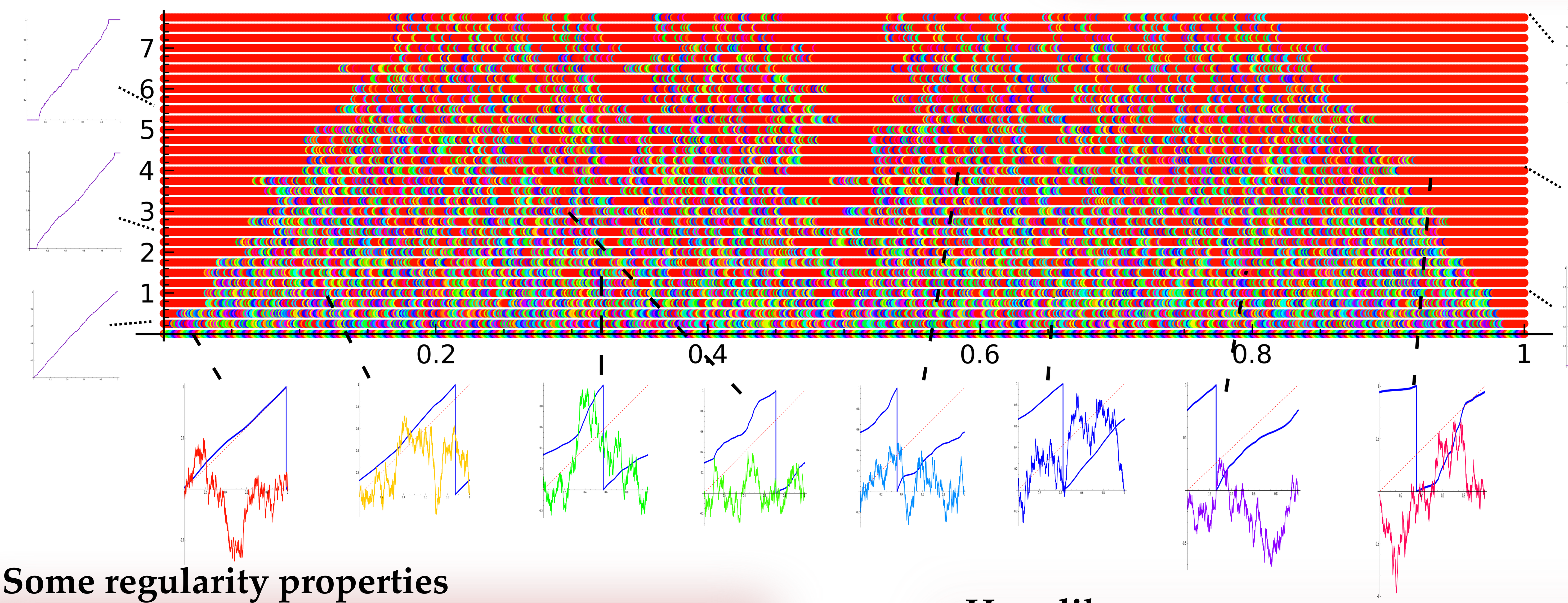
Random circle diffeomorphisms

Michele Triestino (UMPA-ÉNS Lyon)

$$f_\sigma(t) = \frac{\int_0^t e^{\sigma B_s}\, ds}{\int_0^1 e^{\sigma B_s}\, ds} + \lambda$$

Malliavin-Shavgulidze measures

When B is Brownian bridge on $\mathbf{S}^1$ and $\lambda \in \mathbf{S}^1$ is a number uniformely chosen w.r.t. Lebesgue measure, the function f_σ defines a *random diffeomorphism of* $\mathbf{S}^1$, that we call the **Malliavin-Shavgulidze diffeomorphism**.
This gives a family of very nice measures on the group $\mathrm{Diff}^1_+(\mathbf{S}^1)$, called the *Malliavin-Shavgulidze measures*: for any $\sigma > 0$ we have a probability measure which is Radon and gives positive probability to every open set. Moreover, such measures are *Haar-like.*



Some regularity properties

Let f_σ be the Malliavin-Shavgulidze diffeomorphism (cf. the box at the bottom of the page), then $\log f'_\sigma$ is a Brownian bridge on $\mathbf{S}^1$. This implies that $\log f'_\sigma$ is a.s. *as regular as a Brownian motion is*, i.e. $\log f'_\sigma$ is a τ-Hölder function for every $\tau < 1/2$, but it is not 1/2-Hölder. In particular f_σ is not a C^{1+bv}-diffeomorphism.

What about the random dynamics?

The group of circle diffeomorphism is very well understood from the topological point of view, but not that much in *measurable way.* **The big picture below** describes the distribution of the *rotation number* for $0 < \sigma < 8$. We do not know whether for every $\sigma > 0$, the probability that the rotation number of f_σ is *irrational* is positive. Even more interesting, we do not know whether *Denjoy counter-examples* form a set of positive measure. However, we have some positive results when we restrict to diffeomorphisms with rational rotation number:

Theorem (T.). Every periodic orbit is a.s. hyperbolic and the C^1-centralizer of a diffeomorphism with periodic points is a.s. trivial.

Haar-like measures

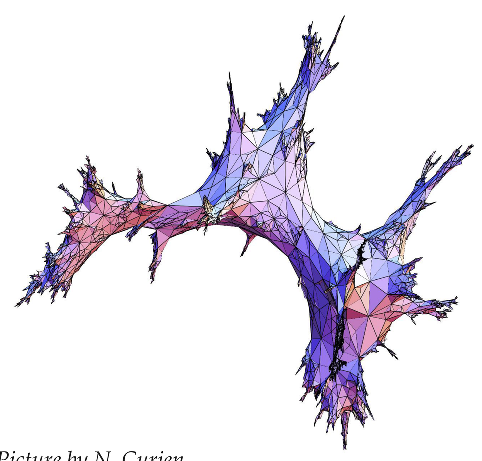
Since $\mathrm{Diff}^1_+(\mathbf{S}^1)$ is not a locally compact topological group, then there is no Haar measure on it. But it turns out that the Malliavin-Shavgulidze measures are *quasi-invariant* measures for the left action of a proper dense subgroup G (containing $\mathrm{Diff}^2_+(\mathbf{S}^1)$). We recall that a measure μ on a standard Borel space (X, μ) is quasi-invariant under the action of a group G if for every $g \in G$, $\mu(E) > 0 \Leftrightarrow \mu(gE) > 0$. More precisely we have:

Shavgulidze's theorem (1996). Fix $\sigma > 0$ and denote by μ_σ the Malliavin-Shavgulidze measure associated to f_σ. For any $g \in G$ we have the Radon-Nykodim cocycle

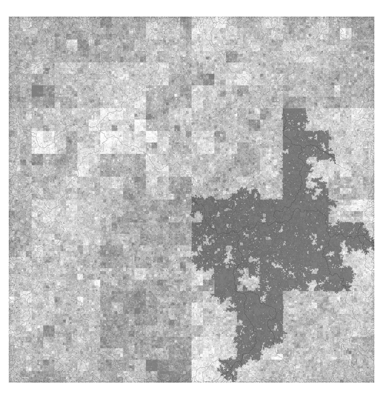
$$\frac{d(L_g)_*\mu_\sigma}{d\mu_\sigma}(f) = \exp\left\{ -\frac{1}{\sigma}\int_0^1 \frac{g''(f(t))}{g'(f(t))}\, df'(t) - \frac{1}{2\sigma}\int_0^1 \left[\frac{g''(f(t))}{g'(f(t))} f'(t)\right]^2 dt \right\}.$$

Quantum gravity

Is it possible to choose **uniformly at random** a conformal metric on a surface?



Picture by N. Curien



Picture by V. Beffara

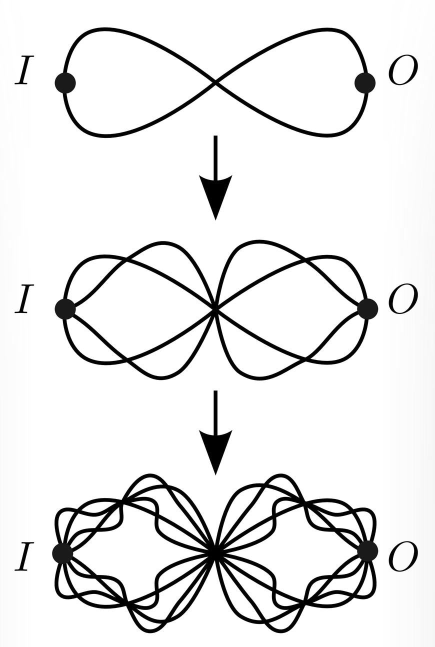
Wandering around the hierarchical eight graph

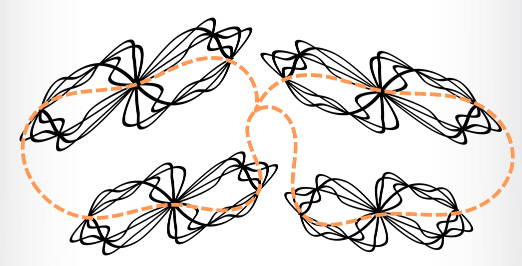
Michele Triestino (UMPA-ÉNS Lyon), joint work w/
M. Khristoforov (Chebyshev Lab of St. Petersburg)
& V. Kleptsyn (CNRS/IRMAR Rennes)

Multiplicative cascades...

The picture of the square above is a sample of a random metric obtained by the **multiplicative cascade** procedure:

- a probability distribution μ on $(0, \infty)$ is given. We will define a sequence of (discontinuous) piecewise flat metrics.
- Take a random variable $\xi_\emptyset$ of law μ: the first metric on the square $Q = Q_\emptyset$ is $\xi_\emptyset |dx|$.
- Cut $Q_\emptyset$ into four little squares Q_0, Q_1, Q_2, Q_3.
- Take four i.i.d. random variables ξ_i of law μ: the second metric is $\xi_i \xi_\emptyset |dx|$ if x belongs to Q_i.
- Keep on dividing the square and multiplying densities...
- **Is there a limit?** It is natural to expect that in general such a limit exists but is degenerate (distances either collapse or blow up).
- **Is there a good rescaling?** If $\lambda > 0$ define μ_λ to be the law of $\lambda \cdot \xi$, where ξ follows the distribution μ. Is there a parameter λ for which the multiplicative cascade procedure converges nontrivially?





...on hierarchical graphs

The first interesting example of convergence of metrics was given by I. Benjamini & O. Schramm, showing this for the interval. Their methods cannot work in 2D, expecially because we lose the uniqueness of the candidate geodesic path: there are so many ways to go from one point to another! Our results deal with an intermediate problem: we keep some features of the 1D world, but we increase the number of possible different paths between two points. For instance, we consider multiplicative cascades on the **hierarchical eight graph** (look at the picture on the left). The key remark is that any good limiting random metric must be *stationary* for the hierarchical construction:

- as a first step, it is enough to consider the random variable IO measuring the distance between the two extremities of the limiting hierarchical eight graph.
- Given a probability measure μ on $(0, \infty)$, we look for a $\lambda > 0$ such that the distance IO is equal in law to the random variable
$$\lambda\xi\left(\min(IO^1, IO^2) + \min(IO^3, IO^4)\right),$$
where ξ follows the law μ, the IO^i's the law IO, and all these variables are independent. Let us give some explanation: we draw for independent realizations of the metric for the hierarchical eight graph and we *paste* this four copies in order to make a bigger hierarchical eight graph, and then we multiply distances by a factor $\lambda\xi$ (look at the picture aside).
- It is then possible to define the distances between *dyadic* points of the hierarchical eight graph.

Main result (KKT). For any non-atomic fully supported probability distribution μ on $(0, \infty)$, there exists a renormalization factor $\lambda_{cr} = \lambda > 0$ so that the multiplicative cascade procedure with law μ_λ on the hierarchical eight graph converges, giving a non-trivial limiting metric.