Minimal C^1-diffeomorphisms of the circle which admit measurable fundamental domains

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Ergodic vs M.F.D.

(X,μ): a probability space,
T: a transformation of X.

μ is said to be quasi-invariant if the push forward $T_*\mu$ is equivalent to μ.
In this case T is called ergodic w.r.t. μ, if a T-invariant Borel subset in X is either null or conull.

A. Katok and M. Herman independently showed the following theorem.

Theorem [KH] [H]

A C^1-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational.

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains (M.F.D. for short).

(X,μ): a probability space,
T: a transformation of X,
μ: quasi-invariant.

A Borel subset C of X is called an M.F.D. if T^nC ($n\in\mathbb{Z}$) is mutually disjoint and the union $\cup_{n\in\mathbb{Z}}T^nC$ is conull.

In this case any Borel function on C can be extended to a T-invariant measurable function on X, and an ergodic component of T is just a single orbit.

The purpose of this poster is to show the following theorem.

Main Theorem [KM]

For any irrational number α, there is a minimal C^1-diffeomorphism of the circle with rotation number α which admits an M.F.D. with respect to the Lebesgue measure.

Cantor set

Suppose $q_1, q_2, \dots$ is a natural number sequence s.t. $q_1 \ge 3$ and $q_n/q_{n+1} \le 1/3$. Then

$$C := \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_n} \;\middle|\; \varepsilon_n = 0 \text{ or } 1 \right\}$$

is a Cantor set.

We regard the circle S^1 as $\mathbb{R}/\mathbb{Z}$.
Suppose R denotes the rotation by α.

Claim

For any irrational number α, we can construct a Cantor set $C \in S^1$ so that $R^nC\cap R^mC = \emptyset$ for any integers $n \neq m$.

idea of proof. If α is badly approximable,

$$C = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^{3^n}} \;\middle|\; \varepsilon_n = 0 \text{ or } 1 \right\} \pmod{\mathbb{Z}}$$

will suffice, since $C+(-C)$ consists of well approximable numbers.
For the case α is well approximable, we can take a sequence (q_n) so rapidly increasing that $C+(-C)$ consists of "better" approximable numbers than $m\alpha$ ($\forall m\in\mathbb{N}$). □

M.F.D. for a Lipschitz homeo.

We will show the weaker version of our Main Theorem:

weaker version

For any irrational number α, there is a minimal Lipschitz homeomorphism of the circle with rotation number α which admits an M.F.D. with respect to the Lebesgue measure.

proof.
C: a Cantor set in S^1 so that $R^nC\cap R^mC=\emptyset$.
Fix a number $D>1$.
$(a_i)_{i\in\mathbb{Z}}$: a sequence of positive numbers s.t. $\sum_{i\in\mathbb{Z}} a_i = 1$ and $1/D \le a_{i+1}/a_i \le D$.

Define a probability measure μ on S^1 by

$$\mu := \sum_{i\in\mathbb{Z}} a_i R^i_* \mu_0.$$

The Radon-Nikodym derivative $\frac{dR_*^{-1}\mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set R^iC, so $\frac{dR_*^{-1}\mu}{d\mu} \in L^\infty(S^1,\mu)$.

Define an orientation preserving homeomorphism h of S^1 by $h(0)=0$ and $h_*Leb=\mu$.
Finally define a homeomorphism F of S^1 by $F := h^{-1}\circ R\circ h$, then

$$\frac{dF_*^{-1}Leb}{d\,Leb} = \frac{dR_*^{-1}\mu}{d\mu}\circ h \in L^\infty(S^1, Leb),$$

i.e. the map F is a Lipschitz homeomorphism.
The set $C' = h^{-1}C$ is a measurable fundamental domain of F. □

Make it C^1

To prove our Main Theorem, it is enough to find a prob. measure μ on C s.t. $g=\frac{dR_*^{-1}\mu}{d\mu}$ is continuous on S^1.

Here we change the direction.

Question

What kind of a continuous function g can be induced from a prob. measure μ on C?

Fix a point $x_0\in C$. For a positive integer i,

$$\begin{aligned} a_i &= (a_i/a_{i-1})\cdots(a_2/a_1)(a_1/a_0)a_0 \\ &= g(R^{i-1}x_0)\cdots g(Rx_0)g(x_0)a_0, \\ a_{-i} &= (a_{-i+1}/a_{-i})^{-1}\cdots(a_{-1}/a_{-2})^{-1}(a_0/a_{-1})^{-1}a_0 \\ &= g(R^{-i}x_0)^{-1}\cdots g(R^{-2}x_0)^{-1}g(R^{-1}x_0)^{-1}a_0. \end{aligned}$$

Set $\phi=\log g$ and

$$\begin{aligned} \phi^{(m)}(x) &= \sum_{i=0}^{m-1}\phi(R^ix) \qquad (m>0), \\ \phi^{(-m)}(x) &= -\sum_{i=1}^{m}\phi(R^{-i}x) \qquad (m>0), \\ \phi^{(0)}(x) &= 0, \end{aligned}$$

then $a_i = \exp(\phi^{(i)}(x_0))a_0$.
To satisfy $\sum_{i\in\mathbb{Z}} a_i = 1$, it suffices to find ϕ so that $\sum_{i\in\mathbb{Z}}\exp(\phi^{(i)}(x_0)) < \infty$.

Construction

Fix an integer $n\in\mathbb{N}$.
Since $R^{-2^n}C,\dots,C,\dots,R^{2^n-1}C$ are disjoint compact sets, there exists an ε-neighbourhood N of C such that $R^{-2^n}N,\dots,N,\dots,R^{2^n-1}N$ are disjoint.

Take a bump function f so that
$\operatorname{supp} f\subset N$,
$f(x) = (3/4)^n$ on C and
$0\le f(x) < (3/4)^n$ on $N\setminus C$.
Define $\phi_n\colon S^1\to\mathbb{R}$ by

$$\phi_n(x) = \begin{cases} -f(R^{-i}x) & x\in R^iN,\ i=0,\dots,2^n-1 \\ f(R^{-i}x) & x\in R^iN,\ i=-2^n,\dots,-1 \\ 0 & \text{otherwise.} \end{cases}$$

$\phi=\sum_{i=1}^{\infty}\phi_n$ converges uniformly, thus ϕ is also continuous.
Carefull calculation shows the following lemma:

Lemma

For $x_0\in C$,

$$\phi_n^{(i)}(x_0) \begin{cases} = -|i|(3/4)^n & \textbf{for } -2^n\le i\le 2^n \\ \le 0 & \textbf{for any } i\in\mathbb{Z}. \end{cases}$$

Therefore, if $2^n\le|i|<2^{n+1}$,
$\phi^{(i)}(x_0) \le \phi^{(i)}_{n+1}(x_0) = -|i|(3/4)^{n+1} \le -2^n\cdot(3/4)^{n+1} = -3/4\cdot(3/2)^n$.
Finally we have: $\sum_{i\in\mathbb{Z}}\exp(\phi^{(i)}(x_0)) \le 1+\sum_{n=0}^{\infty}2^{n+1}\exp(-3/4\cdot(3/2)^n) = M<\infty$.

We define a finite measure $\tilde\mu$ on S^1 by

$$\tilde\mu := \sum_{i\in\mathbb{Z}}(\exp\circ\phi^{(i)}\circ R^{-i})R^i_*\mu_0.$$

Normalize $\tilde\mu$ to obtain a prob. measure μ, namely $\mu := \frac{\tilde\mu}{\int_{S^1}d\tilde\mu}$.

Define h and F in the same manner as before,

$$\frac{dF_*^{-1}Leb}{d\,Leb} = \frac{dR_*^{-1}\mu}{d\mu}\circ h = g\circ h$$

is a continuous function because $g(x) = \exp(\phi(x))$. We proved our Main Theorem. □

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