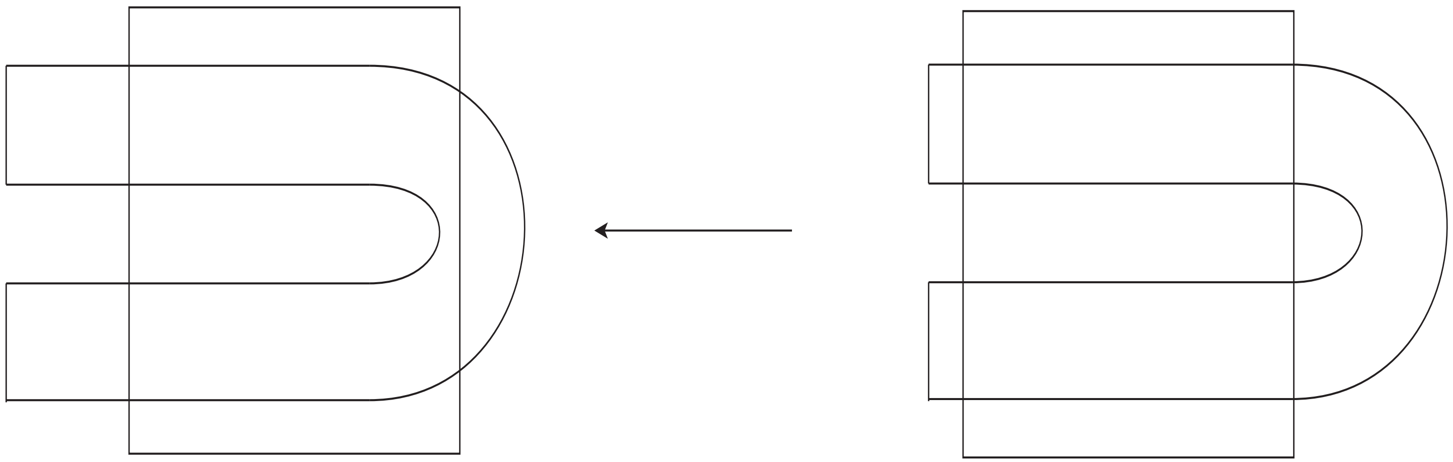
Prevalence of non-uniform hyperbolicity at the first bifurcation of Hénon-like families

Hiroki Takahasi
Department of Mathematics
Faculty of Science & Technology
Keio University
Japan

1. Destruction of Smale's horseshoe



- **What happens at the first bifurcation parameter?**
- **What happens in the sequel of the bifurcation?**

2. The Hénon family

$$f_a\colon (x,y)\in\mathbb{R}^2 \mapsto (1-ax^2+\sqrt{b}y, \pm\sqrt{b}x),$$

$$a\in\mathbb{R}, \quad 0<b\ll 1$$

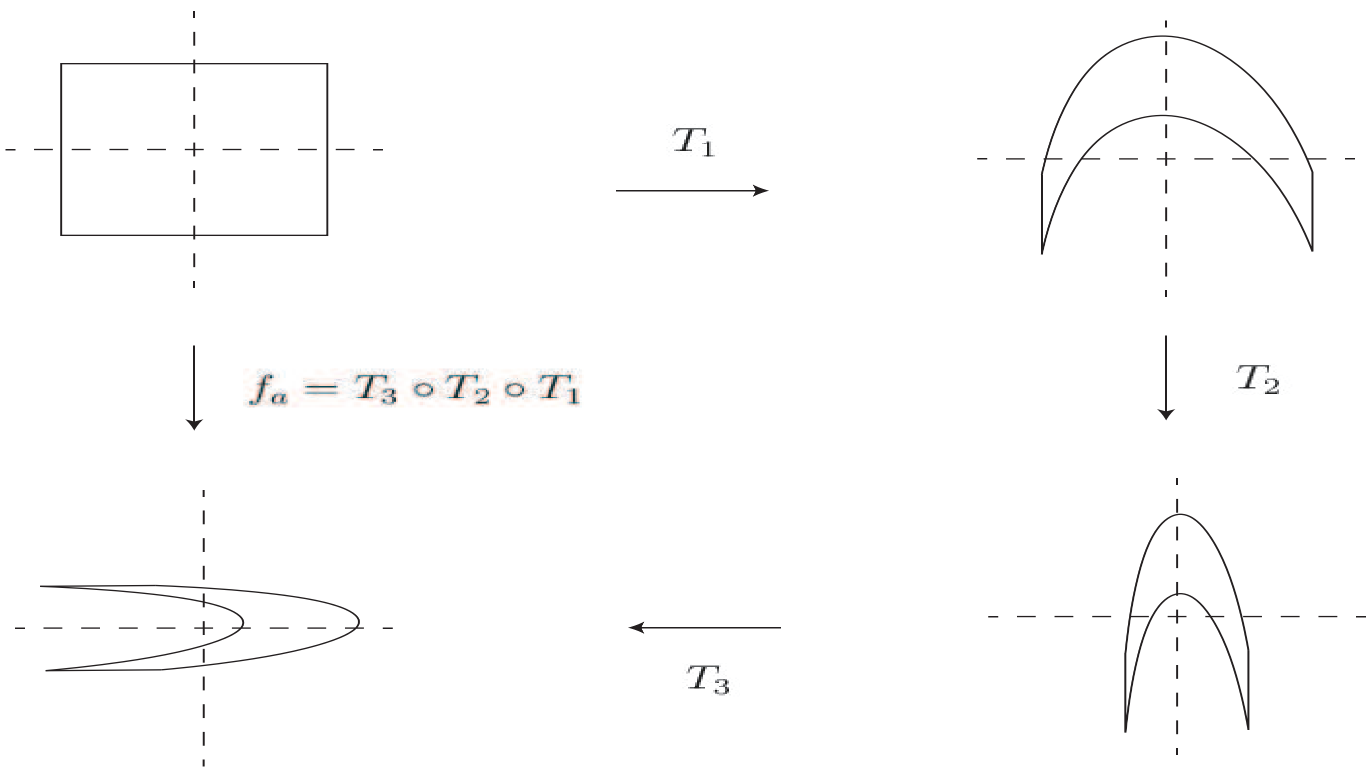
$$\det Df_a = \begin{pmatrix} -2ax & \sqrt{b} \\ \pm\sqrt{b} & 0 \end{pmatrix} = \mp b$$

$$T_1\colon (x,y)\mapsto (x, 1-ax^2+\sqrt{b}y)$$
$$T_2\colon (x,y)\mapsto (\mp\sqrt{b}x, y)$$
$$T_3\colon (x,y)\mapsto (y,-x)$$

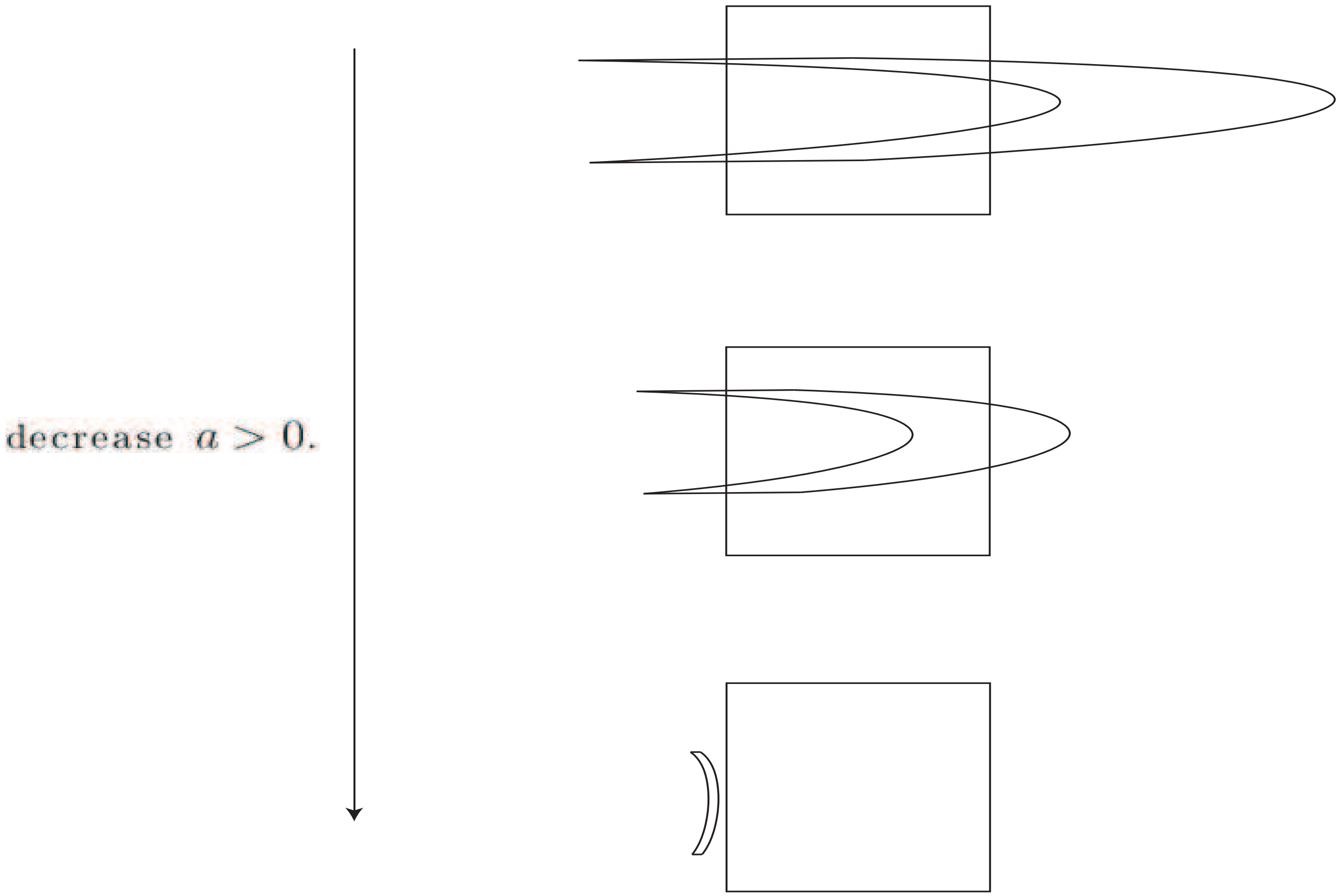
$$f_a = T_3\circ T_2\circ T_1$$

3. The dynamics

$$T_1 \colon (x, y) \mapsto (x, 1 - ax^2 + \sqrt{b}y)$$
$$T_2 \colon (x, y) \mapsto (\mp\sqrt{b}x, y)$$
$$T_3 \colon (x, y) \mapsto (y, -x)$$



4. Destruction of Smale's horseshoe



$\Omega_a := \{x \in \mathbb{R}^2 \colon \forall \text{open neighborhood } U \text{ of } x\ \exists n \in \mathbb{Z} \setminus \{0\} \text{ s.t. } U \cap f_a^n U \neq \emptyset\}.$

non wandering set of f_a (invariant, compact)

5. First bifurcation

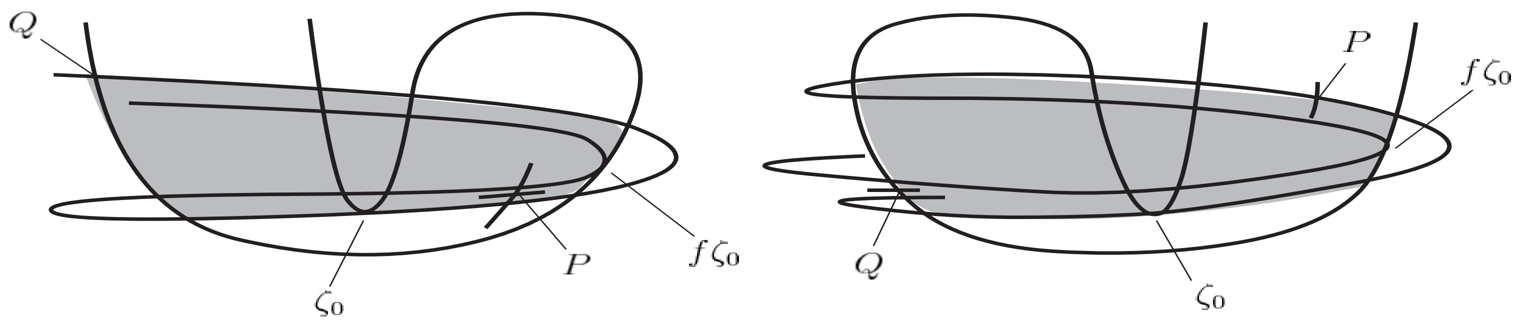
Theorem. (from Devaney & Nitecki 79, Bedford & Smillie 04) *For sufficiently small* $b > 0$ *there exists* $a^* = a^*(b) \in \mathbb{R}$ *near* 2 *such that:*

- *if* $a > a^*$*, then there exist* $C > 0$*,* $\xi \in (0, 1)$*, and for any* $x \in \Omega_a$ *there exists a non-trivial decomposition* $T_x\mathbb{R}^2 = E^s_x \oplus E^u_x$ *with* $D_x f(E^\sigma_x) = E^\sigma_{f_a x}$ $(\sigma = s, u)$ *such that for every* $n \geq 0$*,*

$$\|D_x f_a^n | E^s_x\| \leq C\xi^n \text{ and } \|D_x f_a^{-n} | E^u_x\| \leq C\xi^n.$$

- *there exists a point of quadratic tangency between stable and unstable manifolds of the fixed points of* f_{a^*}*.*

6. Manifold organization for f_{a^*}



There exist two fixed saddles $P \approx (1/2, 0)$, $Q \approx (-1, 0)$. In the orientation preserving case (left),$W^u(Q)$ meets $W^s(Q)$ tangentially. In the orientation reversing case (right), $W^u(P)$ meets $W^s(Q)$ tangentially. The non wandering sets are contained in the shaded regions.

stable manifold of P:

$$W^s(P) := \{x \in \mathbb{R}^2 \colon \lim_{n \to \infty} |f_a^n(x) - P| = 0\}$$

unstable manifold of P:

$$W^u(P) := \{x \in \mathbb{R}^2 \colon \lim_{n \to -\infty} |f_a^n(x) - P| = 0\}$$

7. Emergence of attractors for $a < a^*$

- $\exists$ a sequence $\{S_n\}_n$ of a-intervals at the left of a^* accumulating a^* as $n \to \infty$ s.t. parameters in $\bigcup_n S_n$ correspond to maps having sinks (Gavlirov & Silnikov 72, 73, Alligood & Yorke 83)
- $\exists$ a sequence $\{A_n\}_n$ of positive measure sets at the left of a^* accumulating a^* as $n \to \infty$, s.t. parameters in $\bigcup_n A_n$ correspond to maps admitting (a small scale of) non-uniformly hyperbolic strange attractors (Mora & Viana 93)

8. Prevalent dynamics at the first bifurcation parameter a^*

Theorem A. (Takahasi, [1]) *For sufficiently small* $b > 0$ *there exist* $\varepsilon_0 = \varepsilon_0(b) > 0$ *and a set* $\Delta \subset [a^* - \varepsilon_0, a^*]$ *of* a*-values containing* a^* *such that:*

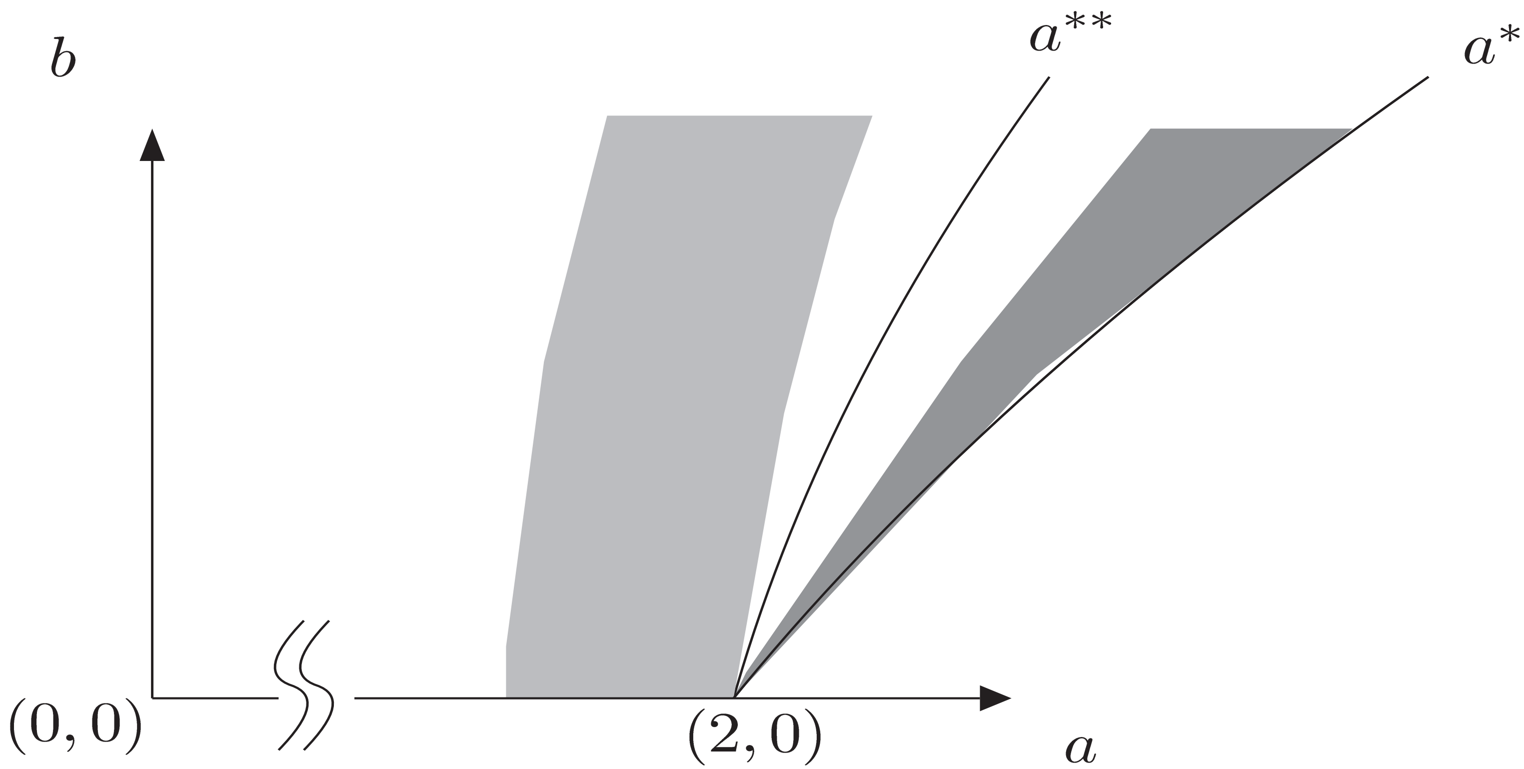
(a) $\lim_{\varepsilon \to +0} (1/\varepsilon)\mathrm{Leb}(\Delta \cap [a^* - \varepsilon, a^*]) = 1$;

(b) if $a \in \Delta$*, then the Lebesgue measure of the set* $\{x \in \mathbb{R}^2 \colon \{f_a^n x\}_{n \in \mathbb{N}}$ *is bounded*$\}$ *is zero. In particular, for Lebesgue a.e.* $x \in \mathbb{R}^2$, $|f_a^n x| \to \infty$ *as* $n \to \infty$*, and thus there is no attractor;*

(c) f_a *is transitive on* Ω_a.

9. Schematic picture in (a, b)-space

The parameter set Δ is contained in the dark region. Benedicks & Carleson's parameter set of positive Lebesgue measure corresponding to non-uniformly hyperbolic strange attractors are contained in the light region.



10. Question

Hyperbolicity of $f_a|\Omega_a$ for $a \in \Delta$?

Definition 1. *f_a is called uniformly hyperbolic if there exist numbers $C > 0$, $\xi \in (0,1)$ and for all $x \in \Omega_a$ a decomposition $T_x\mathbb{R}^2 = E^s_x \oplus E^u_x$ with the invariance property $D_x f_a(E^\sigma_x) = E^\sigma_{f_a x}$ $(\sigma = s, u)$ and satisfying*

$$\|D_x f_a^n | E^s_x\| \leq C\xi^n \quad \forall n \geq 0,$$

and

$$\|D_x f_a^{-n} | E^u_x\| \leq C\xi^n \quad \forall n \geq 0.$$

Remark 1. *If $a > a^*$, then f_a is uniformly hyperbolic.*

Next result: if $a \in \Delta$, then f_a displays a strong form of non-uniform hyperbolicity in terms of Lyapunov exponents of ergodic measures.

11. Regular points

Definition 2. *A point $x \in \Omega_a$ is called regular if there exist number(s) $\lambda_1 > \cdots > \lambda_{r(x)}$ and a decomposition $T_x\mathbb{R}^2 = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for any $v \in E_i(x) \setminus \{0\}$,*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \lambda_i(x) \quad \textit{and}$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det D_x f_a^n| = \sum_{i=1}^{r(x)} \lambda_i(x) \mathrm{dim} E_i(x).$$

- the set of regular points has total probability (from Oseledec's theorem)
- The functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\mathrm{dim}E_i(x)$ are invariant along orbits, and so are constant μ-a.e. if μ is ergodic.

12. Dichotomy on Lyapunov exponents of ergodic measures

For each ergodic μ one of the following holds:

(I) there exist two numbers $\chi^s(\mu) < \chi^u(\mu)$, and for μ-a.e. $x \in \Omega_a$ there exists a decomposition $T_x\mathbb{R}^2 = E^s_x \oplus E^u_x$ such that for any $v^\sigma \in E^\sigma_x \setminus \{0\}$ and $\sigma = s, u$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi^\sigma(\mu) \quad \text{and}$$

$$\int \log |\det Df_a| d\mu = \chi^s(\mu) + \chi^u(\mu);$$

(**II**) there exists $\chi(\mu) \in \mathbb{R}$ such that for μ-a.e. $x \in \Omega_a$ and for any $v \in T_x\mathbb{R}^2 \setminus \{0\}$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi(\mu) \quad \text{and}$$

$$\int \log |\det Df_a| d\mu = 2\chi(\mu).$$

Remark 2. *If $a \in \Delta$ then for each ergodic μ,*

- *(I) holds;*
- $\chi^s(\mu) < 0 \leq \chi^u(\mu)$.

13. Prevalence of non-uniform hyperbolicity

Definition 3. *An ergodic measure* μ *is called* *hyperbolic* *if* μ *has two Lyapunov exponents* $\chi^s(\mu)$, $\chi^u(\mu)$ *with* $\chi^s(\mu) < 0 < \chi^u(\mu)$.

Theorem B. (Takahasi, [2])
The following holds for all $a \in \Delta$:

(a) any f_a*-invariant ergodic measure is hyperbolic;*

(b) for each f_a*-invariant ergodic* μ,

$$\chi^s(\mu) < \frac{1}{3}\log b < 0 < \frac{1}{4}\log 2 < \chi^u(\mu). \tag{1}$$

Remark 3. *(1) does not necessarily imply the uniform hyperbolicity of* f_a*! (For instance,* $a^* \in \Delta$*.)*

14. Proof of Theorem B: dynamics outside of a critical region

$\delta > 0$ small constant, define a critical region

$$I(\delta) := (-\delta, \delta) \times (-b^{\frac{1}{4}}, b^{\frac{1}{4}}).$$

Lemma 1. *For any $\delta > 0$ there exists an open neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(2, 0)$ such that if $(a, b) \in \mathcal{U}$ then the following holds:*
if $n \geq 1$ and $x, f_a x, \ldots, f_a^{n-1} x \in \Omega_a \setminus I(\delta)$, then for any $v \in T_x\mathbb{R}^2 \setminus \{0\}$ with $\mathrm{slope}(v) \leq \sqrt{b}$, $\|D_x f_a^n v\| \geq \delta e^{\frac{99}{100} \log 2 \cdot n} \|v\|$ and
$\mathrm{slope}(D_x f_a^n v) \leq \sqrt{b}$.

Corollary. *If an ergodic μ has two Lyapunov exponents $\chi^s(\mu) < \chi^u(\mu)$ and $\mathrm{supp}(\mu) \cap I(\delta) = \emptyset$, then $\chi^u(\mu) \geq \frac{99}{100} \log 2$.*

15. Proof of Theorem B: dealing with returns to $I(\delta)$

Lemma 2. *Let $a \in \Delta$. Then for any $x \in \Omega_a$ one of the following holds:*

(a) *there exists $\nu_0 \geq 0$ such that for infinitely many $n \geq 0$,*

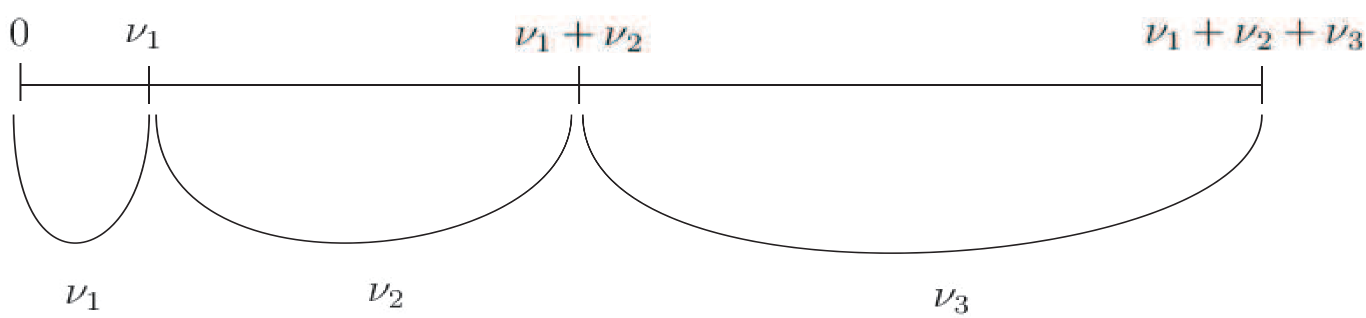
$$\|D_{f_a^{\nu_0+1}x} f_a^n \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq e^{\frac{\log 2}{4}\cdot n};$$

(b) *there exists a sequence $\{\nu_l\}_{l=0}^{\infty}$ of nonnegative integers such that;*

(b-i) *for every $l \geq 0$,*

$$\|D_{f_a^{\nu_0+\cdots+\nu_l+1}x} f_a^{\nu_{l+1}-1} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq e^{\frac{\log 2}{4}(\nu_{l+1}-1)};$$

(b-ii) $\nu_1 > 0$, *and $\nu_{l+1} \geq 2\nu_l$ for every $l \geq 1$.*



16. References

; 29MJ88 %

[1] Takahasi, H.: Prevalent dynamics at the first bifurcation of Hénon-like families. Commun. Math. Phys. **312**, 37-85 (2012)

[2] Takahasi, H.: Prevalence of non-uniform hyperbolicity at the first bifurcation of Hénon-like families. Available at http://arxiv.org/abs/1308.4199