# Groups of uniform homeomorphisms of covering spaces


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## Motivation

The uniform topology is one of basic topologies on function spaces. We study local and global topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric manifolds endowed with the uniform topology. Since the notions of uniform continuity and uniform topology depend on the choice of metrics, we are concerned with dependence of those properties on the behavior of metrics in neighborhoods of ends of manifolds.

## Basic Definitions and Notations

A map $f : (X, d) \to (Y, \varrho)$ between metric spaces is said to be uniformly continuous if

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that if $x, x' \in X$ and $d(x, x') < \delta$ then $\varrho(f(x), f(x')) < \varepsilon$.

The map $f$ is called a uniform homeomorphism if

$f$ is bijective and both $f$ and $f^{-1}$ are uniformly continuous.

A uniform embedding is a uniform homeomorphism onto its image.

For subsets $A, B$ of a metric space $(X, d)$ we say that

$B$ is a uniform neighborhood of $A$ in $(X, d)$ and write $A \subset_u B$

if $B$ contains the $\varepsilon$-neighborhood $O_\varepsilon(A)$ of $A$ in $X$ for some $\varepsilon > 0$.

A subset $A$ of $X$ is said to be uniformly discrete if there exists an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for any distinct points $x, y \in A$.

Suppose $(M, d)$ is a metric $n$-manifold (i.e., a separable topological manifold possibly with boundary assigned a fixed metric). For subsets $X$ and $C$ of $M$, let

$\mathcal{E}^u_*(X, M; C)$ = the space of proper uniform embeddings

$f : (X, d|_X) \to (M, d)$ with $f = \mathrm{id}$ on $X \cap C$.

This space is endowed with the uniform topology induced from the sup-metric

$$d(f, g) = \sup \big\{ d(f(x), g(x)) \mid x \in X \big\} \in [0, \infty] \qquad (f, g \in \mathcal{E}^u_*(X, M; C)).$$

For a subset $A$ of $M$ let

$\mathcal{H}^u_A(M, d)$ = the group of uniform homeomorphisms $h : (M, d) \to (M, d)$ with $h|_A = \mathrm{id}_A$, endowed with the uniform topology,

$\mathcal{H}^u_A(M, d)_b = \{ h \in \mathcal{H}^u_A(M, d) \mid d(h, \mathrm{id}_M) < \infty \}$ (the subgroup of bounded uniform homeomorphisms),

$\mathcal{H}^u_A(M, d)_0$ = the connected component of $\mathrm{id}_M$ in $\mathcal{H}^u_A(M, d)$.

Note that $\mathcal{H}^u_A(M, d)_b$ is an open subgroup of $\mathcal{H}^u_A(M, d)$ and so $\mathcal{H}^u_A(M, d)_0 \subset \mathcal{H}^u_A(M, d)_b$.

## Local deformation property of uniform embeddings

The newt definition is motivated by the Edwards-Kirby local deformation theorem for embeddings of compact spaces [1].

**Definition 1.** For a subset $A$ of $M$ we say that

"$A$ has the local deformation property for uniform embeddings in $(M, d)$"

and write $A$ : (LD)$_M$ if the following holds:

$(*)$ for any tuple $(X, W', W, Z, Y)$ of subsets of $M$ with

$X \subset A,\ X \subset_u W' \subset W \subset M$ and $Z \subset_u Y \subset M$

there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ such that

(1) for each $f \in \mathcal{W}$

(i) $\varphi_0(f) = f$, (ii) $\varphi_1(f) = \mathrm{id}$ on $X$,

(iii) $\varphi_t(f) = f$ on $W - W'$ and $\varphi_t(f)(W) = f(W)$ $(t \in [0, 1])$,

(iv) if $f = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(f) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0, 1])$,

(2) $\varphi_t(i_W) = i_W$ $(t \in [0, 1])$.

We omit the subscript $M$ in the symbol (LD)$_M$ when $A = M$.

The Edwards-Kirby deformation theorem ([1]) can be restated in the next form.

**Edwards-Kirby deformation theorem.**

$K \subset M$ : Relatively compact $\Longrightarrow$ $K$ : (LD)$_M$

**Baisc Properties of (LD).** ([4])

(1) (Invariance under Uniform Homeo's)

$M \cong N$ (uniform homeo) $\quad M$ : (LD) $\Longrightarrow$ $N$ : (LD)

(2) (Restriction) $A \subset B \subset M$ $\quad B$ : (LD)$_M$ $\Longrightarrow$ $A$ : (LD)$_M$

(3) (Additivity) $A \subset_u U \subset M$, $B \subset M$ $\quad U, B$ : (LD)$_M$ $\Longrightarrow$ $A \cup B$ : (LD)$_M$

(4) (Relatively compact subsets) $K \subset M$ : Relatively compact, $A \subset M$

$A$ : (LD)$_M$ $\Longleftrightarrow$ $A \cup K$ : (LD)$_M$

(5) (Neighborhoods of Ends)

$M = K \cup \cup_{i=1}^m L_i$ $\quad K \subset M$ : Relatively compact

$L_i \subset M$ : Closed, $L_i$ : $n$-manifold, $d(L_i, L_j) > 0$ $(i \neq j)$

$M$ : (LD) $\Longleftrightarrow$ $L_i$ : (LD) $(i = 1, \cdots, m)$

(6) $(M, d)$ : (LD) $\Longrightarrow$ $\mathcal{H}^u(M, d)$ : locally contractible

**Example 1.**

**[1]** Metric covering spaces over compact manifolds

The following notion is a natural metric version of Riemannian covering projections.

**Definition 2.** A map $\pi : (X, d) \to (Y, \varrho)$ is called a metric covering projection if it satisfies the following conditions:

$(\natural)_1$ There exists an open cover $\mathcal{U}$ of $Y$ such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of $X$ each of which is mapped isometrically onto $U$ by $\pi$.

$(\natural)_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in $X$.

$(\natural)_3$ $\varrho(\pi(x), \pi(x')) \leq d(x, x')$ for any $x, x' \in X$.

**Theorem 1.** $\pi : (M, d) \to (N, \varrho)$ : a metric covering projection, $N$ : a compact manifold $\Longrightarrow$ $(M, d)$ : (LD)

This theorem can be deduced from the Edwards-Kirby local deformation theorem and the classical Arzela-Ascoli theorem for equi-continuous families of maps

**[2]** Metric manifolds with geometric group actions

In term of covering transformations, Theorem 1 corresponds to the case of free group actions. For the non-free case, we have the following generalization ([4]).

**Definition 3.** $\varPhi : G \times X \to X$ : an action of a discrete group $G$ on a metric space $(X, d)$

(1) $\varPhi$ : locally isometric

$\Longleftrightarrow$ $\forall\, x \in X\ \exists\, \varepsilon > 0$ such that each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$.

(2) $\varPhi$ : locally geometric $\Longleftrightarrow$ $\varPhi$ : proper, cocompact and locally isometric.

**Corollary 1.** $(M, d)$ : a metric manifold

$(M, d)$ admits a locally geometric group action $\Longrightarrow$ $(M, d)$ : (LD)

This follows from Theorem 1 and Additivity of (LD)$_M$.

**[3]** Typical Model spaces

The following model spaces and their ends have the property (LD).

(1) Euclidean $n$-space $\mathbb{R}^n$ (with the standard Euclidean metric)

∘ $\mathbb{R}^n$ admits the canonical geometric action of $\mathbb{Z}^n$

(i) the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$

(ii) the half space $\mathbb{R}^n_{\geq 0} = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_n \geq 0 \}$

(2) Hyperbolic $n$-space $\mathbb{H}^n$

∘ $\mathbb{H}^n$ admits a Riemannian covering projection onto a closed hyperbolic $n$-manifold.

(3) Cylinder $(N \times \mathbb{R}, d)$

( $(N, \varrho)$ : Compact metric $n$-manifold, $d((x, t), (y, s)) = \sqrt{\varrho(x, y)^2 + |t - s|^2}$ )

∘ $\mathbb{Z} \curvearrowright N \times \mathbb{R}$ : $m \cdot (x, t) = (x, t + m)$ : a geometric action

(i) Cylindrical end (Product end) : $(N \times [0, \infty), d)$

## Global deformation property of uniform homeomorphisms

Euclidean $n$-space $\mathbb{R}^n$ has the similarity transformations

$$k_\gamma : \mathbb{R}^n \approx \mathbb{R}^n : \ k_\gamma(x) = \gamma x \qquad (\gamma > 0).$$

Conjugation with these similarity transformations enables us to deduce a global deformation property for uniform embeddings in the Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(0)$ $(r > 0)$ from the local one. Since this global deformation property is preserved by bi-Lipschitz equivalences, we can consider a more general setting of metric spaces with finitely many bi-Lipschitz Euclidean ends ([3]).

**Definition 4.** A bi-Lipschitz $n$-dimensional Euclidean end of a metric space $(X, d)$ means a closed subset $L$ of $X$ which admits a bi-Lipschitz homeomorphism $\theta : \mathbb{R}^n_1 \approx (L, d|_L)$ such that $\theta(\partial \mathbb{R}^n_1) = \mathrm{Fr}_X L$. We say that $L$ is isolated if $d(X - L, \theta(\mathbb{R}^n_r)) \to \infty$ as $r \to \infty$. Let $L' = \theta(\mathbb{R}^n_2)$, $L'' = \theta(\mathbb{R}^n_3)$.

**Theorem 2.** Suppose $X$ is a metric space and $L_1, \cdots, L_m$ are mutually disjoint isolated bi-Lipschitz Euclidean ends of $X$. Let $L' = L'_1 \cup \cdots \cup L'_m$ and $L'' = L''_1 \cup \cdots \cup L''_m$. Then there exists a strong deformation retraction $\varphi$ of $\mathcal{H}^u(X)_b$ onto $\mathcal{H}^u_{L''}(X)_b$ such that

$$\varphi_t(h) = h \ \text{ on } \ h^{-1}(X - L') - L' \ \text{ for any } \ (h, t) \in \mathcal{H}^u(X)_b \times [0, 1].$$

**Example 2.**

**[1]** $\mathcal{H}^u(\mathbb{R}^n)_b \simeq *$ $\qquad (\because\ \mathcal{H}^u(\mathbb{R}^n)_b \searrow \mathcal{H}^u_{\mathbb{R}^n_3}(\mathbb{R}^n) \simeq *)$

**[2]** $N$ : a compact connected 2-manifold with a nonempty boundary

$C = \cup_{i=1}^m C_i$ : a nonempty union of some boundary circles of $N$

$M := N - C$ (a noncompact connected 2-manifold)

$L_i$ : the end of $M$ corresponding to the boundary circle $C_i$

$d$ : a metic on $M$ such that each $L_i$ is an isolated bi-Lipschitz Euclidean end of $(M, d)$

$\Longrightarrow$ $\mathcal{H}^u(M, d)_0 \simeq *$ $\qquad (\because\ \mathcal{H}^u(M, d)_0 \simeq \mathcal{H}^u_{L''}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq *)$

In [2] we showed that $\mathcal{H}^u(\mathbb{R})_b$ is homeomorphic to $\ell_\infty$.

**Conjecture.** $\mathcal{H}^u(\mathbb{R}^n)_b$ is homeomorphic to $\ell_\infty$ for any $n \geq 1$.

## References


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