovich as follows.

Definition 4.1. Let $\mathbb{A}$ be a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ and $\Phi\colon M \to \mathbb{A}^*$ be the moment map defined by $\langle \Phi(x), F\rangle = F(x)$. Let a be a non-trivial idempotent of $QH_*(M,\omega)$. A non-empty fiber $\Phi^{-1}(p)$, $p \in \mathbb{A}^*$ is called *a ζ_a-stem* of $\mathbb{A}$ if all non-empty fibers $\Phi^{-1}(q)$ with $q \neq p$ is strongly ζ_a-null. If a subset of M is a ζ_a-stem of a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$, it is called just *a ζ_a-stem*.

Theorem 4.2. *For every idempotent a of $QH_*(M,\omega)$, every ζ_a-stem is a ζ_a-superheavy subset.*

□

Proof of Theorem 1.1.
Note that $(\mathbb{T}^2, \omega_{\mathbb{T}^2})$ is rational. Consider a momentum map $\Phi \in C^\infty(\mathbb{T}^2)$ such that $\Phi(x) = 0$ if $x \in M \cup L$ and $\Phi(x) > 0$ if $x \notin M \cup L$. Take a real number $\epsilon \neq 0$. Then there exist a positive number δ and an open neighborhood U of $\Phi^{-1}(\epsilon)$ such that $U \subset (\delta, 1-\delta) \times (\delta, 1-\delta)$. Consider a Hamiltonian function $H \in C^\infty(\mathbb{T}^2 \times [0,1])$ such that $H((p,q),t) = p$ for any $p \in [\delta, 1-\delta]$.

Define the free loop $\gamma\colon \mathbb{S}^1 \to \mathbb{T}^2$ by $\gamma(t) = (0,t)$. Let $\alpha \in [\mathbb{S}^1, \mathbb{T}^2]$ be the homotopy class of free loops represented by γ. Then α, U and H satisfy the assumptions of Theorem 3.2, hence U satisfies is strongly ζ_a-null. Thus $M \cup L$ is a ζ_a-stem, hence it is ζ_a-superheavy.

■

5. Proof of Corollary 1.2

We use the following theorem to prove Corollary 1.2.

Theorem 5.1 ([1] Theorem 1.7)**.** *Let (M_1,ω_1) and (M_2,ω_2) be closed symplectic manifolds. Take non-zero idempotents a_1, a_2 of $QH_*(M_1)$, $QH_*(M_1)$, respectively. Assume that for $i = 1, 2$, X_i be a a_i-heavy (respectively, a_i-superheavy) subset. Then the product $X_1 \times X_2$ is $a_1 \otimes a_2$-heavy (respectively, $a_1 \otimes a_2$-superheavy) subset of $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ of $QH_*(M_1 \times M_2)$.*

□

Proof of Corollary 1.2. By Example 2.7 and Theorem 1.1, Theorem 5.1, $C \times (M \cup L)$ is $[C \times (M \cup L)]$-superheavy subset of $(\mathbb{C}P^n \times \mathbb{T}^2, \omega_{FS} \oplus \omega_{\mathbb{T}^2})$. Thus Theorem 2.6 implies that there exists no symplectomorphism f such that $C \times (M \cup L) \cap f(C \times (M \cup L)) = \emptyset$.

■

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Minimal C^1-diffeomorphisms of the circle which admit measurable fundamental domains

Hiroki KODAMA

1. Abstract

This is a joint work with Shigenori Matsumoto (Nihon University).

The concept of ergodicity is important not only for measure preserving dynamical systems but also for systems which admits a natural quasi-invariant measure. Given a probability space (X, μ) and a transformation T of X, μ is said to be *quasi-invariant* if the push forward $T_*\mu$ is equivalent to μ. In this case T is called *ergodic* with respect to μ, if a T-invariant Borel subset in X is either null or conull.

A diffeomorphism of a differentiable manifold always leaves the Riemannian volume (also called the Lebesgue measure) quasi-invariant, and one can ask if a given diffeomorphism is ergodic with respect to the Lebesgue measure (below *ergodic* for short) or not. Answering a question of A. Denjoy [D], A. Katok (see for instance Chapt. 12.7, p. 419, [KH]), and independently M. Herman (Chapt. VII, p. 86, [H]) showed the following theorem.

Theorem 1.1. *A C^1-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational.*

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains. Given a transformation T of a standard probability space (X, μ) leaving μ quasi-invariant, a Borel subset C of X is called a *measurable fundamental domain* if T^nC $(n \in \mathbb{Z})$ is mutually disjoint and the union $\cup_{n\in\mathbb{Z}} T^nC$ is conull. In this case any Borel function on C can be extended to a T-invariant measurable function on X, and an ergodic component of T is just a single orbit. The purpose of this talk is to show the following theorem.

Theorem 1.2 ([KM]). *For any irrational number α, there is a minimal C^1-diffeomorphism of the circle with rotation number α which admits a measurable fundamental domain with respect to the Lebesgue measure.*

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To prove the theorem, first we construct a Lipschitz homeomorphism F with rotation number α which admits a measurable fundamental domain. We regard the circle S^1 as $\mathbb{R}/\mathbb{Z}$. Suppose R denotes the rotation by α.

Claim 1.3. *For any irrational number α, we can construct a Cantor set $C \in S^1$ so that $R^n C \cap R^m C = \emptyset$ for any integers $n \neq m$.*

Admitting this claim, fix a probability measure μ_0 on C without atom such that $\operatorname{supp}(\mu_0) = C$. We also choose a sequence $(a_i)_{i\in\mathbb{Z}}$ of positive numbers satisfying $\sum_{i\in\mathbb{Z}} a_i = 1$. Now we can define a probability measure μ on S^1 by

$$\mu := \sum_{i\in\mathbb{Z}} a_i R^i_* \mu_0. \tag{1.4}$$

The Radon-Nikodym derivative $\frac{dR_*^{-1}\mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set $R^i C$. Now we assume that $\frac{a_{i+1}}{a_i} \in [\frac{1}{D}, D]$ for some $D > 1$, then it follows that $\frac{dR_*^{-1}\mu}{d\mu} \in L^\infty(S^1, \mu)$.

We define a homeomorphism h of S^1 by $h(0) = 0$ and $h(x) = y$ if and only if $Leb[0, x] = \mu[0, y]$, where Leb denotes the Lebesgue measure on S^1; or more briefly, $h_* Leb = \mu$. Finally define a homeomorphism F of S^1 by $F := h^{-1} \circ R \circ h$, then

$$\frac{dF_*^{-1}\, Leb}{d\, Leb} = \frac{dR_*^{-1}\mu}{d\mu} \circ h \in L^\infty(S^1, Leb), \tag{1.5}$$

i.e. the map F is a Lipschitz homeomorphism. The set $C' = h^{-1}C$ is a measurable fundamental domain of F.

To prove Theorem 1.2, it is enough to make the Radon-Nikodym derivative $g = \frac{dR_*^{-1}\mu}{d\mu}$ continuous on S^1. Assume that g is continuous, set $\phi = \log g$ and

$$\begin{aligned} \phi^{(m)}(x) &= \sum_{i=0}^{m-1} \phi(R^i x) \qquad (m > 0), \\ \phi^{(-m)}(x) &= -\sum_{i=1}^{m} \phi(R^{-i} x) \qquad (m > 0), \\ \phi^{(0)}(x) &= 0, \end{aligned} \tag{1.6}$$

then we can conclude that $a_i = \exp(\phi^{(i)}(x_0))a_0$ for any point $x_0 \in C$. Since $\sum_{i\in\mathbb{Z}} a_i = 1$, the sum $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x_0))$ has to be finite.

Fix an integer $n \in \mathbb{N}$. Since $R^{-2^n}C, \ldots, C, \ldots, R^{2^n-1}C$ are disjoint compact sets, for a sufficiently small ε-neighbourhood N of C, $R^{-2^n}N, \ldots,$

$N, \dots, R^{2^n-1}N$ are also disjoint. Take a bump function $f\colon S^1 \to \mathbb{R}$ so that $\operatorname{supp} f \subset N$, $f(x) = (3/4)^n$ for $x \in C$ and $0 \le f(x) < (3/4)^n$ for $x \in N \setminus C$. Define $\phi_n\colon S^1 \to \mathbb{R}$ by

$$\phi_n(x) = \begin{cases} -f(R^{-i}x) & x \in R^iN,\ i = 0, 1, \dots, 2^n - 1 \\ f(R^{-i}x) & x \in R^iN,\ i = -2^n, -2^n + 1, \dots, -1 \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

and $\phi = \sum_{i=1}^{\infty} \phi_n$, then ϕ is a continuous function satisfying

$$\sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty. \tag{1.8}$$

Employing this ϕ, set $\tilde{\mu} = \sum_{i \in \mathbb{Z}} (\exp \circ \phi^{(i)} \circ R^{-i}) R^i_* \mu_0$ and $\mu = \frac{\tilde{\mu}}{\int_{S^1} d\tilde{\mu}}$. The function $F\colon S^1 \to S^1$ constructed from this μ is C^1.

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Configuration spaces of linkages on Riemannian surfaces

Mickaël KOURGANOFF

1. Introduction

A mechanical linkage is a mechanism made of rigid rods linked together by flexible joints, in which some vertices are fixed and others may move. The configuration space of a linkage is the set of all its possible positions.

There has been a lot of work on mechanical linkages. Many papers deal with linkages on the Euclidean plane $\mathbb{R}^2$, but the Definition of linkages extends naturally to any Riemannian manifold.

On the Euclidean plane, Kempe [Kem75] has shown in 1875 that for any algebraic curve $\mathcal{C}$, for any euclidian ball $\mathcal{B} \subseteq \mathbb{R}^2$, there exists a linkage $\mathcal{L}$, and one vertex of this linkage v such that $\mathcal{C} \cap \mathcal{B}$ is exactly the set of the possible positions of v (his proof was flawed, but there is a rather simple way to make it correct, see [Abb08]). In particular, the famous *Peaucellier-Lipkin straight-line motion linkage* (Figure 1) forces a vertex to move on a straight line.

More recently, Kapovich and Millson [KM02] have shown that for any smooth compact manifold without boundary M, there exists a linkage for which the configuration space is diffeomorphic to a finite disjoint union of copies of M. Jordan and Steiner proved a weaker version of this theorem with more elementary techniques [JS99]. Thurston already gave lectures on a similar theorem in the 1980's but never wrote a proof.

When we consider the same linkage on two different Riemannian surfaces, for example on the Euclidean plane and on the sphere, the configuration space may be very different. Therefore, it is natural to ask what the two results above become on surfaces other than the plane. Is there a way of characterizing the curves which may be drawn ? May any smooth compact manifold be seen as the configuration space of some linkage ? As far as we know, it is an open problem whether it is possible in general to force a vertex to move on a geodesic. On the sphere, the answer is easy : just take a fixed vertex linked to a moving vertex by an edge of length $\pi/2$. Some solutions also exist on the hyperbolic plane or on the Minkowski plane.

The analogue of the second result in $\mathbb{RP}^2$ (with the metric induced by the natural covering $\mathbb{S}^2 \longrightarrow \mathbb{RP}^2$) is shown in [KM02] using the methods of Mnëv [Mnë88]. The result also applies to $\mathbb{S}^2$ with slight modifications.

However, it seems difficult to show the analogue of Kempe's theorem on the sphere with these methods, because the projective point of view does not distinguish opposite points on the sphere.

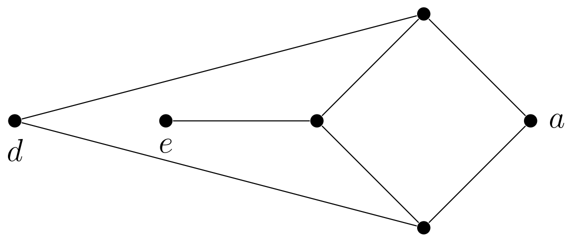

Figure 1: On the plane, the *Peaucellier-Lipkin straight-line motion linkage* forces the point a to move on a straight line. The vertices d and e are fixed to the plane.

2. Main results

Definition 2.1. An *abstract linkage* $\mathcal{L}$ on a Riemannian manifold $\mathcal{N}$ is a graph (V, E) together with :

1. A function $l : E \longrightarrow \mathbb{R}^+$ (which gives the length of each edge) ;
2. A subset $F \subseteq V$ of *fixed vertices* ;
3. A function $\phi_0 : F \longrightarrow \mathcal{N}$ which indicates where the edges of F are fixed.

Definition 2.2. Let $\mathcal{L}$ be an abstract linkage on a manifold $\mathcal{N}$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. A *realization* of a linkage $\mathcal{L}$ on $\mathcal{M}$ is a function $\phi : V \longrightarrow \mathcal{M}$ such that :

1. $\phi|_F = \phi_0$;
2. For each edge $v_1v_2 \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1v_2)$, where δ is the Riemannian distance on $\mathcal{M}$.

Definition 2.3. Let $\mathcal{L}$ be an abstract linkage on a manifold $\mathcal{N}$. Let $W \subseteq V$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. The *partial configuration space of $\mathcal{L}$ on $\mathcal{M}$ with respect to W*, written $\mathcal{E}(W, \mathcal{M})$, is the following set of functions from W to $\mathcal{M}$:

$$\mathcal{E}(W, \mathcal{M}) = \{\phi|_W \mid \phi \text{ realization of } \mathcal{L}\}\,.$$

Definition 2.4. A *semi-algebraic subset of* $(\mathbb{S}^d)^n$ is a set $A \subseteq (\mathbb{S}^d)^n$ such that there exist $N \geq n$, $m \in \mathbb{N}$ and $f : (\mathbb{R}^{d+1})^N = \mathbb{R}^{(d+1)N} \longrightarrow \mathbb{R}^m$ a polynomial such that :

$$A = \left\{ a \in (\mathbb{S}^d)^n \;\middle|\; \exists b \in (\mathbb{S}^d)^{N-n}, f(a,b) = 0 \right\}.$$

A is called an *algebraic subset of* $(\mathbb{S}^d)^n$ when we can choose $N = n$.

In other words, the semi-algebraic subsets of $(\mathbb{S}^d)^n$ are the projections of the algebraic subsets of $(\mathbb{S}^d)^N$ for any $N \geq n$.

Using techniques similar to [KM02], but with different elementary linkages, we proved :

Theorem 2.5 (Kempe's theorem on $\mathbb{S}^d$, $d \geq 2$)**.** *Let $d \geq 2$ and $n \geq 1$. Let A be a semi-algebraic subset of $(\mathbb{S}^d)^n$. Then there exists an abstract linkage $\mathcal{L} = (V, E, l, F, \phi_0)$ and $W \subseteq V$ such that $\mathcal{E}(W, \mathbb{S}^d) = A$.*

Note that for any linkage $\mathcal{L}$, and for any $W \subseteq V$, $\mathcal{E}(W)$ is a semi-algebraic subset of $(\mathbb{S}^d)^n$, so this theorem describes exactly the sets which are partial configuration spaces.

Kempe's original theorem on the plane was only for $n = 1$ and for algebraic subsets of $\mathbb{R}^2$ intersected with an euclidian ball. However, the corresponding theorem for linkages on $\mathbb{R}^2$, semi-algebraic subsets of $\mathbb{R}^2$ and $n \geq 1$ is a direct consequence of Kapovich and Millson's results [KM02]. A similar theorem for linkages in $\mathbb{R}^d$ has been proved by Timothy Good Abbott [Abb08].

Our proof for Theorem 2.5 also gives a new proof for the following result :

Theorem 2.6 (Differential universality theorem on $\mathbb{S}^d$, $d \geq 2$)**.** *Let M be a smooth compact manifold. There exists a linkage $\mathcal{L}$ on $\mathbb{S}^d$ for which $\mathcal{E}(V, \mathbb{S}^d)$ is diffeomorphic the disjoint union of a finite number of copies of M.*

3. Questions

Question 3.1. Is it possible to replace "*a finite number of copies*" by "*one copy*" in Theorem 2.6 ? (This question is also open on the plane.)

Question 3.2. What happens when $\mathbb{S}^d$ is replaced by any Riemannian manifold in Theorems 2.5 and 2.6 ?

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Some remarks on the reconstruction problems of symplectic and cosymplectic manifolds

Agnieszka KOWALIK

1. Introduction

The presentation will contain some results concerning reconstruction problems of symplectic and cosymplectic manifolds.
In both cases following theorems of M. Rubin will be used:

Theorem 1.1 (M. Rubin [2]). *Let X and Y be regular topological spaces and let $\mathrm{H}(X)$, $\mathrm{H}(Y)$ denote groups of all homeomorphisms on X, Y respectively. Let $G \leq \mathrm{H}(X)$ and $H \leq \mathrm{H}(Y)$ be factorizable and non-fixing. Assume that there is an isomorphism $\varphi : G \to H$. Then there is a unique homeomorphism $\tau : X \to Y$ such that for any $g \in G$ one have $\varphi(g) = \tau g \tau^{-1}$.*

Theorem 1.2 (M. Rubin [2]). *Let X, Y be regular topological spaces and let $G \leq \mathrm{H}(X)$, $H \leq \mathrm{H}(X)$. Assume that*

1. *There are $G_1 \leq G$ and $H_1 \leq H$ such that G_1, H_1 are factorizable and non-fixing groups of X and Y respectively.*
2. *For every $x \in X$, $\mathrm{int}\overline{G(x)} \neq \emptyset$ and for every $y \in Y$, $\mathrm{int}\overline{H(y)} \neq \emptyset$.*

Suppose that there is a group isomorphism $\varphi : G \to H$. Then there is a homeomorphism $\tau : X \to Y$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in G$.

In the case of symplectic manifold (M, ω) the symbol $\mathrm{Symp}(M, \omega)$ will stand for the group of all symplectomorphisms on (M, ω). In the case of cosymplectic manifold (M, θ, ω) symbols $\mathrm{Cosymp}(M, \theta, \omega)$, $\mathrm{Ham}(M, \theta, \omega)$, $\mathrm{Grad}(M, \theta, \omega)$ and $\mathrm{Ev}(M, \theta, \omega)$ will stand for the groups of all cosymplectomorphisms and hamiltonian, gradient and evolution cosymplectomorphisms respectively. In both symplectic and cosymplectic cases if G is a group then G_c denotes its subgroup of all compactly supported elements and G_0 denotes its subgroup of all elements that are isotopic with the identity.

2. Main results

Our first result is an extension of theorem of A. Banyaga [1]. This can be done by using Thorem 1.2. Using it we obtain the following:

Theorem 2.1. *Let* (M_i, ω_i) *for* $i = 1, 2$ *be symplectic manifolds and let* $\varphi : \mathrm{Symp}(M_1, \omega_1) \to \mathrm{Symp}(M_2, \omega_2)$ *or* $\varphi : \mathrm{Symp}(M_1, \omega_1)_0 \to \mathrm{Symp}(M_2, \omega_2)_0$ *be an isomorphism. Then there is a unique diffeomorphism* $\tau : M_1 \to M_2$ *such that* $\varphi(f) = \tau f \tau^{-1}$ *for any* $f \in \mathrm{Symp}(M_1, \omega_1)$ *and* $\tau^* \omega_2 = \lambda \omega_1$ *for some constant* λ.

In [1] one can find a similar result, but with stronger assumptions. Namely it must be fullfiled that both M_i are compact or that stmplectic pairing for both ω_i is identically equal to zero.

Our next results deal with cosymplectic manifolds. Among the others we obtain an analogon of above theorem for cosymplectic manifolds.
Our first result is the following:

Proposition 2.2. *Groups* $\mathrm{Ham}(M, \theta, \omega)$ *and* $\mathrm{Grad}(M, \theta, \omega)$ *are factorizable and non-fixing.*

By using above Proposition and Theorem 1.1 we obtain immediately:

Corollary 2.3. *Let* $(M_1, \theta_1, \omega_1)$ *and* $(M_2, \theta_2, \omega_2)$ *be cosymplectic manifolds and let* $G(M_i) = \mathrm{Ham}_c(M_i, \theta_i, \omega_i)$ *or* $G(M_i) = \mathrm{Grad}_c(M_i, \theta_i, \omega_i)$ *for* $i = 1, 2$. *If there is an isomorphism* $\varphi : G(M_1) \to G(M_2)$ *then there is a unique homeomorphism* $\tau : M_1 \to M_2$ *such that for any* $g \in G(M_1)$ *one have* $\varphi(g) = \tau g \tau^{-1}$.

Our next result is the following extension of Theorem of Takens [4] to the cosymplectic case.

Theorem 2.4. *Let* $(M_1, \theta_1, \omega_1)$ *and* $(M_2, \theta_2, \omega_2)$ *be cosymplectic manifolds with complete Reeb vector fields. Let*

$$\tau : (M_1, \theta_1, \omega_1) \to (M_2, \theta_2, \omega_2)$$

be a homeomorphism such that

$$\tau h \tau^{-1} \in \mathrm{Cosymp}(M_2) \Leftrightarrow h \in \mathrm{Cosymp}(M_1)$$

or

$$\tau h \tau^{-1} \in \mathrm{Grad}(M_2, \theta_2, \omega_2) \Leftrightarrow h \in \mathrm{Grad}(M_1, \theta_1, \omega_1),$$

or

$$\tau h \tau^{-1} \in \mathrm{Ev}(M_2, \theta_2, \omega_2) \Leftrightarrow h \in \mathrm{Ev}(M_1, \theta_1, \omega_1).$$

Then τ is a $\mathcal{C}^\infty$ diffeomorphism.

Theorem 2.5. *Let $(M_1, \theta_1, \omega_1)$ and $(M_2, \theta_2, \omega_2)$ be cosymplectic manifolds. Let $G(M_i)$ be either $G(M_i) = \mathrm{Cosymp}(M_i, \theta_i, \omega_i)$ or $G(M_i) = \mathrm{Ev}(M_i, \theta_i, \omega_i)$ or $G(M_i) = \mathrm{Grad}(M_i, \theta_i, \omega_i)$ for $i = 1, 2$. If there exists an isomorphism $\varphi : G(M_1) \to G(M_2)$ then there is a unique smooth diffeomorphism $\tau : M_1 \to M_2$ such that for any $f \in G(M_1)$ there is $\varphi(f) = \tau f \tau^{-1}$ and $\tau^*\omega_1 = \lambda\omega_2$.*

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Generalized Newton transformation and its applications to extrinsic geometry

Wojciech KOZŁOWSKI

Analyzing the study of Riemannian geometry we see that its basic concepts are related with some operators, such as shape, Ricci, Schouten operator, etc. and functions constructed of them, such as mean curvature, scalar curvature, Gauss-Kronecker curvature, etc. The most natural and useful functions are the ones derived from algebraic invariants of these operators, e.g., by taking trace, determinant and in general the r-th symmetric functions σ_r. However, the case $r > 1$ is strongly nonlinear and therefore more complicated. The powerful tool to deal with this problem is the Newton transformation T_r of an endomorphism A (strictly related with the Newton's identities) which, in a sense, enables a linearization of σ_r,

$$(r+1)\sigma_{r+1} = \operatorname{tr}(AT_r).$$

Although this operator appeared in geometry many years ago (see, e.g., [21, 29]), there is a continues increase of applications of this operator in different areas of geometry in the last years (see, among others, [1, 2, 3, 8, 10, 17, 18, 23, 24, 25, 28]).

All these results cause a natural question, what happen if we have a family of operators i.e. how to define the Newton transformation for a family of endomorphisms. A partial answer to this question can be found in the literature (operator T_r and the scalar S_r for even r [5, 15]), nevertheless, we expect that this case is much more subtle. This is because in the case of family of operators we should obtain more natural functions as in the case of one operator and consequently more information about geometry. In order to do this, for any multi–index u and generalized elementary symmetric polynomial σ_u we introduce transformations depending on a system of linear endomorphisms. Since these transformations have properties analogous to the Newton transformation (and in the case of one endomorphism coincides with it) we call this new object *generalized Newton transformation* (GNT) and denote by T_u. The concepts of GNT is based on the variational formula for the r–th symmetric function

$$\frac{d}{d\tau}\sigma_{r+1}(\tau) = \operatorname{tr}\left(T_r \cdot \frac{d}{d\tau}A(\tau)\right),$$

Extended Abstract (joint paper with K. Andrzejewski and K. Niedzialomski) which is crucial in many applications and, as we will show, characterize Newton's transformations. Surprisingly enough, according to knowledge of the authors, GNT has never been investigated before.

1. Generalized Newton transformation (GNT)

Let A be an endomorphism of a p–dimensional vector space V. The *Newton transformation* of A is a system $T = (T_r)_{r=0,1,\dots}$ of endomorphisms of V given by the recurrence relations:

$$\begin{aligned} &T_0 = 1_V, \\ &T_r = \sigma_r 1_V - AT_{r-1}, \quad r = 1, 2, \dots \end{aligned}$$

Here σ_r's are elementary symmetric functions of A. If $r > p$ we put $\sigma_r = 0$. Equivalently, each T_r may be defined by the formula

$$T_r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j.$$

Observe that T_p is the characteristic polynomial of A. Consequently, by Hamilton–Cayley Theorem $T_p = 0$. It follows that $T_r = 0$ for all $r \geq p$.

The Newton transformation satisfies the following relations [21]:

(N1) Symmetric function σ_r is given by the formula

$$r\sigma_r = \operatorname{tr}(AT_{r-1}).$$

(N2) Trace of T_r is equal

$$\operatorname{tr} T_r = (p - r)\sigma_r.$$

(N3) If $A(\tau)$ is a smooth curve in $\operatorname{End}(V)$ such that $A(0) = A$, then

$$\frac{d}{d\tau}\sigma_{r+1}(\tau)_{\tau=0} = \operatorname{tr}\,(\frac{d}{d\tau}A(\tau)_{\tau=0} \cdot T_r), \quad r = 0, 1, \dots, p.$$

Condition (N3) is the starting point to define generalized Newton transformations.

Let V be a p–dimensional vector space (over $\mathbb{R}$) equipped with an inner product $\langle\, , \rangle$. For an endomorphism $A \in \operatorname{End}(V)$, let $A^\top$ denote the adjoint endomorphism, i.e. $\langle Av, w\rangle = \langle v, A^\top w\rangle$ for every $v, w \in V$. The space $\operatorname{End}(V)$ is equipped with an inner product

$$\langle\langle A, B\rangle\rangle = \operatorname{tr}(A^\top B), \quad A, B \in \operatorname{End}(V).$$

Let $\mathbb{N}$ denote the set of nonnegative integers. By $\mathbb{N}(q)$ denote the set of all sequences $u = (u_1, \dots, u_q)$, with $u_j \in \mathbb{N}$. The length $|u|$ of $u \in \mathbb{N}(q)$

is given by $|u| = u_1 + \ldots + u_q$. Denote by $\operatorname{End}^q(V)$ the vector space $\operatorname{End}(V) \times \ldots \times \operatorname{End}(V)$ (q–times). For $\mathbf{A} = (A_1, \ldots, A_q) \in \operatorname{End}^q(V)$, $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$ and $u \in \mathbb{N}(q)$ put

$$\begin{aligned} t^u &= t_1^{u_1} \ldots t_q^{u_q}, \\ t\mathbf{A} &= t_1 A_1 + \ldots + t_q A_q \end{aligned}$$

By a *Newton polynomial* of $\mathbf{A}$ we mean a polynomial $P_{\mathbf{A}} : \mathbb{R}^q \to \mathbb{R}$ of the form $P_{\mathbf{A}}(t) = \det(1_V + t\mathbf{A})$. Expanding $P_{\mathbf{A}}$ we get

$$P_{\mathbf{A}}(t) = \sum_{|u| \le p} \sigma_u t^u,$$

where the coefficients $\sigma_u = \sigma_u(\mathbf{A})$ depend only on $\mathbf{A}$. Observe that $\sigma_{(0,\ldots,0)} = 1$. It is convenient to put $\sigma_u = 0$ for $|u| > p$.

Consider the following (music) convention. For α we define functions $\alpha^\sharp : \mathbb{N}(q) \to \mathbb{N}(q)$ and $\alpha_\flat : \mathbb{N}(q) \to \mathbb{N}(q)$ as follows

$$\begin{aligned} \alpha^\sharp(i_1, \ldots, i_q) &= (i_1, \ldots, i_{\alpha-1}, i_\alpha + 1, i_{\alpha+1}, \ldots, i_q), \\ \alpha_\flat(i_1, \ldots, i_q) &= (i_1, \ldots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \ldots, i_q), \end{aligned}$$

i.e. $\alpha^\sharp$ increases the value of the α–th element by 1 and $\alpha_\flat$ decreases the value of α–th element by 1. It is clear that $\alpha^\sharp$ is the inverse map to $\alpha_\flat$.

Now, we may state the main definition. The *generalized Newton transformation* of $\mathbf{A} = (A_1, \ldots, A_q) \in \operatorname{End}^q(V)$ is a system of endomorphisms $T_u = T_u(\mathbf{A})$, $u \in \mathbb{N}(q)$, satisfying the following condition (generalizing (N3)):

For every smooth curve $\tau \mapsto \mathbf{A}(\tau)$ in $\operatorname{End}^q(V)$ such that $\mathbf{A}(0) = \mathbf{A}$

$$\text{(GNT)} \qquad \begin{aligned} \frac{d}{d\tau}\sigma_u(\tau)_{\tau=0} &= \sum_\alpha \langle\!\langle \frac{d}{d\tau} A_\alpha(\tau)_{\tau=0})^\top | T_{\alpha_\flat(u)} \rangle\!\rangle \\ &= \sum_\alpha \operatorname{tr}\left(\frac{d}{d\tau} A_\alpha(\tau)_{\tau=0} \cdot T_{\alpha_\flat(u)} \right). \end{aligned}$$

From the above definition it is not clear that generalized Newton transformation exists. In order to show the existence of Generalized Newton transformation, we introduce the following notation.

For $q, s \ge 1$ let $\mathbb{N}(q, s)$ be the set of all $q \times s$ matrices, whose entries are elements of $\mathbb{N}$. Clearly, the set $\mathbb{N}(1, s)$ is the set of multi–indices $i = (i_1, \ldots, i_s)$ with $i_1, \ldots, i_s \in \mathbb{N}$, hence $\mathbb{N}(s) = \mathbb{N}(1, s)$. Moreover, every matrix $\mathbf{i} = (i_l^\alpha) \in \mathbb{N}(q, s)$ may be identified with an ordered system $\mathbf{i} = (i^1, \ldots, i^q)$ of multi–indices $i^\alpha = (i_1^\alpha, \ldots, i_s^\alpha)$.

If $i = (i_1, \ldots, i_s) \in \mathbb{N}(s)$ then its *length* is simply the number $|i| = i_1 + \ldots + i_s$. For $\mathbf{i} = (i^1, \ldots, i^q) \in \mathbb{N}(q, s)$ we define its *weight* as an multi–index $|\mathbf{i}| = (|i^1|, \ldots, |i^q|) \in \mathbb{N}(q)$. By the *length* $\|\mathbf{i}\|$ of $\mathbf{i}$ we mean the length of $|\mathbf{i}|$, i.e., $\|\mathbf{i}\| = \sum_\alpha |i^\alpha| = \sum_{\alpha,l} i_l^\alpha$.

Denote by $\mathbb{I}(q,s)$ a subset of $\mathbb{N}(q,s)$ consisting of all matrices $\mathbf{i}$ satisfying the following conditions:

(1) every entry of $\mathbf{i}$ is either 0 or 1,
(2) the length of $\mathbf{i}$ is equal to s,
(3) in every column of $\mathbf{i}$ there is exactly one entry equal to 1, or equivalently $|\mathbf{i}^\top| = (1,\dots,1)$.

We identify $\mathbb{I}(q,0)$ with a set consisting of the zero vector $0 = [0,\dots,0]^\top$.

Let $\mathbf{A} \in \operatorname{End}^q(V)$, $\mathbf{A} = (A_1,\dots,A_q)$, and $\mathbf{i} \in \mathbb{N}(q,s)$. By $\mathbf{A}^{\mathbf{i}}$ we mean an endomorphism (composition of endomorphisms) of the form

$$\mathbf{A}^{\mathbf{i}} = A_1^{i_1^1} A_2^{i_1^2} \dots A_q^{i_1^q} A_1^{i_2^1} \dots A_q^{i_2^q} \dots A_1^{i_s^1} \dots A_q^{i_s^q}.$$

In particular, $\mathbf{A}^0 = 1_V$.

Theorem 1.1. *For every system of endomorphisms $\mathbf{A} = (A_1,\dots,A_q)$, there exists unique generalized Newton transformation $T = (T_u : u \in \mathbb{N}(q))$ of $\mathbf{A}$. Moreover, each T_u is given by the formula*

$$T_u = \sum_{s=0}^{|u|} \sum_{\mathbf{i}\in\mathbb{I}(q,s)} (-1)^{\|\mathbf{i}\|} \sigma_{u-|\mathbf{i}|} \mathbf{A}^{\mathbf{i}}, \tag{1.2}$$

where $\sigma_{u-|\mathbf{i}|} = \sigma_{u-|\mathbf{i}|}(\mathbf{A})$.

As a consequence of above theorem we obtain:

Theorem 1.3 (Generalized Hamilton–Cayley Theorem)**.** *Let $T = (T_u : u \in \mathbb{N}(q))$ be the generalized Newton transformation of $\mathbf{A}$. Then for every $u \in \mathbb{N}(q)$ of length greater or equal to p we have $T_u = 0$.*

Moreover the generalized Newton transformation $T = (T_u : u \in \mathbb{N}(q))$ of $\mathbf{A}$ satisfies the following recurrence relations:

Theorem 1.4.

$$T_0 = 1_V, \qquad \textit{where } 0 = (0,\dots,0), \tag{1.5}$$

$$\begin{aligned} T_u &= \sigma_u 1_V - \sum_\alpha A_\alpha T_{\alpha_\flat(u)} \\ &= \sigma_u 1_V - \sum_\alpha T_{\alpha_\flat(u)} A_\alpha, \end{aligned} \qquad \textit{where } |u| \geq 1. \tag{1.6}$$

2. Applications to extrinsic geometry

Let (M, g) be an oriented Riemannian manifold, D a p–dimensional (transversally oriented) distribution on M. Let q denotes the codimension of D. For each $X \in T_xM$ there is unique decomposition

$$X = X^{\top} + X^{\perp},$$

where $X^{\top} \in D_x$ and $X^{\perp}$ is orthogonal to D_x. Denote by $D^{\perp}$ the bundle of vectors orthogonal to D. Let ∇ be the Levi–Civita connection of g. ∇ induces connections $\nabla^{\top}$ and $\nabla^{\perp}$ in vector bundles D and $D^{\perp}$ over M, respectively.

Let $\pi : P \to M$ be the principal bundle of orthonormal frames (oriented orthonormal frames, respectively) of $D^{\perp}$. Clearly, the structure group G of this bundle is $G = O(q)$ ($G = SO(q)$, respectively).

Every element $(x, e) = (e_1, \ldots, e_q) \in P_x$, $x \in M$, induces the system of endomorphisms $\mathbf{A}(x, e) = (A_1(x, e), \ldots, A_q(x, e))$ of D_x, where $A_\alpha(x, e)$ is the shape operator corresponding to (x, e), i.e.

$$A_\alpha(x, e)(X) = -\left(\nabla_X e_\alpha\right)^{\top}, \quad X \in D_x.$$

Let $T(x, e) = (T_u(x, e))_{u \in \mathbb{N}(q)}$ be the generalized Newton transformation associated with $\mathbf{A}(x, e)$.

The bundle $\pi : P \to M$ and the vector bundles $TM \to M$, $D \to M$, $D^{\perp} \to M$ induce the pull–back bundles

$$E = \pi^{-1}TM, \quad E' = \pi^{-1}D \quad \text{and} \quad E'' = \pi^{-1}D^{\perp} \quad \text{over } P,$$

each with a fiber $(\pi^{-1}TM)_{(x,e)} = T_xM$, $(\pi^{-1}D)_{(x,e)} = D_x$ and $(\pi^{-1}D^{\perp})_{(x,e)} = D_x^{\perp}$, respectively. We have

$$E = E' \oplus E''.$$

Moreover, the connections $\nabla, \nabla^{\top}, \nabla^{\perp}$ of g induce pull–back connections $\nabla^E, \nabla^{E'}$ and $\nabla^{E''}$ in E, E' and E'', respectively.

Define the section $Y_u \in \Gamma(E)$, $u \in \mathbb{N}(q)$ as follows

$$Y_u(x, e) = \sum_{\alpha,\beta} T_{\beta_\flat \alpha_\flat(u)}(x, e)(\nabla_{e_\alpha} e_\beta)^{\top} + \sum_{\alpha} \sigma_{\alpha_\flat(u)}(x, e) e_\alpha. \tag{2.1}$$

Observe that the first component of Y_u is a section of E', whereas the second component is a section of E''. The section Y_u and the vector field $\widehat{Y_u} \in \Gamma(TM)$ obtained from Y_u by integration on the fibers of P play a fundamental role in our considerations.

Lemma 2.2. *The divergence of Y_u is given by the formula*

$$\mathrm{div}_E Y_u = -|u|\sigma_u + \sum_{\alpha,\beta}\Big[\mathrm{tr}\,(R_{\alpha,\beta}T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'}T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha}e_\beta)^\top) \\ - g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\beta)^\top) + \sum_\gamma g((\nabla_{e_\alpha}e_\gamma)^\top, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\gamma}e_\beta)^\top)\Big],$$

where $H_{D^\perp}$ denotes the mean curvature vector of the distribution $D^\perp$.

Put

$$\widehat{\sigma_u}(x) = \int_{P_x}\sigma_u(x,e)\,de = \int_G \sigma_u(x, e_0 a)\,da, \tag{2.3}$$

where (x, e_0) is a fixed element of P_x. We call $\widehat{\sigma_u}$'s *extrinsic curvatures* of a distribution D. Moreover, we define *total extrinsic curvatures*

$$\sigma_u^M = \int_M \widehat{\sigma_u}(x)\,dx. \tag{2.4}$$

Since the projection π in the bundle P is a Riemannian submersion, then by Fubini theorem

$$\sigma_u^M = \int_P \sigma_u(x,e)\,d(x,e).$$

Theorem 2.5. *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, the total extrinsic curvature σ_u^M satisfies*

(2.6)

$$|u|\sigma_u^M = \sum_{\alpha,\beta}\int_P\Big(\mathrm{tr}\,(R_{\alpha,\beta}T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'}T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha}e_\beta)^\top) \\ - g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\beta)^\top) + \sum_\gamma g((T^*_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha}e_\gamma)^\top, (\nabla_{e_\gamma}e_\beta)^\top)\Big),$$

where $H_{D^\perp}$ denotes the mean curvature vector of distribution $D^\perp$.

By Theorem 2.5, we have in particular

$$\sigma^M_{\alpha^\sharp(0,\dots,0)} = 0$$

and

$$2\sigma^M_{\alpha^\sharp\beta^\sharp(0,\dots,0)} = \int_P\Big((\mathrm{Ric}_D)_{\alpha,\beta} - g(H_{D^\perp}, (\nabla_{e_\alpha}e_\beta)^\top) \\ + \sum_\gamma g((\nabla_{e_\alpha}e_\gamma)^\top, (\nabla_{e_\gamma}e_\beta)^\top)\Big), \tag{2.7}$$

where $(\mathrm{Ric}_D)_{\alpha,\beta} = \mathrm{Ric}_D(e_\alpha, e_\beta)$ and Ric_D is the Ricci curvature operator in the direction of D, i.e.,

$$\mathrm{Ric}_D(X,Y) = \sum_i g(R(f_i,X)Y,f_i),$$

where (f_i) is an orthonormal basis of D.

If D is integrable i.e. D defines a foliation $\mathcal{F}$ then above theorems generalized some well known facts:

Corollary 2.8. *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, total extrinsic curvature σ_u^M of a distribution D with totally geodesic normal bundle is of the form*

$$|u|\sigma_u^M = \sum_{\alpha,\beta} \int_P \mathrm{tr}\,(R_{\alpha,\beta} T_{\beta_\flat \alpha_\flat(u)}).$$

Corollary 2.9. *Assume (M,g) is closed and of constant sectional curvature κ. Let $\mathcal{F}$ be a foliation on M with totally geodesic and integrable normal bundle $\mathcal{F}^\perp$. Then the total extrinsic curvatures of $\mathcal{F}$ depend on κ, the volume of M and the dimension of $\mathcal{F}$ only.*

Moreover we may also obtain formulae of Brito and Naveira [13] for mean extrinsic curvature S_r

$$\int_M S_r = \begin{cases} \binom{\frac{p}{2}}{\frac{r}{2}}\binom{q+r-1}{r}\binom{\frac{q+r-1}{2}}{\frac{r}{2}}^{-1}\kappa^{\frac{r}{2}}\mathrm{vol}(M) & \text{for } p \text{ even and } q \text{ odd} \\ 2^r\left(\left(\frac{r}{2}\right)!\right)^{-1}\binom{\frac{q}{2}+\frac{r}{2}-1}{\frac{r}{2}}\binom{\frac{p}{2}}{\frac{r}{2}}\kappa^{\frac{r}{2}}\mathrm{vol}(M) & \text{for } p \text{ and } q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

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On the homeomorphism and diffeomorphism groups fixing a point

Jacek LECH and Ilona MICHALIK

1. Introduction

Let M be a topological metrizable manifold and let $\mathcal{H}(M)$ be the identity component of the group of all compactly supported homeomorphisms of M. By $\mathcal{H}(M,p)$, where $p \in M$, we denote the identity component of the group of all $h \in \mathcal{H}(M)$ with $h(p) = p$.

Definition 1.1. A group G is called *perfect* if it is equal to its own commutator subgroup $[G,G]$, that is $H_1(G) = 0$.

Definition 1.2. A manifold M admits a *compact exhaustion* iff there is a sequence $\{M_i\}_{i=1}^{\infty}$ of compact submanifolds with boundary such that $M_1 \subset \mathrm{Int}M_2 \subset M_2 \subset \ldots$ and $M = \bigcup_{i=1}^{\infty} M_i$.

Theorem 1.3. *[3] Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then $\mathcal{H}(M)$ is perfect.*

The proof of Theorem 1.3 is a consequence of J.N.Mather's paper combined with results of R.D.Edwards and R.C.Kirby. A special case of Theorem 1.3 was already showed by G.M.Fisher.

2. Main results

Definition 2.1. A group is called *bounded* if it is bounded with respect to any bi-invariant metric.

Definition 2.2. For $g \in [G,G]$ the least k such that g is a product of k commutators is called the *commutator length* of g and is denoted by $\mathrm{cl}_G(g)$. For any perfect group G denote by cld_G the commutator length diameter of G, i.e. $\mathrm{cld}_G := \sup_{g\in G}\mathrm{cl}_G(g)$.

Definition 2.3. A group G is called *uniformly perfect* if G is perfect and $\mathrm{cld}_G < \infty$.

DEFINITION 2.4. Let G be a group. A *conjugation-invariant norm* on G is a function $\nu : G \to [0, \infty)$ for every $g, h \in G$ we have

1. $\nu(g) > 0$ if and only if $g \neq e$,
2. $\nu(g^{-1}) = \nu(g)$,
3. $\nu(gh) \leqslant \nu(g) + \nu(h)$,
4. $\nu(hgh^{-1}) = \nu(g)$.

It is easy to see that G is bounded if and only if any conjugation-invariant norm on G is bounded.

Observe that the commutator length cl_G is a conjugation-invariant norm on $[G, G]$, or on G if G is a perfect group.

Proposition 2.5. *Let G be perfect and bounded group. Then G is uniformly perfect.*

Our main results are the following

Theorem 2.6. *1. The groups $\mathcal{H}(\mathbb{R}^n, 0)$ and $\mathcal{H}(\mathbb{R}^n_+, 0)$ are perfect, where $\mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1}$.*

2. Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the group $\mathcal{H}(M, p)$ is perfect.

A similar result was obtained by T.Tsuboi. He proved that $\mathcal{H}([0, 1])$ is perfect by using different argument than that for Theorem 2.6. Next he generalized the result for Lipschitz homeomorphisms and for C^1-diffeomorphisms (resp. C^∞-diffeomorphisms) tangent (resp. infinitely tangent) to the identity at the endpoints. Observe as well that Theorem 2.6 was proved for M closed by K.Fukui in [2]. However, our proof is different than that in [2] and it leads to following theorem.

Theorem 2.7. *The group $\mathcal{H}(\mathbb{R}^n, 0)$ is uniformly perfect and its commutator length diameter is less or equal 2. The same is true for $\mathcal{H}(\mathbb{R}^n_+, 0)$.*

Let $\mathcal{D}^r(M)$ (resp. $\mathcal{D}^r(M, p)$) be the identity component of the group of all compactly supported C^r-diffeomorphisms of M (resp. fixing $p \in M$). It is easy to see that $\mathcal{D}^r(M, p)$ is not perfect for $r \geqslant 1$. Moreover, K.Fukui calculated that $H_1(\mathcal{D}^\infty(\mathbb{R}^n, 0)) = \mathbb{R}$.

Theorem 2.8. *1. $\mathcal{H}(\mathbb{R}^n, 0)$ is bounded group.*

2. Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the *group $\mathcal{H}(M)$ is bounded whenever $\mathcal{H}(M,p)$ is bounded.*

Note that this theorem is no longer true in the C^r category for $r \geqslant 1$. Choose a chart at p. Then there is the epimorphism $\mathcal{D}^r(M,p) \ni f \mapsto \mathrm{jac}_p f \in \mathbb{R}_+$, where $\mathrm{jac}_p f$ is the jacobian of f at p in this chart. From Proposition 1.3 in [1] an abelian group is bounded if and only if it is finite and Lemma 1.10 in [1] implies that $\mathcal{D}^r(M,p)$ is unbounded.

3. Questions

QUESTION 3.1. (1) Let $\mathcal{H}^{Lip}(M,p)$ be the compactly supported identity component of Lipschitz homeomorphism group fixing point p. The question is, whether the group $\mathcal{H}^{Lip}(M,p)$ is perfect or bounded.

(2) Denote by $\mathrm{Symp}(M,\omega;p)$ the compactly supported identity component of symplectomorphism group fixing point p. The problem is to calculate $H_1(\mathrm{Symp}(M,\omega;p))$.

(3) The same questions could be asked for contactomorphism groups and volume preserving diffeomorphism groups.

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Application of a new integral formula for two distributions with singularities on Riemannian manifolds

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1. The Volume of a Tube

1.1. The second fundamental forms of the tubular hypersurfaces

Definition 1.1. Let P be a topologically embedded sub-manifold (possibly with boundary) in a Riemannian manifold M, then a **tube** $T(P,r)$ of radius $r \geq 0$ about P is the set

$$(1.2)\quad \begin{aligned} T(P,r) = \{& m \in M : \text{there exists a geodesic } \xi \text{ of length } L(\xi) \leq r \\ & \text{from } m \text{ meeting } P \text{ orthogonally}\}. \end{aligned}$$

We shall also need a notation closely related to that of tube.

Definition 1.3. We call a hypersurface of the form

$$P_t = \{m \in T(P,r) : \text{distance}(m,P) = t\}$$

the tubular hypersurface at distance t from P.

For $0 < t \leq r$ the tubular hypersurfaces P_t form a natural foliation of the tubular region $T(P,r) - P$.

1.2. The volume of a tube in terms of the infinitesimal change of volume function

We assume that P is topologically embedded submanifold with compact closure of a complete Riemannian manifold M. For all $r \geq 0$ both $T(P,r)$ and P_r are measurable sets. Let

$$\mathrm{V}_P^M(r) = \text{the } n-\text{dimensional volume of } T(P,r),$$

$$\mathrm{A}_P^M(r) = \text{the } (n-1)-\text{dimensional volume of } P_r.$$

It is easy to show that $A_P^M(r)$ is the derivative of $V_P^M(r)$. We use the lemma:

Lemma 1.4. *Suppose that* $\exp_\nu : \{(p,v) \in \nu : \| v \| \leq r\} \longmapsto T(P,r)$ *is a diffeomorphism. Then*

$$A_P^M(r) = r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \mathcal{V}_u(r) du dP.$$

Lemma 1.5. *Suppose that* $\exp_\nu : \{(p,v) \in \nu : \| v \| \leq r\} \longmapsto T(P,r)$ *is a diffeomorphism. Then*

$$\begin{aligned}\frac{d}{dr} V_P^M(r) &= A_P^M(r) \\ &= r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \mathcal{V}_u(r) du dP.\end{aligned}$$

Proofs of this lemmas are available at [3].

2. Riemannian manifolds with singularities

In this section we work with Riemannian geometry of manifolds equipped with a pair of orthogonal plane fields. We want to generalize it to the case of plane fields with singularities, that is defined on a compact manifold except of singular set, the union of submanifolds of lower dimension. Till now, author produced a new integral formula (see[4]) obtained from integration of the divergence of a vector field built from Newton transforms of Weingarten operators applied to the mean curvature vectors of the plane fields under consideration. This formula, in a sense, analogous to the one obtained by Walczak in the 1990 [5].

We get reasonable applications of this formulae leading to provide obstructions for the existence of geometric structures - here, pairs of distributions - satisfying some geometric conditions (for example: being totally geodesic, minimal, umbilical and so on) on some special (locally symmetric, of constant curvature, positively/negatively curved and so on) Riemannian manifolds.

Let M be a Riemannian manifold, $\dim M \geq 3$, equipped with two complementary distributions D_1 and D_2. We assume that

$$p+q=n, \quad \text{where } \ p=\dim D_1, \ \ q=\dim D_2 \ \text{ and } \ n=\dim M.$$

Let us take a local orthonormal frame $e_1,\ldots,e_n$ adapted to D_1 and D_2, i.e., we assume that e_i is tangent to D_1 for $i=1,\ldots,p$ and e_α is tangent to D_2 for $\alpha = p+1,\ldots,n$.
The second fundamental forms B_m of D_m ($m=1,2$) are defined as follows:

$$B_1(X_1,Y_1) = \frac{1}{2}(\nabla_{X_1}Y_1 + \nabla_{Y_1}X_1)^\perp, \quad B_2(X_2,Y_2) = \frac{1}{2}(\nabla_{X_2}Y_2 + \nabla_{Y_2}X_2)^\top$$

for vector fields X_m and Y_m tangent to D_m.

The integrability tensors T_m of D_m $(m = 1, 2)$ are defined as follows:

$$T_1(X_1, Y_1) = \frac{1}{2}[X_1, Y_1]^{\perp}, \quad T_2(X_2, Y_2) = \frac{1}{2}[X_2, Y_2]^{\top}$$

for vector fields X_m and Y_m tangent to D_m.

Then the mean curvature vectors H_m of D_m are given by

$$H_1 = Trace B_1 = \sum_i B_1(e_i, e_i) = \sum_i (\nabla_{e_i} e_i)^{\perp}$$
$$H_2 = Trace B_2 = \sum_{\alpha} B_2(e_{\alpha}, e_{\alpha}) = \sum_i (\nabla_{e_{\alpha}} e_{\alpha})^{\top}.$$

Let us define the Weingarten operators by

$$A_1 : D_1 \times D_2 \to D_1, \qquad \langle A_1(X, N), Y \rangle = \langle B_1(X, Y), N \rangle$$
$$for\ X, Y \in D_1,\ N \in D_2$$
$$A_2 : D_2 \times D_1 \to D_2, \qquad \langle A_2(X', N'), Y' \rangle = \langle B_2(X', Y'), N' \rangle$$
$$for\ X', Y' \in D_2,\ N' \in D_1.$$

Assume now that M has *bounded geometry* (i.e., bounded sectional curvature and injectivity radii r_x, $x \in M$, separated away from zero). Let A be a finite set of singularities(points, closed curve, etc.) on M and codimension$A = n - 2$. Moreover, let $f : M/A \to [0, +\infty)$ be a function defined on M outside a finite set A.
We denote the tube $T(A, r)$ of radius $r \geq 0$ about set A by $N_A(r)$ and $\delta N_A(r)$ as the tubular hypersurface at a distance $r \geq 0$ from A.
We shall also need an one the well-known formula volume$\delta N_{\gamma}(r) \backsimeq L(\gamma) \cdot$ volume$S^{n-1}(r)$, where $\gamma \subset A$ is closed curve, $L(\gamma)$ is length of the the curve γ and $S^{n-1}(r) \subset R^n$ is sphere of radius r. In particular:

- in $\mathbf{R}^2$ we obtain volume $\delta N_{\gamma}(r) = 2r \cdot L(\gamma)$
- in $\mathbf{R}^3$ we obtain volume $\delta N_{\gamma}(r) = \pi r^2 \cdot L(\gamma)$

It leads to the following and useful lemma.

Lemma 2.1.

$$If \ \liminf_{r \to 0^+} \int_{\delta N_{\gamma}(r)} f > 0, then \ \int_M f^2 = 0.$$

These lemma will be used extensively and will allow us to proof the following theorems.

Theorem 2.2. *Let M being a compact Riemannian manifold of dimension $n \geq 3$ and A a finite subset of M. If $\int_M \parallel H_1 \parallel < \infty$ and $\int_M \parallel H_2 \parallel < \infty$ then*

$$\int_M ||B_1||^2 + ||B_2||^2 - ||H_1||^2 - ||H_2||^2 - ||T_1||^2 - ||T_2||^2 = \int_M K(D_1, D_2), \tag{2.3}$$

where $K(D_1, D_2)$ is a generalization on the Ricci curvature equal to the sum

$$\sum_{i,\alpha} < R(e_i, e_\alpha)e_\alpha, e_i >$$

and called the mixed scalar curvature.

Theorem 2.4. *Let M being a compact Riemannian manifold of dimension $n \geq 3$ and A a finite subset of M. If $\int_M \parallel A_1 \parallel < \infty$ and $\int_M \parallel A_2 \parallel < \infty$ then*

$$\begin{aligned}
&\int_M \langle Ric(H_2), H_1 \rangle = \\
&\int_M \langle H_1, (\nabla_{H_2} H_1)^\perp \rangle + \langle H_2, (\nabla_{H_1} H_2)^\top \rangle + \\
&\langle Tr^\perp(\nabla_\bullet T_1)(\bullet, H_2), H_1 \rangle + \langle Tr^\top(\nabla_\bullet T_2)(\bullet, H_1), H_2 \rangle + \\
&\langle A_1^{H_1}, \nabla_\bullet^\top H_2 \rangle + \langle A_2^{H_2}, \nabla_\bullet^\perp H_2 \rangle + \\
&\sum_i \langle A_1\big(H_2, (\nabla_{e_i} H_1)^\perp\big), e_i \rangle + \sum_\alpha \langle A_2\big(H_1, (\nabla_{e_\alpha} H_2)^\top\big), e_\alpha \rangle + \\
&2\sum_i \langle \big(\nabla_{T_1(e_i, H_2)} e_i\big)^\perp, H_1 \rangle + 2\sum_\alpha \langle \big(\nabla_{T_2(e_\alpha, H_1)} e_\alpha\big)^\top, H_2 \rangle - \\
&\langle A_2(H_1, H_2), H_1 \rangle - \langle A_1(H_2, H_1), H_2 \rangle.
\end{aligned} \tag{2.5}$$

Corollary 2.6. *Equality (2.3) holds if and only if $K(D_1, D_2) > 0$.*

Proposition 2.7. *If distributions D_1 and D_2 are totally geodesic and D_2 is the orthogonal complement of D_1, then $H_1 = 0$ and $H_2 = 0$ and we get*

$$\int_M K(D_1, D_2) = \int_M \left(||T_1||^2 + ||T_2||^2\right),$$

where H_m and T_m (m=1, 2) denote mean curvature vectors and integrability tensors of distributions D_m, respectively.

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Almost Contact Confoliations and their Dimensionality Reductions

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1. Introduction

The following two recent results suggest that the topology of codimension one foliations of high dimensional manifolds has two opposite possibilities. Namely, the absence/presence of leafwise symplectic structures could make it dissimilar/similar to the 3-dimensional topology of foliations.

Theorem 1.1 (Meigniez [3]). *Let $\mathcal{F}$ be a transversely oriented codimension one foliation of a closed $(n+3)$-manifold M^{n+3} which is just smooth. Then we can deform $\mathcal{F}$ to a minimal (all leaves dense) foliation $\mathcal{F}'$ such that $T\mathcal{F}'$ is homotopic to $T\mathcal{F}$ as a tangent hyperplane field on M^{n+3} $(n > 0)$.*

Theorem 1.2 (Martínez Torres [2]). *Let $\mathcal{F}$ be an oriented codimension one foliation of an oriented closed $(2n+3)$-manifold M^{2n+3}. Suppose that there exists a closed 2-form ω on M with $\omega^{n+1}|T\mathcal{F} > 0$. Then Donaldson-Auroux approximately holomorphic geometry provides a codimension $2n$ submanifold N^3 such that $\mathcal{G} = \mathcal{F}|N^3$ is a taut foliation and N^3 meets each leaf of $\mathcal{F}$ at a single leaf of $\mathcal{G}$ (i.e., $M^{2n+3}/\mathcal{F} = N^3/\mathcal{G}$).*

Mitsumatsu found another kind of leafwise symplectic foliation which has the same leaf space as a non-taut foliation of a 3-manifold.

Theorem 1.3 (Mitsumatsu [4]). *The Lawson foliation of S^5, which is a leafwise fattening of the Reeb foliation of S^3, admits a leafwise symplectic structure. (It is the restriction of a non-closed 2-form on S^5).*

The Eliashberg-Thurston 3-dimensional confoliation theory discretized the vast whole of foliations into contact structures. In [6], the author defined higher dimensional confoliations by means of almost contact geometry:

Definition 1.4. Let $([\alpha], [\omega])$ be the pair of conformal classes of a 1-form α and a 2-form ω on a closed oriented $(2n+1)$-manifold M^{2n+1}.

1. We say that $([\alpha], [\omega])$ (or $[\alpha]$ itself) is an almost contact structure if

it satisfies $[\alpha] \wedge [\omega]^n > 0$ (for some $[\omega]$).

2. We say that an almost contact structure is a contact structure (resp. a foliation) if $\ker \alpha$ is contact (resp. tangent to a foliation).
3. We say that an almost contact structure is a confoliation if it satisfies $[\alpha \wedge d\alpha^n] \geq 0$. If moreover it belongs to the closure of the space of contact structures or to the space of foliations (in the space of pairs $([\alpha], [\omega])$ with smooth topology), it is called a strict confoliation.

Though we omit the precise construction in this abstract, we fix a situation where we can obtain a family of higher dimensional strict confoliations which goes to a leafwise symplectic foliation (§2).

Outside the Donaldson-Auroux approximately holomorphic geometry, convex hypersurface theory due to Giroux is the most powerful tool in contact topology. In 3-manifold case, it can be considered as a contact version of sutured manifold theory due to Gabai (Honda-Kazez-Matić [1]). Further Honda's category theory regards a contact structure between convex surfaces as a morphism. This author is now trying to generalize this theory in order to understand the (perhaps proper and natural) affinity between high dimensional contact topology and 3-dimensional one (§3).

2. Confoliations

We start with the Thurston-Winkelnkemper-Giroux construction of contact structure on a closed $(2n+1)$-manifold M^{2n+1} equipped with a pagewise exact symplectic open-book structure $\mathcal{O}$. One may say that this construction is an extension (or a fattening) of a contact structure $\ker \mu$ on the binding $\mathcal{B}$ of $\mathcal{O}$ under the presence of exact symplectic filling pages. In [5], the author pointed out that we can further construct a family of contact structures convergent to a foliation $\mathcal{F}$ in the case where the Reeb field of μ is tangent to a Riemannian foliation $\mathcal{G}$ of $\mathcal{B}$ defined by a closed 1-form ν $(n > 1)$. The foliation $\mathcal{F}$ consists of a closed leaf $L = \mathcal{B} \times S^1$, page leaves coiling into L, and a trivial extension of $\mathcal{G}$ also coiling into L. (One might remember the Calabi conjecture theorem of Friedl-Vidussi.)

Theorem 2.1 ([6]). *Assume that the Reeb field of μ is tangent to $\ker \nu$. Suppose moreover that there exists a closed 2-form Ω on $\mathcal{B}$ such that*

1. *$\nu \wedge (d\mu + \varepsilon\Omega)^n > 0$ holds for small $\varepsilon > 0$, and*
2. *Ω extends to a closed 2-form on the page.*

Then we can construct a family of pairs (α_t, ω_t) on M^{2n+1} such that

1. *$\alpha_t \wedge (d\alpha_t)^n > 0$ for $0 \leq t < 1$,*
2. *$\omega_0 = d\alpha_0$, $\ker \alpha_1 = T\mathcal{F}$,*

3. $\alpha_t \wedge \omega_t^n > 0$ *for* $0 \leq t \leq 1$ *,   and*
4. $\omega_1 | \mathcal{F}$ *is leafwise symplectic.*

EXAMPLE 2.2. The Milnor fibration of $x^3 + y^3 + z^3 = 0$ on $\mathbb{C}^3$ naturally defines a pagewise exact symplectic open-book on (small) S^5. In this case Mitsumatsu [4] found the above 2-form Ω. Then the strict confoliations $([\alpha_t], [\omega_t])$ starts with the standard contact structure on S^5 and goes to the Lawson foliation with leafwise symplectic structure.

3. Convex hypersurfaces

Let Σ be a compact oriented hypersurface in a contact manifold. Suppose that, for a suitable representative $\alpha \in [\alpha]$, the sign of $d\alpha^n | T\Sigma$ defines a dividing $\Sigma \setminus \Gamma = \Sigma_+ \cup (-\Sigma_-)$ along a submanifold $\Gamma \subset \Sigma$ into strongly pseudo-convex domains $\Sigma_\pm$. Then Σ is called a convex hypersurface. (One may generalize this notion in various ways, e.g., for leafwise symplectic foliations, one may consider a union of pseudo-convex domains on leaves connected with cylindrical "sutures" transverse to the foliation.)

In [5], the author generalized the Lutz modification of 3-dimensional contact structure by using convex hypersurface theory. In general, this modification changes the contact structure drastically (indeed makes it non-fillable) and produces a convex hypersurface which contains a strongly pseudo-convex domain with disconnected boundary. We call such a domain on a convex hypersurface a Calabi hypersurface.

QUESTION 3.1. Is there any Calabi hypersurface in $S^{2n+3} \subset \mathbb{C}^{n+2}$?

We consider a certain generalization of the convex version of Thurston-Bennequin inequality in higher dimension since the natural generalization of the usual inequality does not hold even locally. On the other hand, if a convex hypersurface violates the inequality, it contains a Calabi hypersurface. Here we notice that, while a surfaces in contact 3-manifolds are smoothly approximated by convex ones, that is not the case with hypersurfaces in higher dimension. Anyway the generalized Lutz modification produces a convex hypersurface which violates the inequality.

The existence problem of convex hypersurfaces is also interesting. First we see that the boundary of the standard neighbourhood of a Legendrian submanifold is naturally convex. Such a convex hypersurface is said to be tubular. On the other hand, the Donaldson-Auroux approximately holomorphic geometry is the main source of non-tubular convex hypersurfaces. For example it provides a pagewise exact symplectic open-book described in §2, and a pair of pages forms a convex hypersurface. (Of course, it is tubular if the page is a cotangent bundle.) Then Theorem 1.2 suggests that a convex hypersurface theory could embody some affinity between high dimensional contact topology and 3-dimensional one.

Further as is described in §1, convex surfaces are objects in Honda's category theory. In [7], the author is trying to generalize this theory to higher dimension. His aim is to show that some quotient of higher dimensional contact category becomes equivalent to Honda's category.

In summary,

Question 3.2. Can we "split" the $2n+3$-dimensional topology of almost contact confoliation into $2n+2$-dimensional (not $2n$-dimensional !) symplectic geometry and 3-dimensional confoliation theory ?

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Dehn surgeries along $(-2, 3, 2s + 1)$-type Pretzel knot with no $\mathbb{R}$-covered foliation and left-orderable groups

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1. Main theorem and its background

A codimension one, transversely oriented foliation $\mathcal{F}$ on a closed 3-manifold M is called a Reebless foliation if $\mathcal{F}$ does not contain a Reeb component. By the theorems of Novikov, Rosenberg, and Palmeira, if M is not homeomorphic to $S^2 \times S^1$ and contains a Reebless foliation, then M has properties that the fundamental group of M is infinite, the universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$ and all leaves of its lifted foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ are homeomorphic to a plane. In this case we can consider a quotient space $\mathcal{T} = \widetilde{M}/\widetilde{\mathcal{F}}$, and $\mathcal{T}$ is called a leaf space of $\mathcal{F}$. The leaf space $\mathcal{T}$ becomes a simply connected 1-manifold, but it might be a non-Hausdorff space. If the leaf space is homeomorphic to $\mathbb{R}$, $\mathcal{F}$ is called an $\mathbb{R}$-covered foliation. The fundamental group $\pi_1(M)$ of M acts on the universal cover $\widetilde{M}$ as deck transformations. Since this action maps a leaf of $\widetilde{\mathcal{F}}$ to a leaf, it induces an action of $\pi_1(M)$ on the leaf space $\mathcal{T}$. In fact, it is known that the action has no global fixed point and it acts on $\mathcal{T}$ as a homeomorphism.

In 2004, J.Jun proved the following theorem.

Theorem 1.1. (J. Jun [10, Theorem 2]) *Let K be a $(-2, 3, 7)$-Pretzel knot in S^3 and $E_K(p/q)$ be a closed manifold obtained by Dehn surgery along K with slope p/q. If $p/q \geqq 10$ and p is odd, then $E_K(p/q)$ does not contain an $\mathbb{R}$-covered foliation.*

Dehn surgery along a knot K in S^3 is a procedure which yields a new closed 3-manifold by digging a solid torus along the knot K and successively attaching a solid torus non-trivially along its boundary. The resultant manifold is determined by the knot K and a rational number ρ which represents a slope of meridian of the attaching solid torus on the boundary torus of the digged sphere. For basic definition and properties of Dehn surgery, see Boyer [1].

We proved the following theorem in [14] which is an extension of Theorem 1.1 to the case of $(-2, 3, 2s + 1)$-type Pretzel knot ($s \geqq 3$).

Theorem 1.2. (Main Theorem) *Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot in S^3 ($s \geqq 3$). If $q > 0$, $p/q \geqq 4s+7$ and p is odd, then $E_{K_s}(p/q)$ does not contain an $\mathbb{R}$-covered foliation.*

In [16], Roberts, Shareshian and Stein proved that there exist infinitely many closed orientable hyperbolic 3-manifolds which do not contain a Reebless foliation. J.Jun also proved in [10] conditions for Dehn surgery slopes that $(-2, 3, 7)$-Pretzel knot K yields closed 3-manifolds which do not contain a Reebless foliation. In [6], Fenley showed that there exist infinitely many closed hyperbolic 3-manifolds which do not admit essential laminations.

These theorems are proved by a similar strategy as follows. Let M be a closed 3-manifold and $\mathcal{F}$ be a Reebless foliation in M. Then, the fundamental group $\pi_1(M)$ acts on the leaf space $\mathcal{T}$ of $\mathcal{F}$ as an orientation preserving homeomorphism which has no global fixed point. By the theorem of Palmeira, $\mathcal{F}$ is determined by its leaf space $\mathcal{T}$. Therefore, for any simply connected 1-manifold $\mathcal{T}$, if there exists a point of $\mathcal{T}$ which is fixed by any action of $\pi_1(M)$ then M cannot contain a Reebless foliation.

In order to use above method to prove our main theorem, we will need an explicit presentation of the fundamental group. Moreover, it is better for proving our theorem that its presentation has simpler form because our investigation of existence of a global fixed point becomes easy if its presentation has fewer generators.

In the next section, we will explain how to get a good presentation of the fundamental group of closed 3-manifold obtained by Dehn surgery along our Pretzel knots K_s. We do not explain the proof of Main theorem here, please see [14].

2. A good presentation of fundamental groups

In the proof of [10], Jun uses the presentation of a knot group of $(-2, 3, 7)$-pretzel knot which obtained by the computer program, SnapPea [18]. Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot in S^3. In order to obtain a good presentation of the knot group of K_s and its meridian-longitude pair, we take the following procedure.

We first notice that K_s is a tunnel number one knot for all $s \geqq 3$ by the theorem of Morimoto, Sakuma and Yokota [13]. A knot K is called a tunnel number one knot if there is an arc τ in S^3 which intersects K only on its endpoints and the closure of $S^3 \setminus (K \cup \tau)$ is homeomorphic to a genus two handlebody. Therefore the knot group of K_s can have a presentation which has two generators and one relator.

It is well known that two groups G and G' are isomorphic if there is a sequence of Tietze transformations such that a presentation of G is transformed into its of G' along this sequence. Although it is generally difficult to find such a sequence, we can find the required sequence by applying the procedure which appeared in the paper of Hilden, Tejada and Toro [7] as follows. At the first step, we obtain the Wirtinger presentation G_1 of the knot K. Then we collapse one crossing of the knot diagram and get a graph Γ which is thought as a resulting object $K \cup \tau$ because the exteriors of Γ and $K \cup \tau$ in S^3 are homeomorphic. We modify Γ with local moves in sequence forward to the shape $S^1 \vee S^1$, and in the same time we modify the presentations by a Tietze transformation which corresponds to each local move. In the sequel we finally obtain the graph which is homeomorphic to $S^1 \vee S^1$ and the corresponding presentation which has two generators and one relator.

In order to apply this procedure to the case of K_s, we add some new local moves which are not treated in [7], and we refer the sequence of modifications which appeared in the paper of Kobayashi [11] to obtain our sequence of modifications. Then we obtain the following presentation.

$$G_{K_s} = \pi_1(S^3 \setminus K_s) = \langle c, l \mid clc\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1} \rangle$$

In order to obtain a presentation of $G_{K_s}(p,q) = \pi_1(E_{K_s}(p/q))$, we have to get a presentation of a meridian-longitude pair. The way of the calculation is as follows. We first fix a meridian c and get a presentation of a longitude L_1 which are compatible with the Wirtinger presentation by using the method which appeared in the book of Burde and Zieschang [3]. Then we continue to modify L_i from $i = 1$ along the steps of sequence of Tietze transformations.

By modifying the last presentation of the longitude slightly, we finally obtain the following presentation.

$$L = \bar{c}^{2s-2} lcl^s cl^s cl \bar{c}^{2s+9}$$

In summary we obtain the following.

Proposition 2.1. *Let K_s be a $(-2, 3, 2s+1)$-type Pretzel knot ($s \geqq 3$). Then the knot group of K_s has a presentation*

$$G_{K_s} = \langle c, l \mid clc\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1} \rangle,$$

and an element which represents the meridian M is c and an element of the longitude L is $\bar{c}^{2s-2} lcl^s cl^s cl \bar{c}^{2s+9}$.

By Proposition 2.1, we obtain a presentation of $G_{K_s}(p,q)$ as follows:

$$G_{K_s}(p,q) = \langle c, l \mid clc\bar{l}\bar{c}\bar{l}^s\bar{c}\bar{l}clcl^{s-1}, M^p L^q \rangle.$$

3. Problems and related topics

We will discuss some related topics and problems in this section.

We first mention about future problems. In Theorem 1.2 we explore the case when the leaf space $\mathcal{T}$ is homeomorphic to $\mathbb{R}$. It is the first problem that we extend Theorem 1.2 to the general case that a leaf space $\mathcal{T}$ is homeomorphic to a simply connected 1-manifold which might not be a Hausdorff space similar to the result of Jun [10]. In this case, the situation of a leaf space $\mathcal{T}$ is very complicated. But we already obtained the explicit presentation of the fundamental group of $E_{K_s}(p/q)$, we are going to investigate the action of the fundamental group to a leaf space referring the discussions used in [16] and [10]. One of other directions of investigations is that we extend Theorem 1.2 to the case for $(-2, 2r+1, 2s+1)$-type Pretzel knot $K_{r,s}$ ($r \geqq 1$, $s \geqq 3$). Although we need the explicit presentation of the fundamental group $\pi_1(S^3 \setminus K_{r,s})$, A.Tran already presented the presentation of $\pi_1(S^3 \setminus K_{r,s})$ with two generators and one relator in [17]. Using this presentation we are going to calculate the meridian-longitude pair which compatible with this presentation, and also extend Theorem 1.2 to the case of $K_{r,s}$ cooperating with him.

Next we discuss about some related topics. We first discuss our result in the viewpoint of Dehn surgery on knots. A knot K in S^3 has a finite or cyclic surgery if the resultant manifold $E_K(p/q)$ obtained by a non-trivial Dehn surgery along K with a slope p/q has a property that its fundamental group is finite or cyclic respectively. Determining and classifying which knots and slopes have a finite or cyclic surgery are an interesting problem. If $E_K(p/q)$ contains a Reebless foliation, we can conclude that $E_K(p/q)$ does not have a finite and cyclic surgery. For example, Delman and Roberts showed that no alternating hyperbolic knot admits a non-trivial finite and cyclic surgery by proving the existence of essential laminations [5]. Our Pretzel knots K_s are in the class of a Montesinos knot. In [9], Ichihara and Jong showed that for a hyperbolic Montesinos knot K if K admits a non-trivial cyclic surgery it must be $(-2, 3, 7)$-pretzel knot and the surgery slope is 18 or 19, and if K admits a non-trivial acyclic finite surgery it must be $(-2, 3, 7)$-pretzel knot and the slope is 17, or $(-2, 3, 9)$-pretzel knot and the slope is 22 or 23. In contrast, by this theorem, infinitely many knots in the family of pretzel knot $\{K_s\}$ which appeared in Theorem 1.2 do not admit cyclic or finite surgery. Then we have following corollary directly.

Corollary 3.1. *There are infinitely many pretzel knots which does not admit finite or cyclic surgery, but they admit Dehn surgery which produces a closed manifold which cannot contain an* $\mathbb{R}$*-covered foliation.*

We had expected that proving the existence of Reebless foliations, especially $\mathbb{R}$-covered foliations, or essential laminations is of use for determining and classifying a non-trivial finite or cyclic surgery on other hyperbolic knots in the same way as [5], but Corollary 3.1 means that in the case of pretzel knots, an $\mathbb{R}$-covered foliation is not of use for it. However, we think there are another applications of non-existence of an $\mathbb{R}$-covered foliation, an approach of Cosmetic surgery conjecture [12] (or see [8]) as an example.

Next we discuss our result in the viewpoint of a left-orderable group. A group G is left-orderable if there exists a total ordering $<$ of the elements of G which is left invariant, meaning that for any elements f, g, h of G, if $f < g$ then $hf < hg$. It is known that a countable group G is left-orderable if and only if there exists a faithful action of G on $\mathbb{R}$, that is, there is no point of $\mathbb{R}$ which fixed by any element of G. By this fact, if a closed 3-manifold M contains an $\mathbb{R}$-covered foliation, the fundamental group of M is left-orderable. The fundamental groups $G_{K_s}(p,q)$ which satisfy the assumptions of Theorem 1.2 do not have a faithful action on $\mathbb{R}$ by the proof of Theorem 1.2. Therefore we conclude the following corollary:

Corollary 3.2. *Let K_s be a $(-2,3,2s+1)$-type Pretzel knot in S^3 $(s \geqq 3)$, $G = G_{K_s}(p,q)$ denotes the fundamental group of the closed manifold which obtained by Dehn surgery along K_s with slope p/q. If $q > 0$, $p/q \geqq 4s+7$ and p is odd, G is not left-orderable.*

Roberts and Shareshian generalize the properties of the fundamental groups treated in [16]. They present conditions when the fundamental groups of a closed manifold obtained by Dehn filling of a once punctured torus bundle is not right-orderable [15, Corollary 1.5]. These are examples of hyperbolic 3-manifolds which has non right-orderable fundamental groups.

Clay and Watson showed the following theorem.

Theorem 3.3. (A. Clay, L. Watson, 2012, [4, Theorem 28]) *Let K_m be a $(-2,3,2m+5)$-type Pretzel knot. If $p/q > 2m+15$ and $m \geqq 0$, the fundamental group $\pi_1(E_{K_m}(p/q))$ is not left-orderable.*

By the fact mentioned before, these fundamental groups do not have a faithful action on $\mathbb{R}$, then these $E_{K_m}(p/q)$ do not admit an $\mathbb{R}$-covered foliation. Although the method of the proof of Theorem 3.3 is different from our strategy, it concludes a stronger result than ours in the sense of an estimation of surgery slopes. By the aspects getting from these results, there are many interaction between a study of $\mathbb{R}$-covered foliations and a study of left-orderability of the fundamental group of a closed 3-manifold, so we think that these objects will be more interesting.

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Foliations via frame bundles

Kamil NIEDZIALOMSKI

1. Introduction

Let (M,g) be a Riemannian manifold. Denote by $L(M)$ and $O(M)$ frame and orthonormal frame bundles over M, respectively. We consider on M the Levi–Civita connection. We can equip these bundles with a Riemannian metric such that the projection $\pi : L(M) \to M$ ($\pi : O(M) \to M$, respectively) is a Riemannian submersion. The classical example is the Sasaki–Mok metric [6, 1, 2]. There are many, so called natural, metrics considered by Kowalski and Sekizawa [3, 4, 5] and by the author [7]. Denote such fixed Riemannian metric by $\bar{g}$.

Assume M is equipped with k–dimensional foliation $\mathcal{F}$. Then $\mathcal{F}$ induces two subbundles $L(\mathcal{F})$ of $L(M)$ and $O(\mathcal{F})$ of $O(M)$ as follows

$$L(\mathcal{F}) = \{u = (u_1, \ldots, u_n) \in L(M) \mid u_1, \ldots, u_k \in T\mathcal{F}\},$$
$$O(\mathcal{F}) = \{u = (u_1, \ldots, u_n) \in O(M) \mid u_1, \ldots, u_k \in T\mathcal{F}\}.$$

Hence $L(\mathcal{F})$ and $O(\mathcal{F})$ are submanifolds of the Riemannian manifolds $(L(M), \bar{g})$ and $O(M), \bar{g})$, respectively.

2. Results

For simplicity denote by P the bundle $L(M)$ or $O(M)$ and by $P(\mathcal{F})$ the corresponding subbundle $L(\mathcal{F})$ or $O(\mathcal{F})$.

The objective is to state the correspondence between the geometry of a foliation $\mathcal{F}$ and the geometry of a submanifold $P(\mathcal{F})$ in P. The approach to the stated problem is the following.

1. The submanifold $P(\mathcal{F})$ of P is the subbbundle with the structure group H of matrices of the form
$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}.$$
This induces the vertical distribution of $P(\mathcal{F})$. The horizontal distribution is induced from the horizontal distribution of P. The aim is to obtain the correspondence between the horizontal lifts to $P(\mathcal{F})$ and P. It appears that it depends on the second fundamental form of $\mathcal{F}$.

2. The horizontal lift X^h to P and $X^{h,\mathcal{F}}$ to $P(\mathcal{F})$ and the formula for the Levi–Civita connection of $(P, \bar{g})$ imply the formula for the connection and second fundamental form of $P(\mathcal{F})$. The scope is to derive the explicit formula for these operators in terms of the connection on M and the second fundamental form of $\mathcal{F}$.
3. The formula for the second fundamental of $P(\mathcal{F})$ determines the extrinsic geometry of this submanifold. It appears that the conditions such as being totally geodesic, minimality, umbilicity of $P(\mathcal{F})$ are related with corresponding conditions of foliation $\mathcal{F}$. We state these correspondences.

3. Further research

The further research, initiated by the author, includes the following two problems:

1. Generalize the results to the case of a single manifold. More precisely, any submanifold N of M induces a subbundle $P(N)$ in $L(M)$ or $O(M)$. We may consider the geometry of the submanifold $P(N)$ and the relation with the geometry of N.
2. The subbundle $L(\mathcal{F})$ of $L(M)$ induced by the foliation $\mathcal{F}$ does not require the integrability of $\mathcal{F}$. Hence, we may consider $L(\mathcal{F})$ if $\mathcal{F}$ is non–integrable distribution. In particular, we may choose $\mathcal{F}$ to be the horizontal distribution $\mathcal{H}^\varphi$ of any submersion $\varphi : M \to N$. Therefore we may lift φ to a map $L\varphi : L(\mathcal{H}^\varphi) \to L(N)$ and study the geometry of $L\varphi$. Partial results of the author show that horizontal conformality of $L\varphi$ is equivalent to horizontal conformality of φ under some additional conditions (such as the restriction on $\bar{g}$).

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The mixed scalar curvature flow and harmonic foliations

Vladimir ROVENSKI

A flow of metrics, g_t, on a manifold is a solution of evolution equation $\partial_t g = S(g)$, where $S(g)$ is a symmetric $(0,2)$-tensor usually related to some kind of curvature. The mixed sectional curvature of a foliated manifold $(M,\mathcal{F})$ regulates the deviation of leaves along the leaf geodesics. (In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics). Let $\{\varepsilon_\alpha, e_i\}_{\alpha\le p, i\le n}$ be a local orthonormal frame on TM adapted to $T\mathcal{F}$ and the orthogonal distribution $\mathcal{D} := T\mathcal{F}^\perp$.

The *mixed scalar curvature* is defined by $\mathrm{Sc}_{\rm mix} = \sum_{i=1}^n \sum_{\alpha=1}^p R(\varepsilon_\alpha, e_i, \varepsilon_\alpha, e_i)$, where R is the Riemannain curvature. For a codimension-one foliation with a unit normal N, we have $\mathrm{Sc}_{\rm mix} = \mathrm{Ric}(N,N)$. For a surface (M^2, g), i.e., $n = p = 1$, we obtain $\mathrm{Sc}_{\rm mix} = K$ – the gaussian curvature.

We study the flow of metrics on a foliation, whose velocity along $\mathcal{D}$ is proportional to $\mathrm{Sc}_{\rm mix}$:

$$\partial_t g = -2(\mathrm{Sc}_{\rm mix}(g) - \Phi)\hat{g}. \tag{1}$$

Here $\Phi\colon M \to \mathbb{R}$ is leaf-wise constant. The *$\mathcal{D}$-truncated metric tensor* $\hat{g}$ is given by $\hat{g}(X_1, X_2) = g(X_1, X_2)$ and $\hat{g}(Y, \cdot\,) = 0$ for $X_i \in \mathcal{D}$, $Y \in T\mathcal{F}$. We show relations of (1) with Burgers equation (the prototype for non-linear advection-diffusion processes in gas and fluid dynamics) and Schrödinger heat equation (which is central to all of quantum mechanics).

Let $h_\mathcal{F}$, h be the second fundamental forms and $H_\mathcal{F}$, H the mean curvature vectors of $T\mathcal{F}$ and the distribution $\mathcal{D}$, respectively. Also denote T the integrability tensor of $\mathcal{D}$. Then, see [2],

$$\mathrm{Sc}_{\rm mix}(g) = \mathrm{div}(H + H_\mathcal{F}) + \|H\|^2 + \|T\|^2 - \|h\|^2 + \|H_\mathcal{F}\|^2 - \|h_\mathcal{F}\|^2. \tag{2}$$

The flow (1) preserves total geodesy (i.e. $h_\mathcal{F} = 0$) and *harmonicity* (i.e. $H_\mathcal{F} = 0$) of foliations and is used to examine the question [1]: *Which foliations admit a metric with a given property of* $\mathrm{Sc}_{\rm mix}$ (*e.g., positive or negative*)? Suppose that the leaves of $\mathcal{F}$ are compact minimal submanifolds. We observe that (1) yields the leaf-wise evolution equation for the vector field H:

$$\partial_t H + \nabla^\mathcal{F} g(H,H) = n\nabla^\mathcal{F}(\mathrm{Div}_\mathcal{F} H) + n\nabla^\mathcal{F}(\|T\|_g^2 - \|h_\mathcal{F}\|_g^2 - n\beta_\mathcal{D}). \tag{3}$$

The function $\beta_{\mathcal D} := n^{-2}\big(n\|h\|^2 - \|H\|^2\big) \ge 0$ is time-independent, it serves as a measure of "*non-umbilicity*" of $\mathcal D$, since $\beta_{\mathcal D} = 0$ for totally umbilical $\mathcal D$. For $\dim \mathcal F = 1$ we have $\beta_{\mathcal D} = n^{-2}\sum_{i<j}(k_i - k_j)^2$, where k_i are the principal curvatures of $\mathcal D$.

Suppose that $H_0 = -n\nabla^{\mathcal F}(\log u_0)$ (leaf-wise conservative) for a function $u_0 > 0$.
If $\|T\|_{g_0} > \|h_{\mathcal F}\|_{g_0}$ then its potential obeys the leaf-wise non-linear Schrödinger heat equation

$$(4)\qquad (1/n)\partial_t u = \Delta_{\mathcal F} u + (\beta_{\mathcal D} + \Phi/n)u - (\Psi/n)u^{-3}, \qquad u(\,\cdot\,,0) = u_0,$$

where $\Psi := u_0^4(\|T\|^2_{g_0} - \|h_{\mathcal F}\|^2_{g_0})$, moreover, the solution obeys $u = \Psi^{1/4}(\|T\|^2_{g_t} - \|h_{\mathcal F}\|^2_{g_t})^{-1/4}$.

If $\Psi \equiv 0$ (e.g., $T(g_0) = 0$ and $h_{\mathcal F}(g_0) = 0$) then (3) reduces to a forced *Burgers equation*

$$(5)\qquad \partial_t H + \nabla^{\mathcal F} g(H,H) = n\nabla^{\mathcal F}(\mathrm{Div}_{\mathcal F} H) - n^2\nabla^{\mathcal F}\beta_{\mathcal D},$$

moreover, the leaf-wise potential function for H may be chosen as a solution of the linear PDE $(1/n)\partial_t u = \Delta_{\mathcal F} u + \beta_{\mathcal D} u$, $u(\cdot,0) = u_0$. The first eigenvalue $\lambda_0 \le 0$ of *Schrödinger operator* $\mathcal H(u) = -\Delta_{\mathcal F} u - \beta_{\mathcal D} u$ corresponds to the unit L_2-norm eigenfunction $e_0 > 0$ (called the *ground state*). Under certain conditions (on any leaf F)
(6)

$$\Phi > -n\beta_{\mathcal D}, \quad |n\lambda_0 + \Phi| \ge \max_F(\|T\|^2_{g_0} - \|h_{\mathcal F}\|^2_{g_0})\big(\max_F(u_0/e_0)/\min_F(u_0/e_0)\big)^4$$

the asymptotic behavior of solutions to (4) is the same as for the linear equation, when (5) has a single-point global attractor: $H_t \to -n\nabla^{\mathcal F}(\log e_0)$ as $t \to \infty$. Using the scalar maximum principle, we show that there exists a positive solution $\tilde u$ of the linear PDE $(1/n)\partial_t\tilde u = \Delta_{\mathcal F}\tilde u + (\beta_{\mathcal D} + \lambda_0)\tilde u$ such that for any $\alpha \in \big(0, \min\{\lambda_1 - \lambda_0, 4|\lambda_0|\}\big)$ and $k \in \mathbb N$ the following hold:

(i) $u = e^{-\lambda_0 t}(\tilde u + \theta(x,t))$, where $\|\theta(\,\cdot\,,t)\|_{C^k} = O(e^{-\alpha t})$ as $t \to \infty$;
(ii) $\nabla^{\mathcal F}(\log u) = \nabla^{\mathcal F}(\log e_0) + \theta_1(x,t)$, where $\|\theta_1(\,\cdot\,,t)\|_{C^k} = O(e^{-\alpha t})$ as $t \to \infty$.

In this case, (1) has a unique global solution g_t $(t \ge 0)$, whose $\mathrm{Sc}_{\rm mix}$ converges exponentially to $n\lambda_0 \le 0$. The metrics are smooth on M when all leaves are compact and have finite holonomy group. After rescaling of metrics on $\mathcal D$, we also obtain convergence to a metric with $\mathrm{Sc}_{\rm mix} > 0$.

Proposition 1. *Let (M,g) be endowed with a harmonic compact foliation $\mathcal F$. Suppose that $\|h_{\mathcal F}\|_g < \|T\|_g$ and $H = -n\nabla^{\mathcal F}(\log u_0)$ for a function $u_0 > 0$.*

(i) *If $\lambda_0 < 0$ then there exists $\mathcal D$-conformal to g metric $\bar g$ with* $\mathrm{Sc}_{\rm mix}(\bar g) < 0$.

(ii) *If* $\lambda_0 > -\frac{1}{n}(\frac{u_0}{\tilde{u}_0 e_0})^4(\|T\|_g^2 - \|h_{\mathcal{F}}\|_g^2)$ *then there is* $\mathcal{D}$*-conformal to* g *metric* $\bar{g}$ *with* $\mathrm{Sc}_{\mathrm{mix}}(\bar{g}) > 0$.

For surfaces of revolution M_t: $[\rho(x,t)\cos\theta, \rho(x,t)\sin\theta, h(x)]$ $(0 \le x \le l$, $|\theta| \le \pi)$ with $(\rho_{,x})^2 + (h_{,x})^2 = 1$, (1) reads as $\partial_t g = -2(K(g) - \Phi)\hat{g}$. This yields the PDE $\partial_t \rho = \rho_{,xx} + \Phi\rho$. For $\Phi = \mathrm{const}$ and appropriate initial and end conditions for ρ, we have the following. If $\Phi < (\pi/l)^2$ then M_t converge to a surface with $K = \Phi$, and if $\Phi = (\pi/l)^2$ then $\lim_{t\to\infty} \rho(x,t) = A\sin(\pi x/l)$, and M_t converge to a surface with $K = \Phi$ (a sphere of radius l/π when $A = l/\pi$).

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Generalizations of a theorem of Herman and a new proof of the simplicity of $\mathrm{Diff}_c^\infty(M)_0$

Tomasz RYBICKI

Let M be a smooth manifold of dimension n. By $\mathrm{Diff}_c^\infty(M)$ we will denote the group of compactly supported diffeomorphisms of M. We shall consider a Lie group structure on $\mathrm{Diff}_c^\infty(M)$ in the sense of the convenient setting of Kriegl and Michor [10]. In particular, we assume that $\mathrm{Diff}_c^\infty(M)$ is endowed with the c^∞-topology [10, Section 4], i.e. the final topology with respect to all smooth curves. For compact M the c^∞-topology on $\mathrm{Diff}^\infty(M)$ coincides with the Whitney C^∞-topology, cf. [10, Theorem 4.11(1)]. In general the c^∞-topology on $\mathrm{Diff}_c^\infty(M)$ is strictly finer than the one induced from the Whitney C^∞-topology, cf. [10, Section 4.26]. The latter coincides with the inductive limit topology $\lim_K \mathrm{Diff}_K^\infty(M)$ where K runs through all compact subsets of M.

Given smooth complete vector fields $X_1, \ldots, X_N$ on M, we consider the map

$$(1) \qquad \begin{gathered} K\colon \mathrm{Diff}_c^\infty(M)^N \to \mathrm{Diff}_c^\infty(M), \\ K(g_1, \ldots, g_N) := [g_1, \exp(X_1)] \circ \cdots \circ [g_N, \exp(X_N)]. \end{gathered}$$

Here $\exp(X)$ denotes the flow of a complete vector field X at time 1, and $[k, h] := k \circ h \circ k^{-1} \circ h^{-1}$ denotes the commutator of two diffeomorphisms k and h. It is readily checked that K is smooth. Indeed, one only has to observe that K maps smooth curves to smooth curves, cf. [10, Section 27.2]. Clearly $K(\mathrm{id}, \ldots, \mathrm{id}) = \mathrm{id}$.

A *smooth local right inverse at the identity for* K consists of an open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_c^\infty(M)$ together with a smooth map

$$\sigma = (\sigma_1, \ldots, \sigma_N)\colon \mathcal{U} \to \mathrm{Diff}_c^\infty(M)^N$$

so that $\sigma(\mathrm{id}) = (\mathrm{id}, \ldots, \mathrm{id})$ and $K \circ \sigma = \mathrm{id}_{\mathcal{U}}$. More explicitly, we require that each $\sigma_i\colon \mathcal{U} \to \mathrm{Diff}_c^\infty(M)$ is smooth with $\sigma_i(\mathrm{id}) = \mathrm{id}$ and, for all $g \in \mathcal{U}$,

$$g = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_N(g), \exp(X_N)].$$

The aim of this talk is to present the following two results which generalize a well-known theorem of Herman for M being the torus [8, 9].

Key words and phrases. diffeomorphism group; perfect group; simple group; fragmentation; convenient calculus; foliation.

Joint with Stefan Haller and Josef Teichmann.

Theorem 1. *Suppose M is a smooth manifold of dimension $n \geq 2$. Then there exist four smooth complete vector fields $X_1, \dots, X_4$ on M so that the map K, see* (1)*, admits a smooth local right inverse at the identity, $N = 4$. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the strong Whitney C^0-topology. If M admits a proper (circle valued) Morse function whose critical points all have index* 0 *or n, then the same statement remains true with three vector fields.*

Particularly, on the manifolds $M = \mathbb{R}^n, S^n, T^n$, $n \geq 2$, or the total space of a compact smooth fiber bundle $M \to S^1$, three commutators are sufficient. At the expense of more commutators, it is possible to gain further control on the vector fields. More precisely, we have:

Theorem 2. *Suppose M is a smooth manifold of dimension $n \geq 2$ and set $N := 6(n+1)$. Then there exist smooth complete vector fields $X_1, \dots, X_N$ on M so that the map K, see* (1)*, admits a smooth local right inverse at the identity. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the strong Whitney C^∞-topology.*

Either of the two theorems implies that $\mathrm{Diff}_c^\infty(M)_o$, the connected component of the identity, is a perfect group, provided M is not $\mathbb{R}$. Our proof rests on Herman's result similarly as that of [17] (see [2]), but is otherwise elementary and different from Thurston's approach. In fact we only need Herman's result in dimension 1.

The perfectness of $\mathrm{Diff}_c^\infty(M)_0$ was already proved by Epstein [5] using ideas of Mather [11, 12] who dealt with the C^r-case, $1 \leq r < \infty$, $r \neq n+1$. The Epstein–Mather proof is based on a sophisticated construction, and uses the Schauder–Tychonov fixed point theorem. The existence of a presentation

$$g = [h_1, k_1] \circ \cdots \circ [h_N, k_N]$$

is guarantied, but without any further control on the factors h_i and k_i. Theorem 1 or 2 actually implies that the universal covering of $\mathrm{Diff}_c^\infty(M)_o$ is a perfect group. This result is known, too, see [17]. Thurston's proof is based on a result of Herman for the torus [8, 9]. Note that the perfectness of $\mathrm{Diff}_c^\infty(M)_o$ implies that this group is simple, see Epstein [4]. The methods used in [4] are elementary and actually work for a rather large class of homeomorphism groups.

One could believe that the phenomenon of smooth perfectness described in Theorems 1 and 2 would be also true for some classical diffeomorphism groups which are simple, e.g. for the Hamiltonian diffeomorphism group of a closed symplectic manifold [1], or for the contactomorphism group of an arbitrary co-oriented contact manifold [15]. However, the available methods seem to be useless for possible proofs of their smooth perfectness. Another open problem related to the above theorems is whether a smooth *global* right inverse at the identity for K would exist. A possible answer in the affirmative seems to be equally difficult. Consequently, it would be difficult to improve Theorems 1 and 2 as they are in any possible direction.

Another essential and important way to generalize the simplicity theorems for $\mathrm{Diff}_c^\infty(M)_o$, where $1 \leq r \leq \infty$, $r \neq n+1$, is to consider the uniform perfectness or, more generally, the boundedness of the groups in question. In particular, we ask if the presentation $g = [h_1, k_1] \circ \cdots \circ [h_N, k_N]$ is available for all $g \in \mathrm{Diff}_c^\infty(M)_o$ with N bounded. This property has been proved in the recent papers by Burago, Ivanov and Polterovich [3], and Tsuboi [18], [19], [20], for a large class of manifolds. For instance, $N = 10$ was obtained in [3] for any closed three dimensional manifold, and then it was improved in [18] to $N = 6$ for any closed odd dimensional manifold. It seems that the methods of [3], [18], [19] and [20] combined with our Theorem 2 would give some analogue of Theorem 1, but certainly not with the presentation (1) and the condition on X_i. Also N could not be smaller in this way. Another advantage of Theorem 1 is that it is valid for all smooth paracompact manifolds. See also [16] for diffeomorphism groups with no restriction of support.

Let $T^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the torus. For $\lambda \in T^n$ we let $R_\lambda \in \mathrm{Diff}^\infty(T^n)$ denote the corresponding rotation. The main ingredient in the proof of Theorems 1 and 2 is the following result of Herman [9, 8].

Theorem 3 (Herman)**.** *There exist $\gamma \in T^n$ so that the smooth map*

$$T^n \times \mathrm{Diff}^\infty(T^n) \to \mathrm{Diff}^\infty(T^n), \qquad (\lambda, g) \mapsto R_\lambda \circ [g, R_\gamma],$$

admits a smooth local right inverse at the identity. Moreover, γ may be chosen arbitrarily close to the identity in T^n.

Herman's result is an application of the Nash–Moser inverse function theorem. When inverting the derivative one is quickly led to solve the linear equation $Y = X - (R_\gamma)^*X$ for given $Y \in C^\infty(T^n, \mathbb{R}^n)$. This is accomplished using Fourier transformation. Here one has to choose γ sufficiently irrational so that tame estimates on the Sobolev norms of X in terms of the Sobolev norms of Y can be obtained. The corresponding small denominator problem can be solved due to a number theoretic result of Khintchine.

We shall make use of the following corollary of Herman's result.

Proposition 1. *There exist smooth vector fields X_1, X_2, X_3 on T^n so that the smooth map* $\mathrm{Diff}^\infty(T^n)^3 \to \mathrm{Diff}^\infty(T^n)$*,*

$$(g_1, g_2, g_3) \mapsto [g_1, \exp(X_1)] \circ [g_2, \exp(X_2)] \circ [g_3, \exp(X_3)],$$

admits a smooth local right inverse at the identity. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the Whitney C^∞-topology.

The following lemma leads to a decomposition of a diffeomorphism into factors which are leaf preserving. If $\mathcal{F}$ is a smooth foliation of M we let $\mathrm{Diff}_c^\infty(M;\mathcal{F})$ denote the group of compactly supported diffeomorphisms preserving the leaves of $\mathcal{F}$. This is a regular Lie group modelled on the convenient vector space of compactly supported smooth vector fields tangential to $\mathcal{F}$.

Lemma 1. *Suppose M_1 and M_2 are two finite dimensional smooth manifolds and set $M := M_1 \times M_2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the foliations with leaves $M_1 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times M_2$ on M, respectively. Then the smooth map*

$$F\colon \mathrm{Diff}_c^\infty(M;\mathcal{F}_1) \times \mathrm{Diff}_c^\infty(M;\mathcal{F}_2) \to \mathrm{Diff}_c^\infty(M), \quad F(g_1,g_2) := g_1 \circ g_2,$$

is a local diffeomorphism at the identity.

Now we need a version of the exponential law.

Lemma 2. *Suppose B and T are finite dimensional smooth manifolds, assume T compact, and let $\mathcal{F}$ denote the foliation with leaves $\{\mathrm{pt}\} \times T$ on $B \times T$. Then the canonical bijection*

$$C_c^\infty(B, \mathrm{Diff}^\infty(T)) \xrightarrow{\cong} \mathrm{Diff}_c^\infty(B \times T;\mathcal{F})$$

is an isomorphism of regular Lie groups.

Another ingredient of the proof is a smooth fragmentation of diffeomorphisms.

Suppose $U \subseteq M$ is an open subset. Every compactly supported diffeomorphism of U can be regarded as a compactly supported diffeomorphism of M by extending it identically outside U. The resulting injective homomorphism $\mathrm{Diff}_c^\infty(U) \to \mathrm{Diff}_c^\infty(M)$ is clearly smooth. Note, however, that a curve in $\mathrm{Diff}_c^\infty(U)$, which is smooth when considered as a curve in $\mathrm{Diff}_c^\infty(M)$, need not be smooth as a curve into $\mathrm{Diff}_c^\infty(U)$. Nevertheless, if there exists a closed subset A of M with $A \subseteq U$ and if the curve has support contained in A, then one can conclude that the curve is also smooth in $\mathrm{Diff}_c^\infty(U)$.

Proposition 2 (Fragmentation)**.** *Let M be a smooth manifold of dimension n, and suppose $U_1, \dots, U_k$ is an open covering of M, ie. $M = U_1 \cup \dots \cup U_k$. Then the smooth map*

$$P\colon \mathrm{Diff}_c^\infty(U_1) \times \dots \times \mathrm{Diff}_c^\infty(U_k) \to \mathrm{Diff}_c^\infty(M), \quad P(g_1,\dots,g_k) := g_1 \circ \dots \circ g_k,$$

admits a smooth local right inverse at the identity.

Proceeding as in [3] permits to reduce the number of commutators considerably, see also [18] and [19].

Proposition 3. *Let M be a smooth manifold of dimension $n \geq 2$ and put $N = 6(n+1)$. Moreover, let U an open subset of M and suppose $\phi \in \mathrm{Diff}^\infty(M)$, not necessarily with compact support, such that the closures of the subsets*

$$U,\ \phi(U),\ \phi^2(U),\ \dots,\ \phi^N(U)$$

are mutually disjoint. Then there exists a smooth complete vector field X on M, a c^∞-open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_c^\infty(U)$, and smooth maps $\varrho_1, \varrho_2 \colon \mathcal{U} \to \mathrm{Diff}_c^\infty(M)$ so that $\varrho_1(\mathrm{id}) = \varrho_2(\mathrm{id}) = \mathrm{id}$ and, for all $g \in \mathcal{U}$,

$$g = [\varrho_1(g), \phi] \circ [\varrho_2(g), \exp(X)].$$

Moreover, the vector field X may be chosen arbitrarily close to zero in the strong Whitney C^∞-topology on M.

Now, by applying the Morse theory ([13], [14]) we get

Lemma 3. *Let M be a smooth manifold of dimension n. Then there exists an open covering $M = U_1 \cup U_2 \cup U_3$ and smooth complete vector fields X_1, X_2, X_3 on M so that $\exp(X_1)(U_1) \subseteq U_2$, $\exp(X_2)(U_2) \subseteq U_3$, and such that the closures of the sets*

$$U_3,\ \ \exp(X_3)(U_3),\ \ \exp(X_3)^2(U_3),\ \ \dots$$

are mutually disjoint. Moreover, the vector fields X_1, X_2, X_3 may be chosen arbitrarily close to zero with respect to the strong Whitney C^0-topology. If M admits a proper (circle valued) Morse function whose critical points all have index 0 or n, then we may, moreover, choose $U_1 = \emptyset$ and $X_1 = 0$.

Theorem 1 is then a consequence of Lemma 3.

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Prevalence of non-uniform hyperbolicity at the first bifurcation of Hénon-like families

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Hyperbolicity and structural stability are key concepts in the development of the theory of dynamical systems. Nowadays, it is known that these two concepts are essentially equivalent to each other, at least for C^1 diffeomorphisms or flows of a compact manifold. Then, a fundamental problem in the bifurcation theory is to study transitions from hyperbolic to non hyperbolic regimes. When the loss of hyperbolicity is due to the formation of a cycle (i.e., a configuration in the phase space involving non-transverse intersections between invariant manifolds), an incredibly rich array of complicated behaviors is unleashed by the unfolding of the cycle (See e.g. [12] and the references therein). Many important aspects of this complexity are poorly understood.

To study bifurcations of diffeomorphisms, we work within a framework set up by Palis: consider arcs of diffeomorphisms losing their hyperbolicity through generic bifurcations, and analyze which dynamical phenomena are more frequently displayed (in the sense of the Lebesgue measure in parameter space) in the sequel of the bifurcation. More precisely, let $\{\varphi_a\}_{a\in\mathbb{R}}$ be a parametrized family of diffeomorphisms which undergoes a first bifurcation at $a = a^*$, i.e., φ_a is hyperbolic for $a > a^*$, and φ_{a^*} has a cycle. We assume $\{\varphi_a\}_{a\in\mathbb{R}}$ unfolds the cycle generically. A dynamical phenomenon $\mathcal{P}$ is *prevalent* at a^* if

$$\liminf_{n\to\infty} \frac{1}{\varepsilon}\mathrm{Leb}\{a \in [a^* - \varepsilon, a^*]\colon \varphi_a \text{ displays } \mathcal{P}\} > 0,$$

where Leb denotes the one-dimensional Lebesgue measure.

Particularly important is the prevalence of hyperbolicity. The pioneering work in this direction is due to Newhouse and Palis [8], on the bifurcation of Morse-Smale diffeomorphisms. The prevalence of hyperbolicity in arcs of surface diffeomorphisms which are not Morse-Smale has been studied in the literature [7, 10, 11, 13, 14]. However, even with all these and other subsequent developments, including [15, 16], we still lack a good understanding as to in which case the hyperbolicity becomes prevalent.

In [7, 10, 11, 13, 14], unfoldings of tangencies of surface diffeomorphisms associated to basic sets have been treated. One key aspect of these models is that the orbit of tangency at the first bifurcation is not contained in the limit set. This implies a global control on new orbits added to the underlying basic set, and moreover allows one to use its invariant foliations to translate dynamical problems to the problem on how two Cantor sets intersect each other. Then, the prevalence of hyperbolicity is related to the Hausdorff dimension of the limit set. This argument is not viable, if the orbit of tangency, responsible for the loss of the stability of the system, is contained in the limit set. Let us call such a first bifurcation an *internal tangency bifurcation.*

We are concerned with an arc $\{f_a\}_{a\in\mathbb{R}}$ of planar diffeomorphisms of the form

$$f_a(x,y) = (1-ax^2,0) + b\cdot\Phi(a,b,x,y), \quad 0<b\ll 1.$$

Here Φ is bounded continuous in (a,b,x,y) and C^4 in (a,x,y). This particular arc, often called an "Hénon-like family", is embedded in generic one-parameter unfoldings of quadratic homoclinic tangencies associated to dissipative saddles [6], and so is relevant in the investigation of structurally unstable surface diffeomorphisms.

Let Ω_a denote the non wandering set of f_a, which is a compact f_a-invariant set. It is known [5] that for sufficiently large $a>0$, f_a is Smale's horseshoe map and Ω_a admits a hyperbolic splitting into uniformly contracting and expanding subspaces. As a decreases, the infimum of the angles between these two subspaces gets smaller, and the hyperbolic splitting disappears at a certain parameter. This first bifurcation is an internal tangency bifurcation. Namely, for sufficiently small $b>0$ there exists a parameter $a^*=a^*(b)$ near 2 with the following properties [1, 2, 3, 5].:

- if $a>a^*$, then Ω_a is a hyperbolic set, i.e., there exist constants $C>0$, $\xi\in(0,1)$ and at each $x\in\Omega_a$ a non-trivial decomposition $T_x\mathbb{R}^2 = E^s_x\oplus E^u_x$ with the invariance property such that $\|D_xf^n_a|E^s_x\|\leq C\xi^n$ and $\|D_xf^{-n}_a|E^u_x\|\leq C\xi^n$ for every $n\geq 0$;
- there is a quadratic tangency between stable and unstable manifolds of the fixed points of f_{a^*}. The orbit of this tangency at $a=a^*$ is accumulated by transverse homoclinic points, and thus it is contained in the limit set.

The orbit of tangency of f_{a^*} is in fact unique, and $\{f_a\}_{a\in\mathbb{R}}$ unfolds this tangency generically. The next theorem gives a partial description of prevalent dynamics at $a=a^*$.

Theorem 1. *For sufficiently small $b>0$ there exist $\varepsilon_0=\varepsilon_0(b)>0$ and a set $\Delta\subset[a^*-\varepsilon_0,a^*]$ of a-values containing a^* with the following properties:*

(a) $\lim\limits_{\varepsilon\to+0}(1/\varepsilon)Leb(\Delta\cap[a^*-\varepsilon,a^*])=1;$

(b) if $a\in\Delta$, then the Lebesgue measure of the set

$$K^+_a := \{x\in\mathbb{R}^2\colon \{f^n_ax\}_{n\in\mathbb{N}} \text{ is bounded}\}$$

is zero. In particular, for Lebesgue almost every $x \in \mathbb{R}^2$, $|f_a^n x| \to \infty$ as $n \to \infty$.

In addition, if $a \in \Delta$ then f_a is transitive on Ω_a. In other words, for "most" diffeomorphisms immediately right after the first bifurcation, the topological dynamics is similar to that of Smale's horseshoe before the bifurcation.

We suspect that the dynamics is non hyperbolic for all, or "most" parameters in Δ. Nevertheless, the proof of the above theorem tells us that the dynamics of f_a, $a \in \Delta$ is fairly structured, and this may yield a weak form of hyperbolicity. A natural question then is the following:

To what extent the dynamics is hyperbolic for $a \in \Delta$?

The main result of this paper gives one answer for this question. For measuring the extent of hyperbolicity we estimate Lyapunov exponents, the asymptotic exponential rates at which nearby orbits are separated (or draw together).

Let us say that a point $x \in \Omega_a$ is *regular* if there exist number(s) $\chi_1 < \cdots < \chi_{r(x)}$ and a decomposition $T_x\mathbb{R}^2 = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for every $v \in E_i(x) \setminus \{0\}$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi_i(x) \quad \text{and}$$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |\det D_x f_a^n| = \sum_{i=1}^{r(x)} \chi_i(x) \mathrm{dim} E_i(x).$$

By the theorem of Oseledec [9], the set of regular points has total probability. If μ is ergodic, then the functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\mathrm{dim}E_i(x)$ are invariant along orbits, and so are constant μ-a.e. From this and the Ergodic Theorem, one of the following holds for each ergodic μ:

- there exist two numbers $\chi^s(\mu) < \chi^u(\mu)$, and for μ-a.e. $x \in \Omega_a$ a decomposition $T_x\mathbb{R}^2 = E_x^s \oplus E_x^u$ such that for any $v^\sigma \in E_x^\sigma \setminus \{0\}$ and $\sigma = s,\, u$,
$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi^\sigma(\mu) \quad \text{and}$$
$$\int \log |\det D f_a| d\mu = \chi^s(\mu) + \chi^u(\mu);$$
- there exists $\chi(\mu) \in \mathbb{R}$ such that for μ-a.e. $x \in \Omega_a$ and all $v \in T_x\mathbb{R}^2 \setminus \{0\}$,
$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi(\mu) \quad \text{and}$$

$$\int \log|\det Df_a|d\mu = 2\chi(\mu).$$

The number(s) $\chi^s(\mu)$ and $\chi^u(\mu)$, or $\chi(\mu)$ is called a *Lyapunov exponent(s)* of μ.

Let $\mathcal{M}^e(f_a)$ denote the set of f_a-invariant Borel probability measures which are ergodic. We call $\mu \in \mathcal{M}^e(f_a)$ a *hyperbolic measure* if μ has two Lyapunov exponents $\chi^s(\mu)$, $\chi^u(\mu)$ with $\chi^s(\mu) < 0 < \chi^u(\mu)$. Our main theorem indicates a strong form of non-uniform hyperbolicty for $a \in \Delta$.

Theorem 2. *For sufficiently small $b > 0$, the following holds for all $a \in \Delta$:*

(a) any $\mu \in \mathcal{M}^e(f_a)$ is a hyperbolic measure;

(b) for each $\mu \in \mathcal{M}^e(f_a)$,

$$\chi^s(\mu) < \frac{1}{3}\log b < 0 < \frac{1}{4}\log 2 < \chi^u(\mu).$$

It must be emphasized that this kind of uniform bounds on Lyapunov exponents of ergodic measures are compatible with the non hyperbolicity of the system, and therefore, Theorem A does not imply the uniform hyperbolicity for $a \in \Delta$. Indeed, $a^* \in \Delta$ and f_{a^*} is genuinely non hyperbolic, due to the existence of tangencies. See [3, 4] for the first examples of non hyperbolic surface diffeomorphisms of this kind. As already mentioned, we suspect that the dynamics is non hyperbolic for all, or “most” parameters in Δ.

Little is known on the prevalence of hyperbolicity at internal tangency bifurcations. The only previously known result in this direction is due to Rios [15], on certain horseshoes in the plane with three branches. However, certain hypotheses in [15] on expansion/contraction rates and curvatures of invariant manifolds near the tangency, are no longer true for $\{f_a\}_{a\in\mathbb{R}}$ due to the strong dissipation.

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Random circle diffeomorphisms

Michele TRIESTINO

1. Introduction

Since the early work of M. Herman [1], we have a very good understanding of the typical features, in the Baire sense, of C^r-diffeomorphisms of the circle $\mathbf{S}^1$.

Here, we propose to depict the landscape in $\mathrm{Diff}^1_+(\mathbf{S}^1)$ from a *measurable* point of view. In fact it is possible to consider a very natural probability measure on the group of C^1-diffeomorphisms of the circle, first introduced by P. Malliavin and E.T. Shavgulidze [3, 5].

Definition 1.1. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and let $(B_s)_{s\in[0,1]}$ be a Brownian bridge on Ω, and $\lambda \in [0,1[$ a uniform random variable independent of B. Then for any $\sigma > 0$, it is possible to define the following random variable f taking values in the space $\mathrm{Diff}^1_+(\mathbf{S}^1)$:

$$f(t) = f_\sigma(t) = \frac{\int_0^t e^{\sigma B_s}ds}{\int_0^1 e^{\sigma B_s}ds} + \lambda\,.$$

The law of f_σ defines a probability Radon measure μ_σ on $\mathrm{Diff}^1_+(\mathbf{S}^1)$ that is called the *Malliavin-Shavgulidze measure.*

The very important property of Malliavin-Shavgulidze measures μ_σ is that they are *Haar-like*: every μ_σ is quasi-invariant under left translations by elements belonging to a dense subgroup (of zero measure!). This implies in particular that every open set has positive measure. We recall that a measure is *quasi-invariant* under a group action if the image of any positive measure set has positive measure.

Theorem 1.2 (Shavgulidze [5])**.** *For any $\sigma > 0$ the measure μ_σ is quasi-invariant under the regular left action of the group of C^1-diffeomorphisms φ with bounded second derivative. When φ is a C^3 diffeomorphism, the Radon-Nykodym cocycle takes the following form:*

$$\frac{d(L_\varphi)_*(\mu_\sigma)}{d\mu_\sigma}(f) = \exp\left\{\frac{1}{\sigma}\int_0^1 \mathcal{S}_\varphi(f(t))\,f'(t)^2\,dt\right\}, \tag{1.3}$$

where $\mathcal{S}_\varphi = D^2 \log D\varphi - \frac{1}{2}(D \log D\varphi)^2$ *denotes the* Schwarzian derivative *of* φ.

QUESTION 1.4. What is the geometrical meaning for the Schwarzian derivative appearing in the expression (1.3)?

By the definition of the measure μ_σ, it follows that the diffeomorphism f is a.s. $C^{1+\alpha}$-regular for any $\alpha < 1/2$. However, Df is a 1/2-Hölder function with probability 0; in particular Df is a.s. not a function of *bounded variation*. Such remarks are very interesting in a dynamical context: there are indeed many results for which the regularities $C^{1+1/2}$ or C^{1+bv} are sharp. Perhaps the first, very well-known example is A. Denjoy's theorem: any C^{1+bv}-circle diffeomorphism without periodic orbits is minimal. Another interesting example is the Godbillon-Vey cocycle gv: S. Hurder and A. Katok showed how to extend the classical definition of gv to $C^{1+1/2+\varepsilon}$-diffeomorphisms [2], but T. Tsuboi explained later that this definition cannot be pursued to lower regularity [6]. Using the Malliavin-Shavgulidze measures and the theory of stochastic integration, it is possible to define gv as an *essential cocycle* on $\mathrm{Diff}^1_+(\mathbf{S}^1)$. However, we do not know whether such cocycle, defined a.e., is not a coboundary (in this measurable setting).

From a dynamical point of view, Malliavin-Shavgulidze measures may allow to *quantify* already known results. For example:

QUESTION 1.5.

1. What is the μ_σ-probability that a diffeomorphism has no periodic orbit?
2. Suppose that for σ sufficiently small, the probability to have no periodic orbit is positive, what is the probability that a diffeomorphism without periodic orbits is not minimal?

Even though such questions arise quite naturally, it is interesting to remark that no other mathematician has thought before about such problems: the Malliavin-Shavgulidze measures were just considered as a very good tool to study the representation theory of the group of smooth circle diffeomorphisms or its Lie algebra. We are very deeply indebted to É. Ghys to have proposed these very beautiful questions.

2. Main results

Our main motivation is to find answers for questions 1.5. Although we have some intuition (and numerical simulations) of what the result should be, we are still rather far away. The following statements are intended as a little step to a global comprehension: we analyse the set of diffeomorphisms *with periodic orbits.* It is worth remarking that in the topological settings, J. Mather and J.-C. Yoccoz gave a very exhaustive description [7].

We define $F_{p/q}$ to be the set of C^1-diffeomorphisms f with rotation number p/q (this implies in particular that f possesses an orbit of period q). It is easy to see that $F_{p/q}$ has non-empty interior and hence positive μ_σ-measure: the subset of *hyperbolic* diffeomorphisms is open and dense. To this purpose, we recall that a diffeomorphism $f \in F_{p/q}$ is hyperbolic if there are finitely many periodic orbits and if $\mathcal{O} = \{x_0, \dots, x_{q-1}\}$ is any q-periodic orbit, its *multiplier*

$$M(\mathcal{O}) = Df(x_0) \cdots Df(x_{q-1})$$

is not equal to 1.

Theorem 2.1. *Let $M > 0$. Then the μ_σ-probability that f has a periodic orbit with multiplier equal to M is zero.*

Corollary 2.2. *The set of hyperbolic diffeomorphisms is of full μ_σ-measure in $F_{p/q}$. In particular a diffeomorphism f has a.s. finitely many periodic orbits and their number is even.*

Corollary 2.3. *The set of diffeomorphisms with trivial C^1-centralizer is of full μ_σ-measure in $F_{p/q}$.*

Actually, Mather and Yoccoz proved that the set of diffeomorphisms with trivial C^1-centralizer in $F_{p/q}$ contains an open dense set.

Proof of corollary 2.3. The proof highly relies on the fact that the random variable f is defined by means of a Brownian bridge.

Take $f \in F_{p/q}$ and suppose f is $C^{1+\alpha}$, for some $\alpha > 0$ and suppose additionally that f has finitely many periodic orbits, each of them hyperbolic. Then by a result of Mather, the C^1-centralizer of f is trivial if and only if its *Mather invariant* has trivial stabilizer. The crucial fact is that the Mather invariant is highly sensible to local modifications of f. This implies that the triviality of the stabilizer of the Mather invariant is an event that depends only on a germ (or tail) filtration of the Brownian bridge B defining f. Then, using *Blumenthal's* 0–1 *law*, we can deduce that such event has zero μ_σ-measure. □

Outline of the proof of theorem 2.1. In order to simplify the details, let us sketch the proof of the theorem for *interval diffeomorphisms*: given a Brownian motion $(B_t)_{t\in[0,1]}$, we can define the random variable f:

$$f(t) = \frac{\int_0^t e^{B_s} ds}{\int_0^1 e^{B_s} ds}.$$

We want to show that for a given $M > 0$, the probability that f has a fixed point t such that $f'(t) = M$ is zero.

It turns out that it is easier to prove that the planar process

$$\left(B_t, B_{f(t)}\right)_{t\in[0,1]}$$

hits the diagonal with zero probability. Indeed, when $f(t) > t$, the variables B_t and $B_{f(t)}$ are almost independent, since for defining $f(t)$, we only have to know how B behaves up to time t (and its geometric average $\int_0^1 e^{B_s} ds$): then, using the Markov property of the Brownian motion, we see that the value of B at time $f(t)$ can be arbitrary. It is actually well known that the planar Brownian motion almost never hits a given point: the tricky proof of this fact (borrowed from [4]) can be adapted to our case. When $f(t) < t$, we can use set $g(t) := 1 - f(1-t)$, which defines an interval diffeomorphism with the same law as f, so that we reduce to the first case. $\square$

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Metric diffusion along compact foliations

Szymon WALCZAK

1. Wasserstein metric

The Wasserstein distance $d_{\mathcal{W}}$ of Borel probability measures μ and ν on Polish space X (complete separable metric space) endowed with a metric d is defined by

$$d_{\mathcal{W}}(\mu,\nu) = \inf \int_{M\times M} d(x,y)d\rho$$

where infimum is taken over all Borel probability measures ρ on $X \times X$ satisfying for any measurable sets $A, B \subset X$

$$\rho(A\times X) = \mu(A),$$
$$\rho(X\times B) = \nu(B).$$

A measure ρ is called *a coupling* of μ and ν. The set $\mathcal{P}(M)$ of all Borel probability measures with finite first moment endowed with $d_{\mathcal{W}}$ is a metric space. Moreover, $d_{\mathcal{W}}$ metrizes the weak-$*$ topology. The metric $d_{\mathcal{W}}$ comes from the Monge-Kantorovich optimal transportation problem [10] [11]. One can find that

Theorem 1.1. *[11] For any two Borel probability there exists a coupling ρ for which the Wasserstein distance is realized.*

One should notice that the Wasserstein distance $d_{\mathcal{W}}(\delta_x, \delta_y)$ of Dirac masses concentrated in points $x, y \in M$ is equal to the distance $d(x,y)$. This fact follows directly from the fact, that $\delta_{(x,y)}$ is the only coupling of δ_x and δ_y.

Let $\Delta^k = \{(t_1,\dots,t_k)\in\mathbb{R}^k : t_j \geq 0, \sum_j t_j = 1\}$.

Proposition 1.2. *The set*

$$\mathcal{D}(M) = \{\mu\in\mathcal{P}(M) : \mu = \sum_{i=1}^{k} t_k\delta_{x_k},\ (t_1,\dots,t_k)\in\Delta^k,\ x_1,\dots,x_k\in M\}$$

is dense in $\mathcal{P}(M)$.

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2. Harmonic measures and heat diffusion

Let $(M, \mathcal{F}, g)$ be a smooth closed oriented foliated manifold equipped with a Riemannian metric g and Laplace-Beltrami operator Δ defined by

$$\Delta f = \mathrm{div} \nabla f.$$

Let $\Delta_{\mathcal{F}}$ be *foliated Laplace-Beltrami operator* [2] [13] given by

$$\Delta_{\mathcal{F}} f(x) = \Delta_{L_x} f(x), \quad x \in M,$$

where L_x is a leaf through x, and Δ_L is Laplace-Beltrami operator on $(L, g|L)$. The operator $\Delta_{\mathcal{F}}$ acts on bounded measurable functions, which are C^2-smooth along the leaves.

Let us recall that a probability measure μ on $(M, \mathcal{F}, g)$ is called *harmonic* if for any $f : M \to \mathbb{R}$

$$\int_M \Delta_{\mathcal{F}} f d\mu = 0.$$

Theorem 2.1. *[8] [1] On any compact foliated Riemannian manifold, harmonic probability measures exist.*

One can associate with the operator $\Delta_{\mathcal{F}}$ the one-parameter semigroup D_t, $t \geq 0$, of *heat diffusion operators* characterized by

$$d_0 = \text{ id}, \; D_{t+s} = D_t \circ D_s, \; \frac{d}{dt} D_t|_{t=0} = \Delta_{\mathcal{F}}.$$

D_t restricted to a leaf $L \in \mathcal{F}$ coincides with the heat diffusion operators on L, which are given by

$$D_t f(x) = \int_{L_x} f(y) p(x, y; t) d \text{ vol}_{L_x}, \tag{2.2}$$

where $p(\cdot, \cdot; t)$ is a *foliated heat kernel* [2] on $(M, \mathcal{F})$. The foliated heat kernel is nonnegative and for any $t > 0$ satisfies

$$\int_{L_x} p(x, y; t) d \text{ vol}_{L_x} = 1.$$

Let μ be a probability measure on M. According to [2, 13], one can define *the diffused measure* $D_t \mu$ by the formula

$$\int f d D_t \mu = \int D_t f d\mu,$$

where f is any bounded measurable function on M. A measure μ is called *diffusion invariant* when $D_t \mu = \mu$ for all $t > 0$.

3. Diffused metric

Let $(M, \mathcal{F}, g)$ be a smooth compact foliated manifold equipped with a Riemannian metric g and carrying foliation $\mathcal{F}$. Let δ_t denotes the Dirac measure at point x. For $t > 0$ the metric

$$D_t d(x, y) = d_{\mathcal{W}}(D_t \delta_x, D_t \delta_y) \tag{3.1}$$

will be called *the metric diffused along the foliation $\mathcal{F}$ at time t*. Since $d_{\mathcal{W}}(\delta_x, \delta_y) = d(x, y)$ for any $x, y \in M$ and $D_0 =$ id, we see that $D_0 d$ coincides the metric d. The metric space $(M, D_t d)$ will be denoted by M_t.

Theorem 3.2. *For any $s, t \geq 0$, metrics $D_t d$ and $D_s d$ are equivalent.*

4. Metric diffusion for compact foliations of dimension one

First, we recall some facts about compact foliations, i.e. foliations with all leaves compact. The topology of the leaf space of a compact foliation $\mathcal{F}$ on a compact manifold M does not have to be Hausdorff. Examples of such foliations were presented by Epstein and Vogt [7], Sullivan [9] and Vogt [12].

The following result describes the topology of a compact foliation in few equivalent conditions. First, denote by $\pi : M \to \mathcal{L}$ the quotient projection defined by $\pi(x) = L_x$, where $\mathcal{L}$ denotes the space of leaves of a foliation $\mathcal{F}$, i.e., a quotient space of the equivalence relation $x \sim y$ if and only if $L_x = L_y$, where L_z denotes the leaf through z.

Theorem 4.1. *[6] The following conditions are equivalent:*

1. *π is a closed map.*
2. *π maps compact sets onto closed sets.*
3. *Each leaf has arbitrarily small saturated neighborhoods.*
4. *$\mathcal{L}$ with quotient topology is Hausdorff.*
5. *If $K \subseteq M$ is compact, then the saturation of K is also compact.*

Let $G_{\mathcal{F}}$ be the set of all points $x \in M$ near which the volume function is bounded, i.e., $x \in G_{\mathcal{F}}$ if and only if there exists an open neighborhood U of x such that the volumes of all leaves passing through U are uniformly bounded. The set $G_{\mathcal{F}}$ is called *the good set* of the foliation $\mathcal{F}$. Due to [5], $G_{\mathcal{F}}$ is open, saturated, and dense in M and all holonomy groups of leaves contained in $G_{\mathcal{F}}$ are finite. The complement $B_{\mathcal{F}} = M \setminus G_{\mathcal{F}}$ of the good set is called *the bad set*. It follows directly from the definition of the good set and Theorem 4.1 that foliations with empty bad set have a volume of leaves commonly bounded.

One of the most important results about compact foliations is the following:

Theorem 4.2. *[4] Let us suppose that M is a smooth compact Riemannian manifold which is foliated by compact foliation of co-dimension one or two. There is an upper bound of the volumes of the leaves of M.*

Let $\mathcal{F}$ be a compact foliation on a compact Riemannian manifold (M, g) with the volume of leaves commonly bounded above. The classical result says that on a compact manifold M the heat is evenly distributed over M while time is tending to infinity. More precisely,

Theorem 4.3. *[3] For any $f \in L^2(M)$, the function $D_t f$ converges uniformly, as t goes to the infinity, to a harmonic function on M. Since M is compact, the limit function is a constant.*

Let $L, L' \in \mathcal{F}$ be two leaves. One can define the metric $\rho_{\,\mathrm{vol}}$ in the space of leaves by

$$\rho_{\,\mathrm{vol}}(L, L') = d_{\mathcal{W}}(\overline{\mathrm{vol}}(L), \overline{\mathrm{vol}}(L')),$$

where $\overline{\mathrm{vol}}(F)$ denotes the normalized volume of the leaf F.

We will now restrict to the compact foliations of dimension 1. We will study the convergence in the Wasserstein-Hausdorff topology of the natural isometric embeddings $\iota : M_t \to \mathcal{P}(M)$ defined by

$$\iota_t(x) = D_t \delta_x.$$

Precisely speeking, $\iota_t(M, D_t d)$ is a compact subset of $\mathcal{P}(M)$, while we define *the Wasserstein-Hausdorff distance of diffused metrics* by

$$d_{\mathcal{WH}}(M_t, M_s) = (d_{\mathcal{W}})_H(\iota_t(M), \iota_s(M)),$$

where $(d_{\mathcal{W}})_H$ denotes the Hausdorff distance of closed subsets of $\mathcal{P}(M)$.

Theorem 4.4. *The Gromov-Hausdorff limit of a diffused foliation with empty bad set is isometric to the space of leaves equipped with the metric $\rho_{\,vol}$.*

The following example visualizes that in the above Theorem the assumption on the bad set is necessary.

Example 4.5. Following [12], let G be a topological group, while $\gamma : [0, 2\pi] \to G$ a closed curve. One can define a one dimensional foliation $\mathcal{F}(\gamma)$ on $S^1 \times G$ filling it by closed curves as follows:

Through a point $(t,x) \in S^1 \times G$ passes a curve

$$[0,2\pi] \ni s \mapsto (s, \gamma(s)\gamma(t)^{-1}x).$$

Leaves of $\mathcal{F}(\gamma)$ are the fibers of a trivial bundle over G with a fiber S^1. Moreover, if G is a Lie group then $\mathcal{F}(\gamma)$ is a C^r-foliation if only γ is a C^r-curve.

Consider as a Lie group a sphere $S^3 = \{(z,w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}$ with multiplication defined by

$$(a,b)\cdot(c,d) = (ac - b\bar{d}, ad + b\bar{c}).$$

The first step is to define, for any $\tau \in (0,1]$, a curve $\gamma_\tau : [0,2\pi] \to S^3$ as follows:

1. if $\tau = \frac{1}{2n+1} - t$, $0 \le t \le \frac{1}{(2n+1)(2n+2)} = a_n$, $n = 0,1,2,\dots$ then
$$\gamma_\tau(s) = (\sqrt{1 - (\frac{t}{a_n})^2}\, e^{ins}, \frac{t}{a_n}\, e^{ins}), \quad s \in [0,2\pi];$$
2. if $\tau = \frac{1}{2n} - t$, $0 \le t \le \frac{1}{2n(2n+1)} = b_n$, $n = 1,2,\dots$ then
$$\gamma_\tau(s) = (\frac{t}{b_n}\, e^{ins}, \sqrt{1 - (\frac{t}{b_n})^2}\, e^{i(n+1)s}), \quad s \in [0,2\pi].$$

One can easily check that the family γ_τ is continuous.

Next step is to foliate $(0,1] \times S^1 \times S^3$ foliating, for given $\tau \in (0,1]$, the set $\{\tau\} \times S^1 \times S^3$ by $\mathcal{F}(\gamma_\tau)$. Directly from the definition of $\mathcal{F}(\gamma_\tau)$, one can see that the length of leaves tends to infinity, and the length of the S^1 component of the vector tangent to a leaf goes to 0 while $\tau \to 0$. Moreover, γ_τ converge tangentially to the left co-sets of closed 1-parameter subgroup

$$H = \{(e^{is}, 0), s \in [0,2\pi]\}.$$

Complementing the foliation of $M = [0,1] \times S^1 \times S^3$ by a foliation of $\{0\} \times S^1 \times S^3$ by leaves of the form

$$\{0\} \times \{t\} \times H \cdot g, \quad g \in S^3, t \in S^1$$

we obtain 1-dimensional foliation $\tilde{\mathcal{F}}$ of $[0,1] \times S^1 \times S^3$ with nonempty bad set.

Now, we introduce a modification of $\tilde{\mathcal{F}}$ to obtain our target foliation.

Let $h : [0,2\pi] \to [0,2\pi]$ be a increasing function with the graph as on the Figure 1

Next, let $\bar{h} : [0,1] \times [0,2\pi] \to [0,2\pi]$ be a smooth homotopy from identity to h, that is $\bar{h}(t,s) = (1-t)s + th(s)$. Define a modificating function $\tilde{h} : [0,1] \times [0,2\pi] \to [0,2\pi]$ by the formula

$$\tilde{h}(t,s) = \begin{cases} \bar{h}(2t,s) & \text{for } t \in [0,\frac{1}{2}], \\ \bar{h}(-2t+2,s) & \text{for } t \in [\frac{1}{2},1]. \end{cases}$$

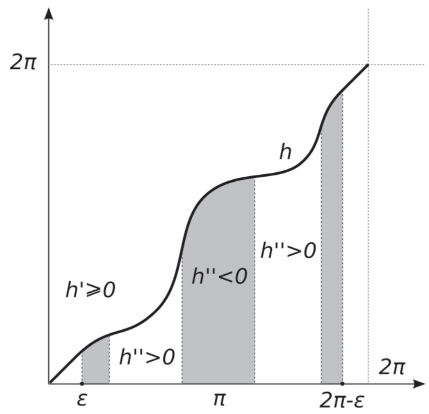

Figure 1: A modificating function.

Having $\tilde{h}$, we define mappings $H_n : [0,1] \times S^1 \times S^3 \to [0,1] \times S^1 \times S^3$ by

$$\tilde{H}_n(\tau, s, x) = \begin{cases} (\tau, \tilde{h}(n(n+1)\tau - n), s), x) & \\ \qquad\qquad \text{for } (\tau, s, x) \in [\frac{1}{2n+2}, \frac{1}{2n+1}] \times S^1 \times S^3, & \\ (\tau, s, x) \qquad \text{otherwise.} & \end{cases}$$

Note that H_n changes $\tilde{\mathcal{F}}$ only on the set

$$(\tau, s, x) \in [\frac{1}{2n+2}, \frac{1}{2n+1}] \times S^1 \times S^3$$

and leaves it unchanged everywhere else.

Let us modify the foliation $\tilde{\mathcal{F}}$ as follows:

For $n_1 = 1$ set $\mathcal{F}_1 = (H_1)_* \tilde{\mathcal{F}}$. Next, choose $\theta_1 > 0$ such that for all $\theta > \theta_1$ and all $p = (\tau, s, x) \in [\frac{1}{2n_1+2}, 1] \times S^1 \times S^3$

$$d_{\mathcal{W}}(D_{\theta_1}\delta_p, \overline{\mathrm{vol}}(L_p)) < \frac{1}{2^{n_1}}.$$

Suppose that we have choosen $n_k > n_{k-1}$ and $\theta_k > \theta_{k-1}$ such that for foliation

$$\mathcal{F}_k = (H_k \circ \cdots \circ H_1)_* \tilde{\mathcal{F}}$$

and all $p = (\tau, s, x) \in [\frac{1}{2(n_k+1)}, 1] \times S^1 \times S^3$

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\mathrm{vol}}(L_p)) < \frac{1}{2^{n_k}}.$$

Let us choose $n_{k+1} > n_k$ for which all leaves of $\mathcal{F}_{k+1} = (H_{k+1})_* \mathcal{F}_k$ passing through $p = (\tau, s, x) \in [0, \frac{1}{n_{k+1}}] \times S^1 \times S^3$ satisfy

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\mathrm{vol}}(L_{(0,s,x)})) < \frac{1}{2^k}.$$

Finally foliation $\mathcal{F}$ as $(\cdots H_n \circ \cdots \circ H_1)_* \tilde{\mathcal{F}}$ and consider the Riemannian metric d induced from $\mathbb{R}^7$ equipped wih $\mathcal{F}$ on M.

Theorem 4.6. *The family of $(M, \mathcal{F}, D_t d)$ does not satisfies the Cauchy condition in Wasserstein-Hausdorff topology. Namely, there exists $\epsilon_0 > 0$ such that for any $T > 0$ one can find $\theta, \lambda > T$ satisfying*

$$d_{\mathcal{WH}}(M_\theta, M_\lambda) > \epsilon_0.$$

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Geometry and Foliations 2013
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LaTeX and graphics

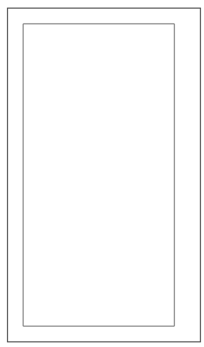
Zofia WALCZAK

Abstract of poster

There are the number of distinct ways of producing graphics each with advantages and disadvantages in terms of flexibility, device independence and ability to include arbitrary TeX text.

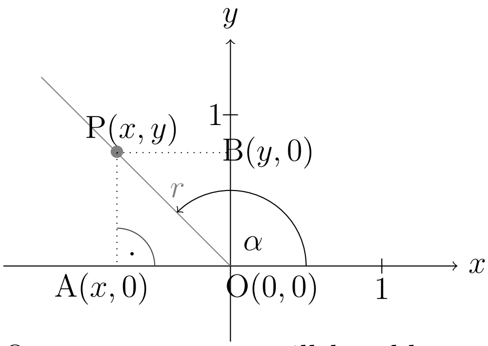
On my poster I will describe various possibilities of embedding graphics in LaTeX document. I will start from *picture* environment provided by L. Lamport with LaTeX2.09 format with the example of very simple picture.

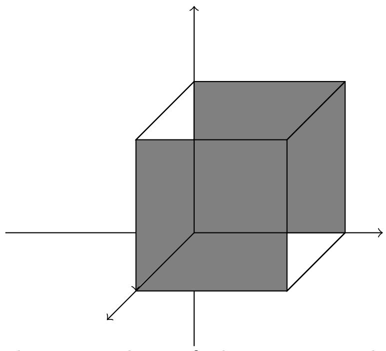
```
\unitlength1cm
\fboxsep2mm
\color{rgray}
\fbox{\color{lgray}
\begin{picture}(2,4)
\put(0,0){\line(1,0){2}}
\put(2,0){\line(0,1){4}}
\put(2,4){\line(-1,0){2}}
\put(0,4){\line(0,-1){4}}
\end{picture}
}
```



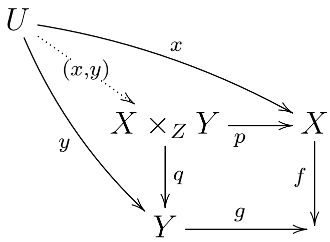
I will also present how to obtain, using not only `tikzpicture` environment, pictures like that.

I plan to show some simple and more complicated pictures produced with TikZ





On my poster one will be able to find examples of diagrams, charts, chemical formulas, music notes and more complicated 3-dimensional graphics in different formats.



My poster will present a short history of creating or embedding graphics to the LaTeX document.

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Groups of uniform homeomorphisms of covering spaces

Tatsuhiko YAGASAKI

1. Introduction

The uniform topology is one of basic topologies on function spaces. In this note we report some results on local and global deformation properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric manifolds endowed with the uniform topology.

Our main goal is to understand local or global topological properties of groups of uniform homeomorphisms of metric manifolds endowed with the uniform topology (for example, local contractibility, homotopy type, local or global topological type as infinite-dimensional manifolds, etc). Since the notions of uniform continuity and uniform topology depend on the choice of metrics, we are also interested in dependence of those topological properties on the behavior of metrics in neighborhoods of ends of manifolds.

In [6] we studied the formal behaviour of local deformation property in the space of uniform embeddings and showed that this property is preserved by the restriction and union of domains of uniform embeddings. This observation reduces our problem to the study of simpler pieces. In [2] A.V. Černavskiĭ considered the case where the manifold M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. Recently, in [5] we treated the class of metric covering spaces over compact manifolds. In this case we can deduce a local deformation theorem for uniform embeddings from the Edwards-Kirby local deformation theorem for embeddings of compact spaces and the classical Arzela-Ascoli theorem for equi-continuous families of maps ([5, Theorem 1.1]). The additivity of local deformation property implies that any metric manifold with a locally geometric group action also has the same local deformation property ([6, Theorem 4.1]).

The local deformation property for uniform embeddings implies the local contractibility of the group of uniform homeomorphisms. Our next aim is to study its global deformation property. The most standard example is the Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric. Its relevant feature is the existence of similarity transformations. This enables us to deduce a global deformation for uniform embeddings in the Euclidean end from the local one. Since this property is preserved by bi-Lipschitz homeomorphisms, we obtain a global deformation theorem for the group of uniform homeomorphisms of any metric manifold with finitely many bi-Lipschitz Euclidean ends ([5, Theorem 1.2]). This implies, for instance, the contractibility of the identity components of the groups of uniform homeomorphisms of $\mathbb{R}^n$ and any non-compact 2-manifold with finitely many bi-Lipschitz Euclidean ends ([5, Example 1.1]).

In the succeeding sections we explain some details of the statements described in this introduction. Section 2 contains local deformation results for uniform embeddings. Section 3 includes global deformation results for uniform homeomorphisms.

2. Local deformation property for uniform embeddings

2.1. Suppose (X, d) is a metric space. For subsets A, B of X we write $A \subset_u B$ and call B a uniform neighborhood of A in X if B contains the ε-neighborhood $O_\varepsilon(A)$ of A in X for some $\varepsilon > 0$.

A map $f : (X, d) \to (Y, \rho)$ between metric spaces is said to be uniformly continuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, x' \in X$ and $d(x, x') < \delta$ then $\rho(f(x), f(x')) < \varepsilon$. The map f is called a uniform homeomorphism if f is bijective and both f and f^{-1} are uniformly continuous. A uniform embedding is a uniform homeomorphism onto its image.

A metric manifold means a separable topological manifold possibly with boundary assigned a fixed metric. Suppose (M, d) is a metric n-manifold. For subsets X and C of M, let $\mathcal{E}^u_*(X, M; C)$ denote the space of proper uniform embeddings $f : (X, d|_X) \to (M, d)$ such that $f = \mathrm{id}$ on $X \cap C$. This space is endowed with the uniform topology induced from the sup-metric

$$d(f, g) = \sup\big\{d(f(x), g(x)) \mid x \in X\big\} \in [0, \infty] \quad (f, g \in \mathcal{E}^u_*(X, M; C)).$$

Definition 2.1. For a subset A of M we say that A has the local deformation property for uniform embeddings in (M, d) and write $A : (\mathrm{LD})_M$ if the following holds:

($*$) for any subset X of A, any uniform neighborhoods $W' \subset W$ of X in (M, d) and any subsets $Z \subset_u Y$ of M there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\phi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ which satisfy the following conditions:

(1) For each $h \in \mathcal{W}$

(i) $\phi_0(h) = h$, (ii) $\phi_1(h) = \mathrm{id}$ on X,

(iii) $\phi_t(h) = h$ on $W - W'$ and $\phi_t(h)(W) = h(W)$ $(t \in [0, 1])$,

(iv) if $h = \mathrm{id}$ on $W \cap \partial M$, then $\phi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0,1])$.

(2) $\phi_t(i_W) = i_W \quad (t \in [0,1])$.

In the case where $A = M$ we omit the subscript M in the symbol $(\mathrm{LD})_M$.

The celebrated Edwards-Kirby local deformation theorem [3] can be restated in the next form.

Theorem 2.2. *(Edwards-Kirby [3]) Any relatively compact subset K of M satisfies the condition $(LD)_M$.*

The condition $(\mathrm{LD})_M$ has the following formal properties:

Proposition 2.3. *([6, Proposition 3.1, Corollary 3.1, Remark 3.2])*

(1) The property (LD) is preserved by any uniform homeomorphism (i.e., if (M,d) is uniformly homeomorphic to (N,ρ), then (M,d) : (LD) $\Longleftrightarrow$ (N,ρ) : (LD).)

(2) (Restriction) (i) Suppose $A \subset B \subset M$. Then, $B : (LD)_M \Longrightarrow A : (LD)_M$.

(ii) Suppose $A \subset_u N \subset M$ and N is an n-manifold. Then, $A : (LD)_N \Longleftrightarrow A : (LD)_M$.

(3) (Additivity) (i) Suppose $A \subset_u U \subset M$ and $B \subset M$. Then, U, $B : (LD)_M \Longrightarrow A \cup B : (LD)_M$.

(ii) Suppose $M = A \cup B$, A, B are n-manifolds and $A - B \subset_u A$. Then, A, $B : (LD) \Longrightarrow M : (LD)$.

(4) Suppose K is a relatively compact subset of M and $A \subset M$. Then, $A : (LD)_M \Longleftrightarrow A \cup K : (LD)_M$.

(5) (Neighborhoods of ends) Suppose $M = K \cup \cup_{i=1}^m L_i$, K is relatively compact, each L_i is an n-manifold and closed in M, and $d(L_i, L_j) > 0$ for any $i \neq j$. Then, $M : (LD) \Longleftrightarrow L_i : (LD)$ $(i = 1, \cdots, m)$.

2.2. Metric covering projections and Geometric group actions

Our next aim is to seek concrete examples which have the local deformation property for uniform embeddings. The following notion is a natural metric version of Riemannian covering projections.

Definition 2.4. A map $\pi : (X,d) \to (Y,\rho)$ between metric spaces is called a metric covering projection if it satisfies the following conditions:

($\natural$)$_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

($\natural$)$_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

($\natural$)$_3$ $\rho(\pi(x), \pi(x')) \leq d(x, x')$ for any $x, x' \in X$.

Here, a subset A of X is said to be uniformly discrete if there exists an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for any distinct points $x, y \in A$. Note that if Y is an n-manifold, then so is X and $\partial X = \pi^{-1}(\partial Y)$. From the Edwards-Kirby local deformation theorem [3] and the Arzela-Ascoli theorem we can deduce the local deformation theorem for uniform embeddings [5, Theorem 1.1].

Theorem 2.5. *If $\pi : (M, d) \to (N, \rho)$ is a metric covering projection and N is a compact metric manifold, then (M, d) satisfies the condition (LD).*

In term of covering transformations, this theorem corresponds to the case of free group actions. For the non-free case, we have the following generalization. Recall that an action Φ of a discrete group G on a metric space X is called geometric if it is proper, cocompact and isometric. (cf. [1, Chapter I.8]). More generally we say that the action Φ is (i) locally isometric if for every $x \in X$ there exists $\varepsilon > 0$ such that each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$, and (ii) locally geometric if it is proper, cocompact and locally isometric.

Corollary 2.6. *([6, Theorem 4.1]) A metric manifold (M, d) satisfies the condition (LD) if it admits a locally geometric group action.*

Example 2.7. The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric admits the canonical geometric action of $\mathbb{Z}^n$ and the associated Riemannian covering projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ onto the flat torus. Therefore, $\mathbb{R}^n$ has the property (LD). From Proposition 2.3 (4) and (3) it follows that the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$ and the half space $\mathbb{R}^n_{\geq 0} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_n \geq 0\}$ also have the property (LD).

3. Groups of uniform homeomorphisms

Suppose (X, d) is a metric space and A is a subset of X. Let $\mathcal{H}^u_A(X, d)$ denote the group of uniform homeomorphisms of (X, d) onto itself which fix A pointwise, endowed with the uniform topology. Let $\mathcal{H}^u_A(X, d)_0$ denote the connected component of the identity map id_X of X in $\mathcal{H}^u_A(X, d)$. We also consider the subgroup

$$\mathcal{H}^u_A(X,d)_b = \{h \in \mathcal{H}^u_A(X,d) \mid d(h, \mathrm{id}_X) < \infty\}.$$

It is easily seen that $\mathcal{H}^u_A(X,d)_0 \subset \mathcal{H}^u_A(X,d)_b$ since the latter is both closed and open in $\mathcal{H}^u_A(X,d)$.

The group $\mathcal{H}^u(M,d)$ is locally contractible if a metric manifold (M,d) satisfies the condition (LD) in Section 2. Hence, our main concern in this section is in the study of its global deformation property.

The most standard model space $\mathbb{R}^n$ has the similarity transformations

$$k_\gamma : \mathbb{R}^n \approx \mathbb{R}^n : \ \ k_\gamma(x) = \gamma x \qquad (\gamma > 0).$$

Conjugation with these similarity transformations enables us to deduce a global deformation property for uniform embeddings in the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$ from the local one. Since this global deformation property is preserved by bi-Lipschitz equivalence, we can transfer to a more general setting of metric spaces with finitely many bi-Lipschitz Euclidean ends.

Recall that a map $h : (X,d) \to (Y,\rho)$ between metric spaces is said to be Lipschitz if there exists a constant $C > 0$ such that $\rho(h(x), h(x')) \leq Cd(x,x')$ for any $x, x' \in X$. The map h is called a bi-Lipschitz homeomorphism if h is bijective and both h and h^{-1} are Lipschitz maps. A bi-Lipschitz n-dimensional Euclidean end of a metric space (X,d) means a closed subset L of X which admits a bi-Lipschitz homeomorphism of pairs, $\theta : (\mathbb{R}^n_1, \partial\mathbb{R}^n_1) \approx ((L, \mathrm{Fr}_X L), d|_L)$ and $d(X - L, L_r) \to \infty$ as $r \to \infty$, where $\mathrm{Fr}_X L$ is the topological frontier of L in X and $L_r = \theta(\mathbb{R}^n_r)$ for $r \geq 1$. We set $L' = \theta(\mathbb{R}^n_2)$ and $L'' = \theta(\mathbb{R}^n_3)$.

Theorem 3.1. *([5, Theorem 1.2]) Suppose X is a metric space and $L_1, \cdots, L_m$ are mutually disjoint bi-Lipschitz Euclidean ends of X. Let $L' = L'_1 \cup \cdots \cup L'_m$ and $L'' = L''_1 \cup \cdots \cup L''_m$. Then there exists a strong deformation retraction ϕ of $\mathcal{H}^u(X)_b$ onto $\mathcal{H}^u_{L''}(X)$ such that*

$$\phi_t(h) = h \ \ \textit{on} \ \ h^{-1}(X - L') - L' \ \ \textit{for any} \ \ (h,t) \in \mathcal{H}^u(X)_b \times [0,1].$$

This theorem leads to the following conclusions.

Example 3.2. (1) $\mathcal{H}^u(\mathbb{R}^n)_b$ is contractible for every $n \geq 1$. In fact, there exists a strong deformation retraction of $\mathcal{H}^u(\mathbb{R}^n)_b$ onto $\mathcal{H}^u_{\mathbb{R}^n_3}(\mathbb{R}^n)$ and the latter is contractible by Alexander's trick.

(2) Suppose N is a compact connected 2-manifold with a nonempty boundary and $C = \cup_{i=1}^m C_i$ is a nonempty union of some boundary circles of N. If the noncompact 2-manifold $M = N - C$ is assigned a metic d such that for each $i = 1, \cdots, m$ the end L_i of M corresponding to the boundary circle

C_i is a bi-Lipschitz Euclidean end of (M, d), then $\mathcal{H}^u(M, d)_0 \simeq \mathcal{H}^u_{L''}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq *$.

We close the section with a question on the topological type of the group $\mathcal{H}^u(\mathbb{R}^n)_b$. In [4] we studied the topological type of $\mathcal{H}^u(\mathbb{R})_b$ as an infinite-dimensional manifold and showed that it is homeomorphic to ℓ_∞. Example 3.2 (1) leads to the following conjecture.

Conjecture 3.3. *$\mathcal{H}^u(\mathbb{R}^n)_b$ is homeomorphic to ℓ_∞ for any $n \geq 1$.*

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Non-wandering, recurrence, p.a.p. and *R*-closed properties for flows and foliations

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1. Introduction

In the recent papers [6]-[11], we study pointwise almost periodic (p.a.p), recurrent, non-wandering, and R-closed properties for flows. By a flow, we mean a continuous action of a topological group G on X. Also we define these notions for decompositions (in particular foliations). Using them, we study codimension one and two foliations. By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a set X such that $X = \sqcup \mathcal{F}$. In this talk, we survey these results. The following relations for decompositions of compact Hausdorff spaces hold [6][10]:

$$R\text{-closed} \Rightarrow \text{pointwise almost periodic} \Rightarrow \text{recurrent} \Rightarrow \text{non-wandering}.$$

Let $\mathcal{F}$ be a decomposition of a topological space X. An element L of $\mathcal{F}$ is said to be recurrent if either it is compact or $\overline{L} - L$ is not closed. An element L of $\mathcal{F}$ is non-wandering if it is contained in the closure of the union of recurrent elements. A decomposition $\mathcal{F}$ is said to be recurrent (resp. non-wandering) if so is each element of $\mathcal{F}$. We call that $\mathcal{F}$ is pointwise almost periodic (p.a.p.) if the set of all closures of elements of $\mathcal{F}$ also is a decomposition. Then denote by $\hat{\mathcal{F}}$ the decomposition of closures and by $M/\hat{\mathcal{F}}$ the quotient space, called the orbit class space. For any $x \in X$, denote by L_x the element of $\mathcal{F}$ containing x. $\mathcal{F}$ is R-closed if $R := \{(x, y) \mid y \in \overline{L_x}\}$ is closed. For a subset $A \subseteq X$, A is saturated if $A = \sqcup_{x \in A} L_x$. A decomposition $\mathcal{F}$ is upper semicontinuous (usc) if each element of $\mathcal{F}$ is both closed and compact and, for any $L \in \mathcal{F}$ and for any open neighborhood U of L, there is a saturated neighborhood of L contained in U. The following relations for p.a.p. decompositions of compact metrizable spaces hold[9]:

$$\mathcal{F} : R\text{-closed} \Longleftrightarrow \hat{\mathcal{F}} : R\text{-closed} \Longleftrightarrow \hat{\mathcal{F}} : \text{usc} \Longleftrightarrow M/\hat{\mathcal{F}} : \text{Hausdorff}$$

By a continuum we mean a compact connected metrizable space. A continuum $A \subset X$ is said to be annular if it has a neighbourhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components each of which is homeomorphic to an annulus. We call that

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a subset $C \subset X$ is a circloid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. We say that a minimal set $\mathcal{M}$ on a surface homeomorphism $f : S \to S$ is an extension of a Cantor set if there are a surface homeomorphism $\widetilde{f} : S \to S$ and a surjective continuous map $p : S \to S$ which is homotopic to the identity such that $p \circ f = \widetilde{f} \circ p$ and $p(\mathcal{S})$ is a Cantor set which is a minimal set of $\widetilde{f}$.

In [5], it has shown that a weakly almost-periodic orientation-preserving homeomorphism on $\mathbb{S}^2$ which is not periodic has exactly two fixed points and the closure of each regular orbit is annular continuum. From now on, let f an R-closed homeomorphism on a connected orientable closed surface S. Now we state the results for R-closed homeomorphisms.

Proposition 1.1. *[7] If $S = \mathbb{S}^2$ and f is not periodic but orientation-preserving (resp. reversing), then the minimal sets of f (resp. f^2) are exactly two fixed points and other circloids and $\mathbb{S}^2/\hat{f} \cong [0, 1]$.*

Theorem 1.2. *[7] If S has genus more than one, then each minimal set of f is either a periodic orbit or an extension of a Cantor set.*

In [3], it has shown that an invariant continuum K of a non-wandering homeomorphism of a compact orientable surface satisfies one of the following holds: (1) f has a periodic point in K; (2) K is annular; (3) $K = S = \mathbb{T}^2$. Moreover, it has shown [2] that a minimal set $\mathcal{M} \neq \mathbb{T}^2$ of a non-wandering toral homeomorphism satisfies one of the following holds: (1) $\mathcal{M}$ is a periodic orbit; (2) $\mathcal{M}$ is the orbit of a periodic circloid; (3) $\mathcal{M}$ is the extension of a Cantor set.

Theorem 1.3. *[7] If $S = \mathbb{T}^2$ and f is neither minimal nor periodic, then either each minimal set of f is a finite disjoint union of essential circloids or there is a minimal set which is an extension of a Cantor set.*

Recall that a subset U of a topological space is locally connected if every point of U admits a neighbourhood basis consisting of open connected subsets.

Theorem 1.4. *[8] The suspension v_f of f satisfies one of the following conditions:*
1) each orbit closure of v_f is toral.
2) there is a minimal set which is not locally connected.

We state the results for $\mathbb{R}$-actions. Let v be a continuous $\mathbb{R}$-action on a connected orientable closed surface S. Denote by LD the union of locally dense orbits.

Theorem 1.5. *[11] The $\mathbb{R}$-action v is non-wandering if and only if* $\overline{LD \sqcup \mathrm{Per}(v)} \cup \mathrm{Sing}(v) = S$. *In particular, if v is non-wandering, then* $\mathrm{Per}(v)$ *is open and there are no exceptional orbits.*

In [1] and [4], it is showed that the following properties are equivalent for an action of a finitely generated group G on either a compact zero-dimensional space or a graph X: (1) (G, X) is pointwise recurrent. (2) (G, X) is pointwise almost periodic. (3) (G, X) is R-closed. Now we state $\mathbb{R}$-actions on surfaces.

Theorem 1.6. *[6] The following are equivalent:*
1) v is pointwise recurrent.
2) v is pointwise almost periodic.
3) v is either minimal or pointwise periodic.

Theorem 1.7. *[6] Suppose v is neither identical nor minimal. Then v is R-closed if and only if v consists of periodic orbits and at most two centers.*

We state the results for foliations.

Theorem 1.8. *[6] Let $\mathcal{F}$ a continuous codimension one foliation on a closed connected manifold. The following are equivalent:*
1) $\mathcal{F}$ is pointwise almost periodic.
2) $\mathcal{F}$ is R-closed.
3) $\mathcal{F}$ is minimal or compact.

Note that, for a closed connected manifold M, the set of codimension two foliations on M which are minimal or compact is a proper subset of the set of R-closed codimension two foliations on M [9].

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The minimal volume orientable hyperbolic 3-manifold with 4 cusps

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1. Introduction

For hyperbolic 3-manifolds, their volumes are known to be topological invariants. The structure of the set of the volumes of hyperbolic 3-manifolds is known by the Jørgensen-Thurston theorem: The set of the volumes of orientable hyperbolic 3-manifolds is a well-ordered set of the type ω^ω with respect to the order of $\mathbb{R}$. The volume of an orientable hyperbolic 3-manifold with n-cusps corresponds to an n-fold limit ordinal.

This theorem gives rise to the problem of determining the minimal volume orientable hyperbolic 3-manifolds with n cusps. The answers are known in the cases where $0 \leq n \leq 2$. Agol [1] conjectured which manifolds have the minimal volume in the cases where $n \geq 3$. We present the result that we determined it in the case where $n = 4$.

Theorem 1.1. *The minimal volume orientable hyperbolic 3-manifold with 4 cusps is homeomorphic to the* 8^4_2 *link complement. Its volume is* $7.32... = 2V_8$*, where* V_8 *is the volume of the ideal regular octahedron.*

2. Outline of Proof

8^4_2 link complement is obtained from two ideal regular octahedra by gluing along the faces. Hence we need a lower bound on the volume of an orientable hyperbolic 3-manifold with 4 cusps. The proof relies on Agol's argument used to determine the minimal volume hyperbolic 3-manifolds with 2 cusps [1].

Let M be a finite volume hyperbolic 3-manifold, and let X be a (non-necessarily connected) essential surface in M. After we cut M along X, the relative JSJ decomposition can be performed. The obtained components are characteristic or hyperbolic. The union of hyperbolic components is called the *guts* of $M - X$. The guts admit another hyperbolic metric with totally geodesic boundary. Then we can obtain a lower bound of the volume of M.

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Theorem 2.1 (Agol-Storm-Thurston [2]). *Let L be the guts of $M - X$. Then* $\mathrm{vol}(M) \geq \mathrm{vol}(L) \geq \frac{V_8}{2}|\chi(\partial L)|$, *where* $\mathrm{vol}(L)$ *is defined with respect to the hyperbolic metric of L with totally geodesic boundary.*

Therefore it is sufficient that we estimate the Euler characteristic of the boundary of guts. Let M be a finite volume hyperbolic 3-manifold with 4 cusps. At first, we construct an essential surface X such that the guts of $M - X$ have 4 torus or annular cusps. We need to estimate the volume of a hyperbolic manifold L with totally geodesic boundary and 4 cusps. Purely homological arguement shows that L has a non-separating essential surface. Beginning from this surface, we construct an essential surface Y in L such that $\chi(\partial(\text{guts of } L - Y)) \leq -4$.

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