Generalized Newton transformation and its applications to extrinsic geometry

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1 Motivation

Analyzing the study of Riemannian geometry we see that its basic concepts are related with some operators, such as shape, Ricci, Schouten operator, etc. and functions constructed of them, such as mean curvature, scalar curvature, Gauss-Kronecker curvature, etc. The most natural and useful functions are the ones derived from algebraic invariants of these operators, e.g., by taking trace, determinant and in general the r-th symmetric functions σ_r. However, the case $r > 1$ is strongly nonlinear and therefore more complicated. The powerful tool to deal with this problem is the Newton transformation T_r of an endomorphism A (strictly related with the Newton's identities) which, in a sense, enables a linearization of σ_r,

$$(r+1)\sigma_{r+1} = \mathrm{tr}\,(AT_r).$$

2 Newton transformation and Generalized Newton transformation

Let A be an endomorphism of a p–dimensional vector space V. The *Newton transformation* of A is a system $T = (T_r)_{r=0,1,\ldots}$ of endomorphisms of V given by the recurrence relations:

$$T_0 = 1_V,$$
$$T_r = \sigma_r 1_V - AT_{r-1}, \quad r = 1, 2, \ldots$$

Here σ_r's are elementary symmetric functions of A. If $r > p$ we put $\sigma_r = 0$. Equivalently, each T_r may be defined by the formula

$$T_r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j.$$

Observe that T_p is the characteristic polynomial of A. Consequently, by Hamilton–Cayley Theorem $T_p = 0$. It follows that $T_r = 0$ for all $r \geq p$. The Newton transformation satisfies the following relations:

(N1) Symmetric function σ_r is given by the formula

$$r\sigma_r = \mathrm{tr}\,(AT_{r-1}).$$

(N2) Trace of T_r is equal

$$\mathrm{tr}\,T_r = (p-r)\sigma_r.$$

(N3) If $A(\tau)$ is a smooth curve in $\mathrm{End}\,(V)$ such that $A(0) = A$, then for $r = 0, 1, \ldots, p$

$$\frac{d}{d\tau}\sigma_{r+1}(\tau)_{\tau=0} = \mathrm{tr}\,(\frac{d}{d\tau}A(\tau)_{\tau=0} \cdot T_r).$$

Condition (N3) is the starting point to define generalized Newton transformations.

Let V be a p–dimensional vector space (over $\mathbb{R}$) equipped with an inner product $\langle\,,\rangle$. For an endomorphism $A \in \mathrm{End}\,(V)$, let $A^\top$ denote the adjoint endomorphism.

Let $\mathbb{N}$ denote the set of nonnegative integers. By $\mathbb{N}(q)$ denote the set of all sequences $u = (u_1, \ldots, u_q)$, with $u_j \in \mathbb{N}$. The length $|u|$ of $u \in \mathbb{N}(q)$ is given by $|u| = u_1 + \ldots + u_q$. Denote by $\mathrm{End}^{\,q}(V)$ the vector space $\mathrm{End}\,(V) \times \ldots \times \mathrm{End}\,(V)$ (q–times). For $\mathbf{A} = (A_1, \ldots, A_q) \in \mathrm{End}^{\,q}(V)$, $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$ and $u \in \mathbb{N}(q)$ put

$$t^u = t_1^{u_1} \ldots t_q^{u_q},$$
$$t\mathbf{A} = t_1 A_1 + \ldots + t_q A_q$$

By a *Newton polynomial* of $\mathbf{A}$ we mean a polynomial $P_{\mathbf{A}} : \mathbb{R}^q \to \mathbb{R}$ of the form $P_{\mathbf{A}}(t) = \det(1_V + t\mathbf{A})$. Expanding $P_{\mathbf{A}}$ we get

$$P_{\mathbf{A}}(t) = \sum_{|u| \leq p} \sigma_u t^u,$$

where the coefficients $\sigma_u = \sigma_u(\mathbf{A})$ depend only on $\mathbf{A}$. Observe that $\sigma_{(0,\ldots,0)} = 1$. It is convenient to put $\sigma_u = 0$ for $|u| > p$.

Consider the following (music) convention. For α we define functions $\alpha^\sharp : \mathbb{N}(q) \to \mathbb{N}(q)$ and $\alpha_\flat : \mathbb{N}(q) \to \mathbb{N}(q)$ as follows

$$\alpha^\sharp(i_1, \ldots, i_q) = (i_1, \ldots, i_{\alpha-1}, i_\alpha + 1, i_{\alpha+1}, \ldots, i_q),$$
$$\alpha_\flat(i_1, \ldots, i_q) = (i_1, \ldots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \ldots, i_q),$$

i.e. $\alpha^\sharp$ increases the value of the α–th element by 1 and $\alpha_\flat$ decreases the value of α–th element by 1. It is clear that $\alpha^\sharp$ is the inverse map to $\alpha_\flat$.

Now, we may state the main definition. The *generalized Newton transformation* of $\mathbf{A} = (A_1, \ldots, A_q) \in \mathrm{End}^{\,q}(V)$ is a system of endomorphisms $T_u = T_u(\mathbf{A})$, $u \in \mathbb{N}(q)$, satisfying the following condition (generalizing (N3)):

For every smooth curve $\tau \mapsto \mathbf{A}(\tau)$ in $\mathrm{End}^{\,q}(V)$ such that $\mathbf{A}(0) = \mathbf{A}$

$$\begin{aligned}\frac{d}{d\tau}\sigma_u(\tau)_{\tau=0} &= \sum_\alpha \langle\!\langle \frac{d}{d\tau}A_\alpha(\tau)_{\tau=0})^\top | T_{\alpha_\flat(u)} \rangle\!\rangle \\ &= \sum_\alpha \mathrm{tr}\,\left(\frac{d}{d\tau}A_\alpha(\tau)_{\tau=0} \cdot T_{\alpha_\flat(u)}\right).\end{aligned} \qquad \text{(GNT)}$$

Theorem 2.1 (Generalized Hamilton–Cayley Theorem) *Let $T = (T_u : u \in \mathbb{N}(q))$ be the generalized Newton transformation of $\mathbf{A}$. Then for every $u \in \mathbb{N}(q)$ of length greater or equal to p we have $T_u = 0$.*

Moreover the generalized Newton transformation $T = (T_u : u \in \mathbb{N}(q))$ of $\mathbf{A}$ satisfies the following recurrence relations:

Theorem 2.2

$$\begin{aligned} T_0 &= 1_V, && \textit{where } 0 = (0, \ldots, 0) \\ T_u &= \sigma_u 1_V - \sum_\alpha A_\alpha T_{\alpha_\flat(u)} && \textit{where } |u| \geq 1. \\ &= \sigma_u 1_V - \sum_\alpha T_{\alpha_\flat(u)} A_\alpha, \end{aligned}$$

For $q, s \geq 1$ let $\mathbb{N}(q, s)$ be the set of all $q \times s$ matrices, whose entries are elements of $\mathbb{N}$. Clearly, the set $\mathbb{N}(1, s)$ is the set of multi–indices $i = (i_1, \ldots, i_s)$ with $i_1, \ldots, i_s \in \mathbb{N}$, hence $\mathbb{N}(s) = \mathbb{N}(1, s)$. Moreover, every matrix $\mathbf{i} = (i_l^\alpha) \in \mathbb{N}(q, s)$ may be identified with an ordered system $\mathbf{i} = (i^1, \ldots, i^q)$ of multi–indices $i^\alpha = (i_1^\alpha, \ldots, i_s^\alpha)$.

If $i = (i_1, \ldots, i_s) \in \mathbb{N}(s)$ then its *length* is simply the number $|i| = i_1 + \ldots + i_s$. For $\mathbf{i} = (i^1, \ldots, i^q) \in \mathbb{N}(q, s)$ we define its *weight* as an multi–index $|\mathbf{i}| = (|i^1|, \ldots, |i^q|) \in \mathbb{N}(q)$. By the *length* $\|\mathbf{i}\|$ of $\mathbf{i}$ we mean the length of $|\mathbf{i}|$, i.e., $\|\mathbf{i}\| = \sum_\alpha |i^\alpha| = \sum_{\alpha,l} i_l^\alpha$.

Denote by $\mathbb{I}(q, s)$ a subset of $\mathbb{N}(q, s)$ consisting of all matrices $\mathbf{i}$ satisfying the following conditions:

1. every entry of $\mathbf{i}$ is either 0 or 1,
2. the length of $\mathbf{i}$ is equal to s,
3. in every column of $\mathbf{i}$ there is exactly one entry equal to 1, or equivalently $|\mathbf{i}^\top| = (1, \ldots, 1)$.

We identify $\mathbb{I}(q, 0)$ with a set consisting of the zero vector $0 = [0, \ldots, 0]^\top$.

Let $\mathbf{A} \in \mathrm{End}^{\,q}(V)$, $\mathbf{A} = (A_1, \ldots, A_q)$, and $\mathbf{i} \in \mathbb{N}(q, s)$. By $\mathbf{A}^{\mathbf{i}}$ we mean an endomorphism (composition of endomorphisms) of the form

$$\mathbf{A}^{\mathbf{i}} = A_1^{i_1^1} A_2^{i_1^2} \ldots A_q^{i_1^q} A_1^{i_2^1} \ldots A_q^{i_2^q} \ldots A_1^{i_s^1} \ldots A_q^{i_s^q}.$$

In particular, $\mathbf{A}^0 = 1_V$.

Theorem 2.3 *For every system of endomorphisms $\mathbf{A} = (A_1, \ldots, A_q)$, there exists unique generalized Newton transformation $T = (T_u : u \in \mathbb{N}(q))$ of $\mathbf{A}$. Moreover, each T_u is given by the formula*

$$T_u = \sum_{s=0}^{|u|} \sum_{\mathbf{i} \in \mathbb{I}(q,s)} (-1)^{\|\mathbf{i}\|} \sigma_{u-|\mathbf{i}|} \mathbf{A}^{\mathbf{i}},$$

where $\sigma_{u-|\mathbf{i}|} = \sigma_{u-|\mathbf{i}|}(\mathbf{A})$.

3 Applications to extrinsic geometry

Let (M, g) be an oriented Riemannian manifold, D a p–dimensional (transversally oriented) distribution on M. Let q denotes the codimension of D. For each $X \in T_xM$ there is unique decomposition

$$X = X^\top + X^\perp,$$

where $X^\top \in D_x$ and $X^\perp$ is orthogonal to D_x. Denote by $D^\perp$ the bundle of vectors orthogonal to D. Let ∇ be the Levi–Civita connection of g. ∇ induces connections $\nabla^\top$ and $\nabla^\perp$ in vector bundles D and $D^\perp$ over M, respectively.

Let $\pi : P \to M$ be the principal bundle of orthonormal frames (oriented orthonormal frames, respectively) of $D^\perp$. Clearly, the structure group G of this bundle is $G = O(q)$ ($G = SO(q)$, respectively).

Every element $(x, e) = (e_1, \ldots, e_q) \in P_x$, $x \in M$, induces the system of endomorphisms $\mathbf{A}(x, e) = (A_1(x, e), \ldots, A_q(x, e))$ of D_x, where $A_\alpha(x, e)$ is the shape operator corresponding to (x, e), i.e.

$$A_\alpha(x, e)(X) = -\left(\nabla_X e_\alpha\right)^\top, \quad X \in D_x.$$

Let $T(x, e) = (T_u(x, e))_{u \in \mathbb{N}(q)}$ be the generalized Newton transformation associated with $\mathbf{A}(x, e)$.

Define the section $Y_u \in \Gamma(E)$, $u \in \mathbb{N}(q)$ as follows

$$Y_u(x, e) = \sum_{\alpha,\beta} T_{\beta_\flat\alpha_\flat(u)}(x, e)(\nabla_{e_\alpha} e_\beta)^\top + \sum_\alpha \sigma_{\alpha_\flat(u)}(x, e)e_\alpha.$$

Lemma 3.1 *The divergence of Y_u is given by the formula*

$$\begin{aligned}\mathrm{div}_E Y_u = -|u|\sigma_u + \sum_{\alpha,\beta} \Big[&\mathrm{tr}\,(R_{\alpha,\beta} T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'} T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha} e_\beta)^\top) \\ &- g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha} e_\beta)^\top) + \sum_\gamma g((\nabla_{e_\alpha} e_\gamma)^\top, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\gamma} e_\beta)^\top)\Big],\end{aligned}$$

where $H_{D^\perp}$ denotes the mean curvature vector of the distribution $D^\perp$.

Put

$$\widehat{\sigma_u}(x) = \int_{P_x} \sigma_u(x, e)\, de = \int_G \sigma_u(x, e_0 a)\, da,$$

where (x, e_0) is a fixed element of P_x. We call $\widehat{\sigma_u}$'s *extrinsic curvatures* of a distribution D. Moreover, we define *total extrinsic curvatures*

$$\sigma_u^M = \int_M \widehat{\sigma_u}(x)\, dx.$$

Theorem 3.2 *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, the total extrinsic curvature σ_u^M satisfies*

$$\begin{aligned}|u|\sigma_u^M = \sum_{\alpha,\beta} \int_P \Big(&\mathrm{tr}\,(R_{\alpha,\beta} T_{\beta_\flat\alpha_\flat(u)}) + g(\mathrm{div}_{E'} T^*_{\beta_\flat\alpha_\flat(u)}, (\nabla_{e_\alpha} e_\beta)^\top) \\ &- g(H_{D^\perp}, T_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha} e_\beta)^\top) + \sum_\gamma g((T^*_{\beta_\flat\alpha_\flat(u)}(\nabla_{e_\alpha} e_\gamma)^\top, (\nabla_{e_\gamma} e_\beta)^\top)\Big),\end{aligned}$$

where $H_{D^\perp}$ denotes the mean curvature vector of distribution $D^\perp$.

By Theorem 3.2, we have in particular

$$\sigma^M_{\alpha^\sharp(0,\ldots,0)} = 0$$

and

$$2\sigma^M_{\alpha^\sharp\beta^\sharp(0,\ldots,0)} = \int_P \Big((\mathrm{Ric}_D)_{\alpha,\beta} - g(H_{D^\perp}, (\nabla_{e_\alpha} e_\beta)^\top) + \sum_\gamma g((\nabla_{e_\alpha} e_\gamma)^\top, (\nabla_{e_\gamma} e_\beta)^\top)\Big),$$

where $(\mathrm{Ric}_D)_{\alpha,\beta} = \mathrm{Ric}_D(e_\alpha, e_\beta)$ and Ric_D is the Ricci curvature operator in the direction of D, i.e.,

$$\mathrm{Ric}_D(X, Y) = \sum_i g(R(f_i, X)Y, f_i),$$

where (f_i) is an orthonormal basis of D.

If D is integrable i.e. D defines a foliation $\mathcal{F}$ then above theorems generalized some well known facts:

Corollary 3.3 *Assume M is closed. Then, for any $u \in \mathbb{N}(q)$, total extrinsic curvature σ_u^M of a distribution D with totally geodesic normal bundle is of the form*

$$|u|\sigma_u^M = \sum_{\alpha,\beta} \int_P \mathrm{tr}\,(R_{\alpha,\beta} T_{\beta_\flat\alpha_\flat(u)}).$$

Corollary 3.4 *Assume (M, g) is closed and of constant sectional curvature κ. Let $\mathcal{F}$ be a foliation on M with totally geodesic and integrable normal bundle $\mathcal{F}^\perp$. Then the total extrinsic curvatures of $\mathcal{F}$ depend on κ, the volume of M and the dimension of $\mathcal{F}$ only.*

Moreover we may also obtain formulae of Brito and Naveira for mean extrinsic curvature S_r

$$\int_M S_r = \begin{cases} \binom{\frac{p}{2}}{\frac{r}{2}}\binom{q+r-1}{r}\binom{\frac{q+r-1}{2}}{\frac{r}{2}}^{-1} \kappa^{\frac{r}{2}}\mathrm{vol}(M) & \text{for } p \text{ even and } q \text{ odd} \\ 2^r\left(\left(\frac{r}{2}\right)!\right)^{-1}\binom{\frac{q}{2}+\frac{r}{2}-1}{\frac{r}{2}}\binom{\frac{p}{2}}{\frac{r}{2}}\kappa^{\frac{r}{2}}\mathrm{vol}(M) & \text{for } p \text{ and } q \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

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