The mixed scalar curvature flow and harmonic foliations

Vladimir Rovenski (University of Haifa)

Introduction

1. Geometry of foliations. Let (M^{n+p}, g) be a connected closed (i.e., compact without a boundary) Riemannian manifold, endowed with a p-dimensional foliation $\mathcal{F}$, i.e., a partition into submanifolds (called leaves) of the same dimension p. We have the g-orthogonal decomposition $TM = \mathcal{D}_\mathcal{F} \oplus \mathcal{D}$, where the distribution $\mathcal{D}_\mathcal{F}$ is tangent to $\mathcal{F}$. Denote by $(\,\cdot\,)^\mathcal{F}$ and $(\,\cdot\,)^\perp$ projections onto $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}$, respectively.

The second fundamental tensor and the mean curvature vector field of $\mathcal{F}$ are defined by

$$h_\mathcal{F}(X,Y) = (\nabla_X Y)^\perp, \quad H_\mathcal{F} = \mathrm{Tr}_g\, h \quad (X,Y \in \mathcal{D}_\mathcal{F}).$$

Here ∇ is the *Levi-Civita connection* of g. Totally geodesic ($h_\mathcal{F} = 0$) and harmonic ($H_\mathcal{F} = 0$) foliations were investigated by many geometers. A foliation is

geometrically taut, if there is a metric making $\mathcal{F}$ harmonic. D. Sullivan provided a *topological tautness* condition for for geometric tautness.
The integrability and 2nd fundamental tensors of $\mathcal{D}$ are

$$h(X,Y) = (1/2)\,(\nabla_X Y + \nabla_Y X)^{\mathcal{F}},$$
$$T(X,Y) = (1/2)\,[X,\,Y]^{\mathcal{F}} \quad (X,Y \in \mathcal{D}).$$

The mean curvature vector of $\mathcal{D}$ is $H = \operatorname{Tr}_g h$.

The principal problem of geometry of foliations [1]: *Given a foliation $\mathcal{F}$ on M and a geometric property (P), does there exist a Riemannian metric g on M such that $\mathcal{F}$ enjoys (P) with respect to g?*

2. The mixed scalar curvature. The geometrical sense of the mixed curvature follows from the fact that certain components of the curvature tensor regulate the deviation of leaves along the leaf geodesics. Let $\{e_i,\,\varepsilon_\alpha\}_{i\le n,\,\alpha\le p}$ be a local orthonormal frame on TM adapted to $\mathcal{D}$ and $\mathcal{D}_{\mathcal{F}}$. The *mixed scalar curvature*, see [1, 3], is defined by

$$\operatorname{Sc}_{\,\rm mix} = \sum\nolimits_{i=1}^{n} \sum\nolimits_{\alpha=1}^{p} R(\varepsilon_\alpha, e_i, \varepsilon_\alpha, e_i).$$

If either $\mathcal{D}$ or $\mathcal{D}_{\mathcal{F}}$ is one-dimensional and tangent to a unit vector field N, then $\operatorname{Sc}_{\,\rm mix}$ is simply the Ricci curvature $\operatorname{Ric}(N,N)$. On a foliated surface (M^2, g) this coincides with the gaussian curvature: $\operatorname{Sc}_{\,\rm mix} =$

$\mathrm{Ric}(N, N) = K$. Recall the formula, see [3]:

$$\mathrm{Sc}_{\mathrm{mix}} = \mathrm{div}(H + H_{\mathcal{F}}) + |H|^2 + |H_{\mathcal{F}}|^2 + |T|^2 - |h|^2 - |h_{\mathcal{F}}|^2 \tag{1}$$

which yields decomposition criteria for foliations under constraints on the sign of $\mathrm{Sc}_{\mathrm{mix}}$.

The basic question that we address in the work is: *Which foliations admit a metric with a given property of* $\mathrm{Sc}_{\mathrm{mix}}$ *(e.g., positive or negative)*?

3. Flows of metrics on foliations. A *flow of metrics* on a manifold is a solution g_t of a differential equation $\partial_t g = S(g)$, where $S(g)$ is a symmetric $(0, 2)$-tensor related to some kind of curvature.

The notion of the $\mathcal{F}$*-truncated* $(r, 2)$-tensor field $\hat{S}$ (where $r = 0, 1$) is helpful: $\hat{S}(X_1, X_2) = S(X_1^{\perp}, X_2^{\perp})$. The $\mathcal{F}$*-truncated metric* $\hat{g}$ is an example.

For $\mathcal{D}$*-conformal* flows of metrics we have

$$\hat{S}(g) = s(g)\,\hat{g}, \quad \text{where} \quad s(g) \quad \text{is smooth.}$$

Author and Walczak [1] studied flows of metrics that depend on the extrinsic geometry of foliations, and posed the question:

Given a geometric property (P)*, can one find an* $\mathcal{F}$*-truncated flow* $\partial_t g = \hat{S}(g)$ *on a foliation* $(M, \mathcal{F})$ *such that the solution metrics* g_t $(t \geq 0)$ *converge to a metric for which* $\mathcal{F}$ *enjoys* (P)?

In aim to prescribe the sign of $\mathrm{Sc}_{\,\mathrm{mix}}$, we study the *mixed scalar curvature flow* of metrics g_t:

$$\partial_t g = -2\,(\mathrm{Sc}_{\,\mathrm{mix}}(g) - \Phi)\,\hat{g}. \tag{2}$$

Here $\Phi : M \to \mathbb{R}$ is leaf-wise constant. The flow (2) preserves harmonic and totally geodesic foliations.

Example 1. (a) For (M^2, g_0) of Gaussian curvature K, with a geodesic vector field N, (2) has the view

$$\partial_t g = -2\,(K(g) - \Phi)\,\hat{g}.$$

It "looks like" the normalized Ricci flow on surfaces. For $\Phi = \mathrm{const}$, we get $\partial_t k = K_{,x}$ and Burgers PDE

$$\partial_t k = k_{,xx} - (k^2)_{,x}\,.$$

(b) For Hopf fibrations $\pi : (S^{\,2m+1}, g_{\mathrm{can}}) \to \mathbb{C}P^m$ of a sphere with fiber S^1, the distribution $\mathcal{D}$ is non-integrable while $h = 0$. By (1), $\mathrm{Sc}_{\,\mathrm{mix}} = 2\,m$. Thus, g_{can} is a fixed point of (2) with $\Phi = 2m$.

We study the question:

Given (M, g) *with a harmonic foliation* $\mathcal{F}$*, when do solution metrics* g_t *of (2) converge to the limit metric* $\bar{g}$ *with* $\mathrm{Sc}_{\,\mathrm{mix}}(\bar{g})$ *positive or negative*?

Throughout the work everything (manifolds, foliations, etc.) is assumed to be smooth and oriented. We also assume that the leaves of $\mathcal{F}$ are compact.

1 Preliminaries

Let g_t be a smooth family of metrics on M and $S = \partial_t g$. Since the difference of two connections is a tensor, $\partial_t \nabla^t$ is a $(1,2)$-tensor on (M, g_t), it is given by

$$2\, g_t((\partial_t \nabla^t)(X,Y), Z) = \\ = (\nabla^t_X S)(Y,Z) + (\nabla^t_Y S)(X,Z) - (\nabla^t_Z S)(X,Y)$$

for t-independent vector fields $X, Y, Z \in \Gamma(TM)$. If $\mathcal{D}$-conformal tensor $S = s(g)\hat{g}$, for short, we write

$$\partial_t g = s\, \hat{g}\,. \tag{3}$$

Lemma 1. *For variations* (3) *we have*

$$\partial_t h_{\mathcal{F}} = -s\, h_{\mathcal{F}}\,, \qquad \partial_t H_{\mathcal{F}} = -s\, H_{\mathcal{F}}\,.$$

Thus, (3) *preserve harmonic (totally geodesic)* $\mathcal{F}$.

Let $A_N, T^\sharp_N$ be dual operators on $\mathcal{D}$ to tensors h, T.

Lemma 2. *For* (3) *we have (with* $N \in \mathcal{D}_{\mathcal{F}}$*)*

$$\partial_t A_N = -\frac{1}{2}\, N(s)\, \widehat{\rm id}\,, \;\; \partial_t T^\sharp_N = -s T^\sharp_N, \;\; \partial_t H = -\frac{n}{2} \nabla^{\mathcal{F}} s.$$

Based on the "linear algebra" inequality $n\,|h\,|^2 \ge |H|^2$ we define the measure of "non-umbilicity" of $\mathcal{D}$:

$$\boldsymbol{\beta} := n^{-2}\big(n\,|h|^2 - |H|^2\big) \ge 0.$$

For $p = 1$, we have $\boldsymbol{\beta} = n^{-2} \sum_{i<j} (k_i - k_j)^2$, where k_i are the principal curvatures of $\mathcal{D}$.

By Lemma 2, (3) preserves the property $\boldsymbol{\beta}=0$ and the domain $U=\{x\in M:\ |T|^2_{g_0}-|h_{\mathcal F}|^2_{g_0}\ne 0\}$. The function $\delta_t=|T|^2_{g_t}-|h_{\mathcal F}|^2_{g_t}$ is nonzero on U.

Proposition 1 (**Conservation laws**). *Let the metrics g_t $(t\ge 0)$ solve (3), and $H_0=-n\nabla^{\mathcal F}\log u_0$ for a function $u_0>0$ on M. Then $\boldsymbol{\beta}$ and the vector field $H_t-\frac{n}{4}\nabla^{\mathcal F}(\log|\delta_t|)$ on U are t-independent.*

Proof. Using Lemma 2 and $\hat g(H,\,\cdot\,)=0$, we find

$$\partial_t\,|h|^2=\partial_t\sum\nolimits_\alpha \operatorname{Tr}(A^2_{\varepsilon_\alpha})=-g(\nabla s,H),$$
$$\partial_t g(H,H)=-n\,g(\nabla s,H).$$

Hence, $n\,\partial_t\boldsymbol{\beta}=0$, that is $\boldsymbol{\beta}$ doesn't depend on t.

For any $f\in C^1(M)$ and $N\in\mathcal{D}_{\mathcal F}$, we find

$$g(\nabla^{\mathcal F}(\partial_t f),N)=\partial_t N(f)=g(\partial_t(\nabla^{\mathcal F}f),N).$$

Hence $\nabla^{\mathcal F}(\partial_t f)=\partial_t(\nabla^{\mathcal F}f)$. By Lemmas 1–2, we find

$$\partial_t\,|T|^2=-\partial_t\sum\nolimits_\alpha \operatorname{Tr}\big((T^\sharp_{\varepsilon_\alpha})^2\big)=-2\,s\,|T|^2.$$
$$\partial_t|h_{\mathcal F}|^2=-2\,s\,|h_{\mathcal F}|^2.$$

We have $\partial_t\log|\delta_t|=-2\,s$ on U. (If $T=0$ or $h_{\mathcal F}=0$ at $x\in M$ then $T(x)_{g_t}=0$ or $h_{\mathcal F}(x)_{g_t}=0$ for $t\ge 0$). Using $\nabla^{\mathcal F}\partial_t=\partial_t\nabla^{\mathcal F}$, we obtain on U

$$\partial_t H_t=(n/4)\,\partial_t\nabla^{\mathcal F}\log|\delta_t|.\quad\square$$

2 Main results

Next lemma allows us to reduce (2) to the leaf-wise PDE (with space derivatives along $\mathcal{F}$ only).

From (1), using $\operatorname{div} H = \operatorname{div}_{\mathcal{F}} H - |H|_g^2$, we obtain.

Lemma 3. *For a harmonic foliation* $\mathcal{F}$, (1) *reads*

$$\mathrm{Sc}_{\mathrm{mix}} = \operatorname{div}_{\mathcal{F}} H - \frac{1}{n}|H|^2 + |T|^2 - |h_{\mathcal{F}}|^2 - n\,\boldsymbol{\beta}\,.$$

Denote $\nabla^{\mathcal{F}} f := (\nabla f)^{\mathcal{F}}$. For a vector field X and a function u on M, define t-independent functions: $\operatorname{div}_{\mathcal{F}} X = \sum_\alpha g(\nabla_\alpha X, \varepsilon_\alpha)$ and $\Delta_{\mathcal{F}}\, u = \operatorname{div}_{\mathcal{F}} \nabla^{\mathcal{F}} u$. The leaf-wise Schrödinger operator $\mathcal{H}$ is given by

$$\mathcal{H}(u) = -\Delta_{\mathcal{F}}\, u - \boldsymbol{\beta}\, u. \tag{4}$$

The spectrum of $\mathcal{H}$ is an infinite sequence of isolated real eigenvalues $\lambda_0 \le \lambda_1 \le \dots \le \lambda_j \le \dots$ (leaf-wise constant), and $\lim_{j\to\infty} \lambda_j = \infty$. Fix an orthonormal basis of eigenfunctions $\{e_j\} \subset L_2$, i.e., $\mathcal{H}(e_j) = \lambda_j e_j$.

Remark 1. If the leaf $F(x)$ through $x \in M$ is compact then $\lambda_1 > \lambda_0$ and e_0 (the *ground state*) may be chosen positive. The *fundamental gap* $\lambda_1 - \lambda_0$ of $\mathcal{H}$ has many mathematical and physical implications.

Proposition 2 (Short-time existence/uniqueness)**.** *Let* $\mathcal{F}$ *be a harmonic foliation on a closed manifold* (M, g_0)*. Then the linearization of (2) is the* *leaf-wise parabolic PDE, hence (2) has a unique solution g_t defined on a positive time interval $[0, t_0)$ and smooth on the leaves.*

A smooth function $f(t,x)$ on $(0,\infty)\times F$ converges to $\bar f(x)$ as $t\to\infty$ in C^∞, if it converges in C^k-norm for any $k\ge 0$. It converges *exponentially fast* if $\exists\,\omega>0$ (the *exponential rate*) such that $\lim_{t\to\infty} e^{\omega t}|f(t,\cdot)-\bar f|_{C^k}=0$ for any $k\ge 0$.

Define a function $\Psi := u_0^4\,(|T|^2_{g_0}-|h_{\mathcal F}|^2_{g_0})$.

Proposition 3. *Let $\mathcal F$ be a harmonic foliation on (M,g_0) and metrics g_t $(0\le t<t_0)$ solve (2). Then the leaf-wise Burgers type equation holds*

$$\partial_t H+\nabla^{\mathcal F}|H|^2=n\nabla^{\mathcal F}\mathrm{div}_{\mathcal F}H+n\nabla^{\mathcal F}(|T|^2-|h_{\mathcal F}|^2-n\boldsymbol\beta) \tag{5}$$

If $H_0=-n\nabla^{\mathcal F}\log u_0$ for a function $u_0>0$ then $H_t=-n\nabla^{\mathcal F}\log u$ for $u: M\times[0,t_0)$, moreover,

(i) if $\Psi>0$ then $u=\Psi^{1/4}\big(|T|^2_{g_t}-|h_{\mathcal F}|^2_{g_t}\big)^{-1/4}$, and the non-linear PDE is satisfied

$$\frac1n\,\partial_t u=\Delta_{\mathcal F}\,u+(\boldsymbol\beta+\frac{\Phi}{n})u-\frac{\Psi}{n}\,u^{-3},\quad u(\cdot,0)=u_0 \tag{6}$$

(ii) if $\Psi\equiv 0$ then H obeys a forced leaf-wise Burgers equation (a consequence of (5)), and u (i.e., the potential for H) may be chosen as a solution of

$$(1/n)\,\partial_t u=\Delta_{\mathcal F}\,u+\boldsymbol\beta\,u,\quad u(\,\cdot\,,0)=u_0.$$

Based on Proposition 3(ii), we obtain the following.

Theorem 1. *Let $\mathcal{F}$ be a totally geodesic compact foliation with integrable orthogonal distribution on (M, g_0), and $H_0 = -n\nabla^{\mathcal{F}} \log u_0$ for $u_0 > 0$. Then (2) has a unique global solution g_t $(t \geq 0)$ smooth on the leaves. If $\Phi = n\,\lambda_0$ then, as $t \to \infty$, the metrics g_t converge in C^∞ with the exponential rate $n(\lambda_1 - \lambda_0)$ to the limit metric $\bar{g}$ and*

$$\mathrm{Sc}_{\mathrm{mix}}(\bar{g}) = n\lambda_0, \qquad \bar{H} = -n\nabla^{\mathcal{F}} \log e_0.$$

Moreover, if the leaves have finite holonomy group, then all g_t and $\bar{g}$ are smooth on M.

The central result of the work is the following.

Theorem 2. *Let $\mathcal{F}$ be a harmonic compact foliation on (M, g_0) with $|T|^2_{g_0} - |h_{\mathcal{F}}|^2_{g_0} \neq 0$. Assume $H_0 = -n\nabla^{\mathcal{F}}(\log u_0)$ for a function $u_0 > 0$, and*

$$\Phi > -n\,\boldsymbol{\beta}, \qquad |n\lambda_0 + \Phi| \geq \max_F \Big||T|^2_{g_0} - |h_{\mathcal{F}}|^2_{g_0}\Big| \Big(\max_F(\frac{u_0}{e_0}) \,/\, \min_F(\frac{u_0}{e_0})\Big)^4. \quad (7)$$

Then (2) has a unique global solution $g_t = g_t^\perp + \hat{g}$ $(t \geq 0)$ smooth on any leaf F, and if leaves have finite holonomy, then g_t are smooth on M,

$$\mathrm{Sc}_{\mathrm{mix}} \overset{t\to\infty}{\to} n\lambda_0 + \Phi, \;\; H \overset{t\to\infty}{\to} -n\nabla^{\mathcal{F}} \log e_0, \;\; h_{\mathcal{F}} \overset{t\to\infty}{\to} 0$$

and convergence in C^∞ *with the exponential rate* $n\,\alpha$ *for any* $\alpha\in(0,\,\min\{\lambda_1-\lambda_0,\,4\,|\lambda_0+\Phi/n|\})$*.*

The fibers of a fibre bundle has trivial holonomy. Let u obeys (6) on a leaf F. Define $\tilde u_0=u^0_0+\int_0^\infty q_0(\tau)\,{\rm d}\tau$, where $u^0_0=\int_F u_0\,e_0\,{\rm dx}$ and $q_0(\tau)=-e^{(\lambda_0+\Phi/n)\tau}\int_F(\Psi/n)\,u^{-3}e_0\,{\rm dx}$.

Corollary 1. *Let* $\mathcal F$ *be a harmonic compact foliation on* (M,g) *with* $|T|^2-|h_{\mathcal F}|^2\ne 0$*. Suppose that* $H=-n\nabla^{\mathcal F}\log u_0$ *for a function* $u_0>0$*.*

(i) *If* $\lambda_0<0$ *then there exists* $\mathcal D$*-conformal to* g *metric* $\bar g$ *with* ${\rm Sc}_{\rm mix}(\bar g)<0$*.*

(ii) *If* $\lambda_0>-\frac1n(\frac{u_0}{\tilde u_0e_0})^4\Big||T|^2-|h_{\mathcal F}|^2\Big|$ *then there exists* $\mathcal D$*-conformal to* g *metric* $\bar g$ *with* ${\rm Sc}_{\rm mix}(\bar g)>0$*.*

3 Applications to warped products

Consider $M=[0,l]\times\bar M^n$ with the warped product metric $g={\rm dx}^2+\varphi^2(x)\bar g$. The fibers $\{x\}\times\bar M$ are totally umbilical with a unit normal $N=\partial_x$. We have

$$K(N,X)=-\varphi_{,xx}/\varphi\quad\text{for } X\in\mathcal D\quad(\text{when}\ \ \varphi\ne 0).$$

Now, let a family of warped product metrics $g_t={\rm dx}^2+\varphi^2(t,x)\bar g$ solves (2) on M. This yields

$$\begin{aligned}\partial_t\varphi=n\,\varphi_{,xx}+\Phi\,\varphi,\quad \varphi(0,\cdot)=\varphi_0,\\ \varphi(t,0)=\mu_0(t),\quad \varphi(t,l)=\mu_1(t)\,,\end{aligned}\qquad(8)$$

where $\mu_j(t) \geq 0,\ j = 0, 1$. The Cauchy's problem (8) has a unique classical solution φ for all $t \geq 0$.

We study convergence of a solution to a stationary state $\tilde{\varphi}$ which solves the Cauchy's problem

$$n\,\tilde{\varphi}_{,xx} + \Phi\,\tilde{\varphi} = 0, \quad \tilde{\varphi}(0) = \tilde{\mu}_0, \quad \tilde{\varphi}(l) = \tilde{\mu}_1. \quad (9)$$

For $\Phi < n(\frac{\pi}{l})^2$, its solution exists and does not depend on φ_0. For $\Phi = n(\frac{\pi}{l})^2$, (9) is solvable when $\tilde{\mu}_1 = -\tilde{\mu}_0$, in this case, $\tilde{\varphi} = C \sin(\frac{\pi}{l}\,x) + \tilde{\mu}_0 \cos(\frac{\pi}{l}\,x)$ for $C > 0$.

Assume that $\mu_j(t)$ are continuously differentiable on $[0, \infty)$, and there exist $\lim_{t\to\infty} \mu_j(t) = \tilde{\mu}_j$ and $\lim_{t\to\infty} \mu'_j(t) = 0$. Denote $\delta_j(t) := \mu_j(t) - \tilde{\mu}_j$.

Theorem 3. *Let the warped product metrics g_t on $[0, l] \times \bar{M}^n$ solve (2). If $\Phi > n(\pi/l)^2$ then g_t diverge as $t \to \infty$, otherwise g_t converge uniformly for $[0, l]$ to the limit metric $g_\infty = \mathrm{dx}^2 + \varphi^{\,2}_{\infty}(x)\bar{g}$ such that $K_{g_\infty}(N, X) = \Phi$ for $X \in \mathcal{D}$. Moreover,*

(i) if $\Phi < n(\pi/l)^2$ then φ_{∞} solves (9).

(ii) if $\Phi = n(\pi/l)^2$ and conditions on $[0, l]$ hold

$$\tilde{\mu}_j = 0, \quad \int_0^\infty |\delta_j(\tau)|\,\mathrm{d}\tau < \infty, \quad \int_0^\infty |\delta'_j(\tau)|\,\mathrm{d}\tau < \infty$$

then $\varphi_\infty = (v^0_1 + \int_0^\infty f_1(\tau)\,\mathrm{d}\tau)\,e_1$.

4 Proofs

Proof of Proposition 3. By Proposition 2, (2) admits a unit local smooth solution g_t $(0\le t<t_0)$. The functions $\mathrm{Sc}_{\,\rm mix}, H$, $|T|$ and $|h_{\mathcal F}|$ etc. are then uniquely determined for $0\le t<t_0$. From Lemma 2 with $s=-2\,(\mathrm{Sc}_{\,\rm mix}-\Phi)$ and Lemma 3 we obtain (5).

(**i**) By Proposition 1(ii), $H_t-(n/4)\nabla^{\mathcal F}\!\log|\delta_t|=X$ for a vector field X on M. Since H_0 is conservative, $X=-\frac{n}{4}\,\nabla^{\mathcal F}\!\log\psi$ for a function $\psi>0$ on M. Hence,

$$H_t=-n\nabla^{\mathcal F}\log\,(\psi^{1/4}\delta_t^{-1/4})$$

and one may take $\psi=\Psi:=u_0^4\,\delta_0$ by conditions. Define $u:=\Psi^{1/4}\delta_t^{-1/4}$ on $U\times[0,t_0)$ and calculate

$$\partial_t\log|\delta_t|=-4\,\partial_t\log(\delta_t^{-1/4})=-4\,\partial_t\log u.$$

From the above and Lemma 3 we then obtain

$$\partial_t\log u=-\frac{1}{4}\,\partial_t\log|\delta_t|=\frac{s}{2}=-\mathrm{Sc}_{\rm mix}(g_t)+\Phi$$
$$=n\Delta_{\mathcal F}\log u+n\,|\nabla^{\mathcal F}\!\log u|^2+n\,\boldsymbol\beta+\Phi-\Psi\,u^{-4}.$$

Substituting

$$\partial_t\log u=u^{-1}\partial_t u,\quad \nabla^{\mathcal F}\log u=u^{-1}\nabla^{\mathcal F}u,$$
$$\Delta_{\mathcal F}\,\log u=u^{-1}\Delta_{\mathcal F}\,u-u^{-2}g(\nabla^{\mathcal F}u,\nabla^{\mathcal F}u),$$

we find that u solves (6). (**ii**) See the proof in [2]. □

Proof of Theorem 2. (**i**) By Proposition 2, there exists a unique local solution g_t on $M \times [0, t_0)$. By Proposition 3(ii), H obeys (5), and $H = -n\nabla^{\mathcal{F}}(\log u)$ for a positive function u satisfying (6) with $u(\cdot, 0) = u_0$. Note that conditions (7) yield

$$|n\lambda_0 + \Phi| \geq \max_F \left\{\delta_0(u_0/e_0)^4\right\}/\min_F(u_0/e_0)^4$$

that is (11) with $\boldsymbol{\beta} = \boldsymbol{\beta} + \Phi/n > 0$.

By Theorem 4, one may leaf-wise smoothly extend a solution of (6) on $M \times [0, \infty)$, hence $H_t(x)$ is defined for all $t \geq 0$ and is smooth on the leaves.

By Theorem 5(i), $u(\,\cdot\,, t) \to \infty$ as $t \to \infty$ with the exponential rate $n\alpha$ for any $\alpha \in (0, \min\{\lambda_1 - \lambda_0,\, 4\,|\lambda_0 + \Phi/n|\})$. Hence, the function $\delta_t^2 = \Psi\, u^{-4}$ is leaf-wise smooth, and $|T|_{g_t} \to 0$ and $|h_{\mathcal{F}}|_{g_t} \to 0$ as $t \to \infty$. By Theorem 5(ii), $H_t = -n\nabla^{\mathcal{F}}\log u$ approaches in C^∞, as $t \to \infty$, to the vector field $\bar{H} = -n\nabla^{\mathcal{F}}\log e_0$ and $\operatorname{div} H_t$ approaches to the smooth function $-n\Delta_{\mathcal{F}}\log e_0$. Recall that

$$-\Delta_{\mathcal{F}}\, e_0 - (\boldsymbol{\beta} + \Phi/n)e_0 = (\lambda_0 + \Phi/n)e_0.$$

Thus, $\operatorname{div} H_t - g(H_t, H_t)/n \to n\lambda_0 + n\,\boldsymbol{\beta}$ as $t \to \infty$. By Lemma 3, we find a smooth on leaves function $\mathrm{Sc}_{\,\mathrm{mix}}(x, t)$ which approaches exponentially to $n\lambda_0 + \Phi$ as $t \to \infty$. The leaf-wise smooth solution to (2) is $g_t = g_0 \exp(-2\int_0^t (\mathrm{Sc}_{\mathrm{mix}}(x, \tau) - \Phi)\, \mathrm{d}\tau)$ $(t \geq 0)$.

(**ii**) The smoothness of g_t on M follows from the finite holonomy assumption. $\square$

Proof of Corollary 1. Metrics g_t of Theorem 2 diverge as $t\to\infty$ with the exponential rate $\mu=n\lambda_0+\Phi$. Consider $\mathcal{D}$-conformal metrics $\bar g_t=e^{-\mu\,t}g_t^\perp+\hat g$. By Lemma 2, $\bar H_t=H_t$. Then $v=e^{\mu\,t}u$ converges as $t\to\infty$ to $\tilde u_0 e_0$, where $\tilde u_0=(\tilde u,e_0)_0=u_0^0+\int_0^\infty q_0(\tau)\,{\rm d}\tau$, see Theorem 5.

Denote $\bar\delta_t:=|T|^2_{\bar g_t}-|h_{\mathcal F}|^2_{\bar g_t}$. Then $v^4(\cdot,t)\bar\delta_t=\Psi$ for all t. Hence, $\bar\delta_t=e^{-2\mu\,t}\delta_t$, it converges as $t\to\infty$ to $u_0^4/(\tilde u_0 e_0)^4\delta_0$, and $\bar g_t$ converge to the metric $\bar g_\infty=(\tilde u_0 e_0/u_0)^2 g_0^\perp+\hat g$. By Lemma 3, ${\rm Sc}_{\rm mix}(\bar g_t)={\rm Sc}_{\rm mix}(g_t)-\delta_t+\bar\delta_t$ converges to $n\lambda_0+u_0^4/(\tilde u_0 e_0)^4\delta_0$. $\square$

5 The non-linear Schrödinger heat equation

The section is important for proofs of main results.

Let (F^p,g) be a closed manifold (e.g., a leaf of $\mathcal F$). Spaces over F will be denoted without writing (F).

For a function $\boldsymbol\beta$ on F, the Schrödinger operator (4) is defined in H^2, it is self-adjoint and bounded from below. Any $u\in L_2$ is expanded into series

$$u(x)=\sum\nolimits_j c_j\,e_j(x),\ \ c_j=(u,e_j)_0=\int_F u(x)\,e_j(x)\,{\rm d}x\,.$$

Proposition 4. *The eigenspace of $\mathcal{H}$, corresponding to the least eigenvalue, λ_0, is one-dimensional, and it contains a positive smooth eigenfunction, e_0.*

After scaling in time by n and replacing $\Psi \to n\Psi$ and $\boldsymbol{\beta}+\Phi/n\to\boldsymbol{\beta}$, (6) reads as the Cauchy's problem:

$$\partial_t u = -\mathcal{H}(u) - \Psi(x)\, u^{-3}, \quad u(x,0) = u_0(x), \quad (10)$$

where $\boldsymbol{\beta}(x) > 0$, and $\Psi(x) \geq 0$ for any $x \in F$. It has a unique smooth solution u for $t \in [0, t_0)$.

Let $\boldsymbol{\beta}(x) \geq \boldsymbol{\beta}^- > 0$, and we get $\lambda_0 \leq -\boldsymbol{\beta}^- < 0$.

Denote $\mathcal{C}_\infty = F \times (0,\infty)$ and define the quantities

$$\Psi^+ = \max_{x\in F} \Psi(x)/e_0^4(x), \quad u_0^- = \min_{x\in F} u_0(x)/e_0(x).$$

Theorem 4 (Long-time existence)**.** *The Cauchy's problem (10) with the condition, see* (7)*,*

$$(u_0^-)^4 \geq \Psi^+/|\lambda_0| \qquad (11)$$

has a unique smooth solution $u > 0$ in $\mathcal{C}_\infty$.

Theorem 5 (Asymptotic behavior)**.** *Let $u > 0$ be a smooth solution on $\mathcal{C}_\infty$ of (10), and* (11) *is satisfied. Then there exists solution $\tilde{u} > 0$ on $\mathcal{C}_\infty$ of*

$$\partial_t \tilde{u} = \Delta\tilde{u} + (\boldsymbol{\beta}(x) + \lambda_0)\tilde{u} \qquad (12)$$

such that $\forall\alpha\in\big(0, \min\{\lambda_1 - \lambda_0,\, 4|\lambda_0|\}\big)$ and $k \in \mathbb{N}$:

1. $u=e^{-\lambda_0 t}(\tilde{u}+\theta(\cdot,t))$, $|\theta(\cdot,t)|_{C^k}=O(e^{-\alpha t})$ *as* $t\to\infty$

2. $\nabla\log u = \nabla\log e_0+\theta_1(\cdot,t)$, $\|\theta_1(\cdot,t)\|_{C^k} = O(e^{-\alpha t})$ *as* $t \to \infty$.

References

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E-mail: rovenski @ math.haifa.ac.il