Generalizations of a theorem of Herman and a new proof of the simplicity of $\mathrm{Diff}^\infty_c(M)_0$

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Let M be a smooth manifold of dimension n. By $\mathrm{Diff}^\infty_c(M)$ we will denote the group of compactly supported diffeomorphisms of M. We shall consider a Lie group structure on $\mathrm{Diff}^\infty_c(M)$ in the sense of the convenient setting of Kriegl and Michor [10]. In particular, we assume that $\mathrm{Diff}^\infty_c(M)$ is endowed with the c^∞-topology [10, Section 4], i.e. the final topology with respect to all smooth curves. For compact M the c^∞-topology on $\mathrm{Diff}^\infty(M)$ coincides with the Whitney C^∞-topology, cf. [10, Theorem 4.11(1)]. In general the c^∞-topology on $\mathrm{Diff}^\infty_c(M)$ is strictly finer than the one induced from the Whitney C^∞-topology, cf. [10, Section 4.26]. The latter coincides with the inductive limit topology $\lim_K \mathrm{Diff}^\infty_K(M)$ where K runs through all compact subsets of M.

Given smooth complete vector fields $X_1, \dots, X_N$ on M, we consider the map

$$K\colon \mathrm{Diff}^\infty_c(M)^N \to \mathrm{Diff}^\infty_c(M), \qquad (1)$$
$$K(g_1, \dots, g_N) := [g_1, \exp(X_1)] \circ \cdots \circ [g_N, \exp(X_N)].$$

Here $\exp(X)$ denotes the flow of a complete vector field X at time 1, and $[k,h] := k \circ h \circ k^{-1} \circ h^{-1}$ denotes the commutator of two diffeomorphisms k and h. It is readily checked that K is smooth. Indeed, one only has to observe that K maps smooth curves to smooth curves, cf. [10, Section 27.2]. Clearly $K(\mathrm{id}, \dots, \mathrm{id}) = \mathrm{id}$.

A *smooth local right inverse at the identity for* K consists of an open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^\infty_c(M)$ together with a smooth map

$$\sigma = (\sigma_1, \dots, \sigma_N)\colon \mathcal{U} \to \mathrm{Diff}^\infty_c(M)^N$$

so that $\sigma(\mathrm{id}) = (\mathrm{id}, \dots, \mathrm{id})$ and $K \circ \sigma = \mathrm{id}_{\mathcal{U}}$. More explicitly, we require that each $\sigma_i\colon \mathcal{U} \to \mathrm{Diff}^\infty_c(M)$ is smooth with $\sigma_i(\mathrm{id}) = \mathrm{id}$ and, for all $g \in \mathcal{U}$,

$$g = [\sigma_1(g), \exp(X_1)] \circ \cdots \circ [\sigma_N(g), \exp(X_N)].$$

We presents two results which generalize a well-known theorem of Herman for M being the torus [8, 9].

Theorem 1

Suppose M is a smooth manifold of dimension $n \geq 2$. Then there exist four smooth complete vector fields $X_1, \dots, X_4$ on M so that the map K, see (1), admits a smooth local right inverse at the identity, $N = 4$. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the strong Whitney C^0-topology. If M admits a proper (circle valued) Morse function whose critical points all have index 0 or n, then the same statement remains true with three vector fields.

Particularly, on the manifolds $M = \mathbb{R}^n, S^n, T^n$, $n \geq 2$, or the total space of a compact smooth fiber bundle $M \to S^1$, three commutators are sufficient. At the expense of more commutators, it is possible to gain further control on the vector fields. More precisely, we have:

Theorem 2

Suppose M is a smooth manifold of dimension $n \geq 2$ and set $N := 6(n+1)$. Then there exist smooth complete vector fields $X_1, \dots, X_N$ on M so that the map K, see (1), admits a smooth local right inverse at the identity. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the strong Whitney C^∞-topology.

Either of the two theorems implies that $\mathrm{Diff}^\infty_c(M)_o$, the connected component of the identity, is a perfect group, provided M is not $\mathbb{R}$. Our proof rests on Herman's result similarly as that of [17] (see [2]), but is otherwise elementary and different from Thurston's approach. In fact we only need Herman's result in dimension 1.

The perfectness of $\mathrm{Diff}^\infty_c(M)_0$ was already proved by Epstein [5] using ideas of Mather [11, 12] who dealt with the C^r-case, $1 \leq r < \infty$, $r \neq n+1$. The Epstein–Mather proof is based on a sophisticated construction, and uses the Schauder–Tychonov fixed point theorem. The existence of a presentation

$$g = [h_1, k_1] \circ \cdots \circ [h_N, k_N]$$

is guarantied, but without any further control on the factors h_i and k_i. Theorem 1 or 2 actually implies that the universal covering of $\mathrm{Diff}^\infty_c(M)_o$ is a perfect group. This result is known, too, see [17]. Thurston's proof is based on a result of Herman for the torus [8, 9]. Note that the perfectness of $\mathrm{Diff}^\infty_c(M)_o$ implies that this group is simple, see Epstein [4]. The methods used in [4] are elementary and actually work for a rather large class of homeomorphism groups.

One could believe that the phenomenon of smooth perfectness described in Theorems 1 and 2 would be also true for some classical diffeomorphism groups which are simple, e.g. for the Hamiltonian diffeomorphism group of a closed symplectic manifold [1], or for the contactomorphism group of an arbitrary co-oriented contact manifold [15]. However, the available methods seem to be useless for possible proofs of their smooth perfectness. Another open problem related to the above theorems is whether a smooth *global* right inverse at the identity for K would exist. A possible answer in the affirmative seems to be equally difficult. Consequently, it would be difficult to improve Theorems 1 and 2 as they are in any possible direction.

Another essential and important way to generalize the simplicity theorems for $\mathrm{Diff}^r_c(M)_o$, where $1 \leq r \leq \infty$, $r \neq n+1$, is to consider the uniform perfectness or, more generally, the boundedness of the groups in question. In particular, we ask if the presentation $g = [h_1, k_1] \circ \cdots \circ [h_N, k_N]$ is available for all $g \in \mathrm{Diff}^\infty_c(M)_o$ with N bounded. This property has been proved in the recent papers by Burago, Ivanov and Polterovich [3], and Tsuboi [18], [19], [20], for a large class of manifolds. For instance, $N = 10$ was obtained in [3] for any closed three dimensional manifold, and then it was improved in [18] to $N = 6$ for any closed odd dimensional manifold. It seems that the methods of [3], [18], [19] and [20] combined with our Theorem 2 would give some analogue of Theorem 1, but certainly not with the presentation (1) and the condition on X_i. Also N could not be smaller in this way. Another advantage of Theorem 1 is that it is valid for all smooth paracompact manifolds. See also [16] for diffeomorphism groups with no restriction of support.

Let $T^n := \mathbb{R}^n/\mathbb{Z}^n$ denote the torus. For $\lambda \in T^n$ we let $R_\lambda \in \mathrm{Diff}^\infty(T^n)$ denote the corresponding rotation. The main ingredient in the proof of Theorems 1 and 2 is the following result of Herman [9, 8].

Theorem 3 (Herman)

There exist $\gamma \in T^n$ so that the smooth map

$$T^n \times \mathrm{Diff}^\infty(T^n) \to \mathrm{Diff}^\infty(T^n), \qquad (\lambda, g) \mapsto R_\lambda \circ [g, R_\gamma],$$

admits a smooth local right inverse at the identity. Moreover, γ may be chosen arbitrarily close to the identity in T^n.

Herman's result is an application of the Nash–Moser inverse function theorem. When inverting the derivative one is quickly led to solve the linear equation $Y = X - (R_\gamma)^* X$ for given $Y \in C^\infty(T^n, \mathbb{R}^n)$. This is accomplished using Fourier transformation. Here one has to choose γ sufficiently irrational so that tame estimates on the Sobolev norms of X in terms of the Sobolev norms of Y can be obtained. The corresponding small denominator problem can be solved due to a number theoretic result of Khintchine.

We shall make use of the following corollary of Herman's result.

Proposition 4

There exist smooth vector fields X_1, X_2, X_3 on T^n so that the smooth map $\mathrm{Diff}^\infty(T^n)^3 \to \mathrm{Diff}^\infty(T^n)$,

$$(g_1, g_2, g_3) \mapsto [g_1, \exp(X_1)] \circ [g_2, \exp(X_2)] \circ [g_3, \exp(X_3)],$$

admits a smooth local right inverse at the identity. Moreover, the vector fields X_i may be chosen arbitrarily close to zero with respect to the Whitney C^∞-topology.

The following lemma leads to a decomposition of a diffeomorphism into factors which are leaf preserving. If $\mathcal{F}$ is a smooth foliation of M we let $\mathrm{Diff}^\infty_c(M; \mathcal{F})$ denote the group of compactly supported diffeomorphisms preserving the leaves of $\mathcal{F}$. This is a regular Lie group modelled on the convenient vector space of compactly supported smooth vector fields tangential to $\mathcal{F}$.

Lemma 5

Suppose M_1 and M_2 are two finite dimensional smooth manifolds and set $M := M_1 \times M_2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the foliations with leaves $M_1 \times \{pt\}$ and $\{pt\} \times M_2$ on M, respectively. Then the smooth map

$$F\colon \mathrm{Diff}^\infty_c(M; \mathcal{F}_1) \times \mathrm{Diff}^\infty_c(M; \mathcal{F}_2) \to \mathrm{Diff}^\infty_c(M), \quad F(g_1, g_2) := g_1 \circ g_2,$$

is a local diffeomorphism at the identity.

Now we need a version of the exponential law.

Lemma 6

Suppose B and T are finite dimensional smooth manifolds, assume T compact, and let $\mathcal{F}$ denote the foliation with leaves $\{pt\} \times T$ on $B \times T$. Then the canonical bijection

$$C^\infty_c(B, \mathrm{Diff}^\infty(T)) \xrightarrow{\cong} \mathrm{Diff}^\infty_c(B \times T; \mathcal{F})$$

is an isomorphism of regular Lie groups.

Another ingredient of the proof is a smooth fragmentation of diffeomorphisms.

Suppose $U \subseteq M$ is an open subset. Every compactly supported diffeomorphism of U can be regarded as a compactly supported diffeomorphism of M by extending it identically outside U. The resulting injective homomorphism $\mathrm{Diff}^\infty_c(U) \to \mathrm{Diff}^\infty_c(M)$ is clearly smooth. Note, however, that a curve in $\mathrm{Diff}^\infty_c(U)$, which is smooth when considered as a curve in $\mathrm{Diff}^\infty_c(M)$, need not be smooth as a curve into $\mathrm{Diff}^\infty_c(U)$. Nevertheless, if there exists a closed subset A of M with $A \subseteq U$ and if the curve has support contained in A, then one can conclude that the curve is also smooth in $\mathrm{Diff}^\infty_c(U)$.

Proposition 7 (Fragmentation)

Let M be a smooth manifold of dimension n, and suppose $U_1, \dots, U_k$ is an open covering of M, ie. $M = U_1 \cup \dots \cup U_k$. Then the smooth map

$$P\colon \mathrm{Diff}^\infty_c(U_1) \times \cdots \times \mathrm{Diff}^\infty_c(U_k) \to \mathrm{Diff}^\infty_c(M), \quad P(g_1, \dots, g_k) := g_1 \circ \cdots \circ g_k,$$

admits a smooth local right inverse at the identity.

Proceeding as in [3] permits to reduce the number of commutators considerably, see also [18] and [19].

Proposition 8

Let M be a smooth manifold of dimension $n \geq 2$ and put $N = 6(n+1)$. Moreover, let U an open subset of M and suppose $\phi \in \mathrm{Diff}^\infty(M)$, not necessarily with compact support, such that the closures of the subsets

$$U,\ \phi(U),\ \phi^2(U),\ \dots,\ \phi^N(U)$$

are mutually disjoint. Then there exists a smooth complete vector field X on M, a c^∞-open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^\infty_c(U)$, and smooth maps $\varrho_1, \varrho_2\colon \mathcal{U} \to \mathrm{Diff}^\infty_c(M)$ so that $\varrho_1(\mathrm{id}) = \varrho_2(\mathrm{id}) = \mathrm{id}$ and, for all $g \in \mathcal{U}$,

$$g = [\varrho_1(g), \phi] \circ [\varrho_2(g), \exp(X)].$$

Moreover, the vector field X may be chosen arbitrarily close to zero in the strong Whitney C^∞-topology on M.

Now, by applying the Morse theory ([13], [14]) we get

Lemma 9

Let M be a smooth manifold of dimension n. Then there exists an open covering $M = U_1 \cup U_2 \cup U_3$ and smooth complete vector fields X_1, X_2, X_3 on M so that $\exp(X_1)(U_1) \subseteq U_2$, $\exp(X_2)(U_2) \subseteq U_3$, and such that the closures of the sets

$$U_3,\quad \exp(X_3)(U_3),\quad \exp(X_3)^2(U_3),\ \dots$$

are mutually disjoint. Moreover, the vector fields X_1, X_2, X_3 may be chosen arbitrarily close to zero with respect to the strong Whitney C^0-topology. If M admits a proper (circle valued) Morse function whose critical points all have index 0 or n, then we may, moreover, choose $U_1 = \emptyset$ and $X_1 = 0$.

Theorem 1 is then a consequence of Lemma 9.

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