Lie foliations transversely modeled on nilpotent Lie algebras

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1 Preliminaries

Let $\mathcal{F}$ be a transversely orientable codimension q smooth foliation of a closed orientable smooth manifold M. Let $\mathfrak{g}$ be a q-dimensional real Lie algebra.

Definition

$\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation if $^\exists$ a $\mathfrak{g}$-valued non-singular Maurer-Cartan form $\omega\colon TM \to \mathfrak{g}$ such that $\mathrm{Ker}(\omega) = T\mathcal{F}$.

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation of a closed manifold M. Then we have the following structure theorem proved by P. Molino.

Theorem (Molino '82)

(i) $\overline{\mathcal{F}} = \{\overline{L}\}_{L\in\mathcal{F}}$ is a foliation of M.

(ii) $M/\overline{\mathcal{F}}$ is a closed manifold and $\pi\colon M \to M/\overline{\mathcal{F}}$ is a locally trivial fibration.

(iii) $^\exists$ Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is uniquely determined by $\mathcal{F}$ such that, for each fiber F, $\mathcal{F}|_F$ is a Lie $\mathfrak{h}$-foliation.

- $\mathfrak{h}$ is called the structure Lie algebra of $\mathcal{F}$
- If $\mathcal{F}$ is flow, then $\mathfrak{h}$ is abelian, i.e. $\mathfrak{h} \cong \mathbb{R}^m$ for $^\exists m$ (by Caron and Carrière's Theorem '80).

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Let $m \leq \dim \mathfrak{g}$ be a non-negative integer.

Definition

- $(\mathfrak{g}, \mathfrak{h})$ is realizable if $^\exists$ a closed manifold M and $^\exists$ a Lie $\mathfrak{g}$-foliation of M such that the structure Lie algebra is $\mathfrak{h}$.
- $(\mathfrak{g}, m)$ is realizable if $^\exists$ a closed manifold M and $^\exists$ a Lie $\mathfrak{g}$-flow on M such that the structure Lie algebra is $\mathbb{R}^m$.

Question

(i) Which pair $(\mathfrak{g}, \mathfrak{h})$ is realizable ?

(ii) Which pair $(\mathfrak{g}, m)$ is realizable?

2 Main Theorem

Theorem A

Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra. Suppose that $\mathfrak{g}$ has a rational structure.
Then $(\mathfrak{g}, m)$ is realizable if and only if $m \leq \dim \mathfrak{c}(\mathfrak{g})$.

Theorem B

Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of $\mathfrak{g}$.
Then $(\mathfrak{g}, \mathfrak{h})$ is realizable if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the quotient Lie algebra $\mathfrak{h}\backslash\mathfrak{g}$ has a rational structure.

3 Sketch of Proof

Key Theorem

Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$.
Let $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_k \subset \mathfrak{g}$ be ideals of $\mathfrak{g}$ with $\dim \mathfrak{g}_j = l_j$.

Theorem (Mal'cev '62)

$^\exists$ a basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$ such that

(i) $1 \leq {}^\forall l \leq q$, $\mathfrak{h}_l = \langle X_1, \ldots, X_l\rangle_{\mathbb{K}} \subset \mathfrak{g}$ is an ideal

(ii) $1 \leq {}^\forall j \leq k$, $\mathfrak{h}_{l_j} = \mathfrak{g}_j$.

- A basis satisfying (i) and (ii) is called a strong Mal'cev basis through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.
- A strong Mal'cev basis of $\mathfrak{g}$ through $\{0\}$ is called a strong Mal'cev basis of $\mathfrak{g}$.

Proof of Theorem A.

"only if" part

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a closed manifold M with the structure Lie algebra $\mathbb{R}^m$. Fix a basis $\{e_1, \ldots, e_q\}$ of $\mathfrak{g}$ and fix transverse vector fields $\bar{X}_1, \ldots, \bar{X}_q$ of $\mathcal{F}$ such that $\omega(\bar{X}_i) = e_i$. Then $\langle \bar{X}_1 \ldots, \bar{X}_q\rangle_{\mathbb{R}} \cong \mathfrak{g}$ as a Lie algebra.
Since G is unimodular, $\mathcal{F}$ is unimodular. Thus there exist central transverse vector fields $\bar{Y}_1, \ldots, \bar{Y}_m$ such that $\{\bar{Y}_1, \ldots, \bar{Y}_m\}$ is linearly independent everywhere.

Let f_{ij} be the basic functions such that $\bar{Y}_j = \sum_{i=1}^{q} f_{ij}\bar{X}_i$. Since $\mathfrak{g}$ is nilpotent, we can prove that each f_{ij} is constant. Therefore $\bar{Y}_i \in \langle \bar{X}_1, \ldots, \bar{X}_q\rangle_{\mathbb{R}} \cong \mathfrak{g}$.
Hence $\bar{Y}_i \in \mathfrak{c}(\langle \bar{X}_1 \ldots, \bar{X}_q\rangle_{\mathbb{R}}) \cong \mathfrak{c}(\mathfrak{g})$ and $m \leq \dim \mathfrak{c}(\mathfrak{g})$.

"if" part

Let l be the dimension of $\mathfrak{c}(\mathfrak{g})$. Let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ has a rational structure, we can take a strong Mal'cev basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$ through $\mathfrak{c}(\mathfrak{g})$ such that

$$\Delta = \exp \mathbb{Z}X_1 \cdots \exp \mathbb{Z}X_q$$

is a uniform lattice of G.
Fix $b_1, \ldots, b_l \in \mathbb{R}$ such that

(i) $b_1, \ldots b_m, 1$ are linearly independent over $\mathbb{Q}$

(ii) $b_{m+1}, \ldots, b_l \in \mathbb{Q}$.

$X = \sum_{i=1}^{l} b_i X_i \in \mathfrak{c}(\mathfrak{g})$.

Define $D\colon G \times \mathbb{R} \to G$ and $h\colon \Delta \times \mathbb{Z} \to G$ by

$$D(g,t) = g \cdot \exp tX$$
$$h = D|_{\Delta\times\mathbb{Z}}.$$

Then (D, h) defines a Lie $\mathfrak{g}$-flow on $M = (\Delta \times \mathbb{Z})\backslash(G \times \mathbb{R})$.
By the choice of $b_1, \ldots b_l$, the dimension of the structure Lie algebra of $\mathcal{F}$ is equal to m. □

Proof of Theorem B.

"only if" part

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation of a closed manifold M. Let Γ be the holonomy group of $\mathcal{F}$ and H be the subgroup of G with its Lie algebra $\mathfrak{h}$. Then $H = \overline{\Gamma}_e$.
Since $\Gamma \subset G$ is uniform, $\overline{\Gamma} \subset G$ is a closed uniform subgroup. Hence $H = \overline{\Gamma}_e \subset G$ is a normal subgroup. Thus $\mathfrak{h} \subset \mathfrak{g}$ is an ideal.
Since

$$M/\overline{\mathcal{F}} \cong \overline{\Gamma}\backslash G \cong (\overline{\Gamma}_e\backslash\overline{\Gamma})\backslash(\overline{\Gamma}_e\backslash G),$$

$\overline{\Gamma}_e\backslash\overline{\Gamma} \subset \overline{\Gamma}_e\backslash G$ is a uniform and discrete subgroup. Hence $\mathfrak{h}\backslash\mathfrak{g}$ has a rational structure.

"if" part

Let $p\colon \mathfrak{g} \to \mathfrak{h}\backslash\mathfrak{g}$ be the natural projection. Since $\mathfrak{h}\backslash\mathfrak{g}$ has a rational structure, there exists a strong Mal'cev basis $\{p(Z_1), \ldots, p(Z_{q-m})\}$ of $\mathfrak{h}\backslash\mathfrak{g}$ with integer structure constants such that

$$\exp \mathbb{Z}p(Z_1) \cdots \exp \mathbb{Z}p(Z_{q-m})$$

is a uniform lattice of $H\backslash G$.
Let $\{X_1, \ldots, X_q\}$ be a strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{h}$. Then $\{X_1, \ldots, X_m, Z_1, \ldots Z_{q-m}\}$ is also a strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{h}$.
Fix $a \in \mathbb{R} - \mathbb{Q}$.
Define

$$\Gamma = \langle \exp X_1, \exp aX_1, \ldots, \exp X_m, \exp aX_m, \exp Z_1, \ldots, \exp Z_{q-m}\rangle.$$

Then $\overline{\Gamma}_e = H$.
Since Γ is a finitely generated uniform subgroup of the simply connected nilpotent Lie group G, we can construct a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M such that the holonomy group is Γ.
Since $\overline{\Gamma}_e = H$, the structure Lie algebra of $\mathcal{F}$ is $\mathfrak{h}$. □

4 Corollaries

Corollary 1

$^\forall$ nilpotent Lie algebra $\mathfrak{g}$, $^\exists$ a minimal Lie $\mathfrak{g}$-foliation of a closed manifold.

∵) Since $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}\backslash\mathfrak{g} = \{0\}$ has the rational structure, $(\mathfrak{g}, \mathfrak{g})$ is realizable. □

Corollary 2

$^\exists$ a nilpotent Lie algebra $\mathfrak{g}$ which has no rational structures and $^\exists m$ such that $(\mathfrak{g}, m)$ is realizable.

Sketch of proof.

Fix a nilpotent Lie algebra $\mathfrak{g}'$ with no rational structures and fix a Lie $\mathfrak{g}'$-foliation of a closed manifold M. Then there exists a simply connected nilpotent Lie group G'' with a uniform lattice and there exists a submersion homomorphism $f\colon G'' \to G'$.
Since f is a submersion homomorphism, the induced homomorphism $f_*\colon \mathfrak{g}'' \to \mathfrak{g}'$ is a surjective homomorphism.
Fix a strong Mal'cev basis $\{X_1, \ldots, X_{q''}\}$ of $\mathfrak{g}''$ through $\mathrm{Ker}(f_*)$.
Take $\mathfrak{h}_0 = \{0\}$ and $\mathfrak{h}_k = \langle X_1, \ldots, X_k\rangle_{\mathbb{R}} \subset \mathfrak{g}''$. Then we have the sequence of nilpotent Lie algebras

$$\mathfrak{g}'' = \mathfrak{g}''/\mathfrak{h}_0 \xrightarrow{p_0} \mathfrak{g}''/\mathfrak{h}_1 \xrightarrow{p_1} \ldots \xrightarrow{p_{l-1}} \mathfrak{g}''/\mathfrak{h}_l \xrightarrow{f_*} \mathfrak{g}'.$$

Since $\mathfrak{g}''$ has a rational structure and $\mathfrak{g}'$ has no rational structures, there exists k such that $\mathfrak{g}''/\mathfrak{h}_k$ has a rational structure and $\mathfrak{g}''/\mathfrak{h}_{k+1}$ has no rational structures.
Since $\widetilde{p_k}\colon G''/H_k \to G''/H_{k+1}$ is a submersion homomorphism and $\dim \mathrm{Ker}(\widetilde{p_k}) = 1$, $\widetilde{p_k}$ defines a Lie $\mathfrak{g}''/\mathfrak{h}_{k+1}$-flow on a closed manifold $\Delta\backslash(G''/H_k)$. □

5 Nilpotent Lie algebras with no rational structures

Example (Chao '62)

Let $c_{ij}^k, 1 \leq i, j \leq m, 1 \leq k \leq n$ be real numbers such that $c_{ij}^k = -c_{ji}^k$. Assume that c_{ij}^k are algebraically independent over $\mathbb{Q}$.
Let $\mathfrak{g} = \langle X_1, \ldots, X_m, Y_1, \ldots, Y_n\rangle_{\mathbb{R}}$ with the products $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k Y_k$ for $i, j = 1, \ldots, m$ and all other products being zero.
Then $\mathfrak{g}$ is a nilpotent Lie algebra.
This Lie algebra $\mathfrak{g}$ has no rational structures if $(n/2)(m^2 - m) > m^2 + n^2$.

Proposition

Let $\mathfrak{g}$ be the Lie algebra defined above.
If $(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2$, then $\mathfrak{g}$ cannot be realized as Lie flows.

Remark

The inequality $(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2$ holds if $n = 4$ and $m \geq 8$. Thus, for any $q \geq 12$, there exists a q-dimensional nilpotent Lie algebra $\mathfrak{g}$ with no rational structures such that $\mathfrak{g}$ cannot be realized as Lie flows.