The minimal volume orientable hyperbolic 3-manifold with 4 cusps

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The volume of a hyperbolic 3-manifold is a topological invariant. The structure of the set of the volumes of hyperbolic 3-manifolds is known by the Jørgensen-Thurston theorem.

Theorem (Jørgensen-Thurston [6, Ch. 6])
The set of the volumes of orientable hyperbolic 3-manifolds is a well-ordered set of the type ω^ω with respect to the order of $\mathbb{R}$. The volume of an orientable hyperbolic 3-manifold with n-cusps corresponds to an n-fold limit ordinal.

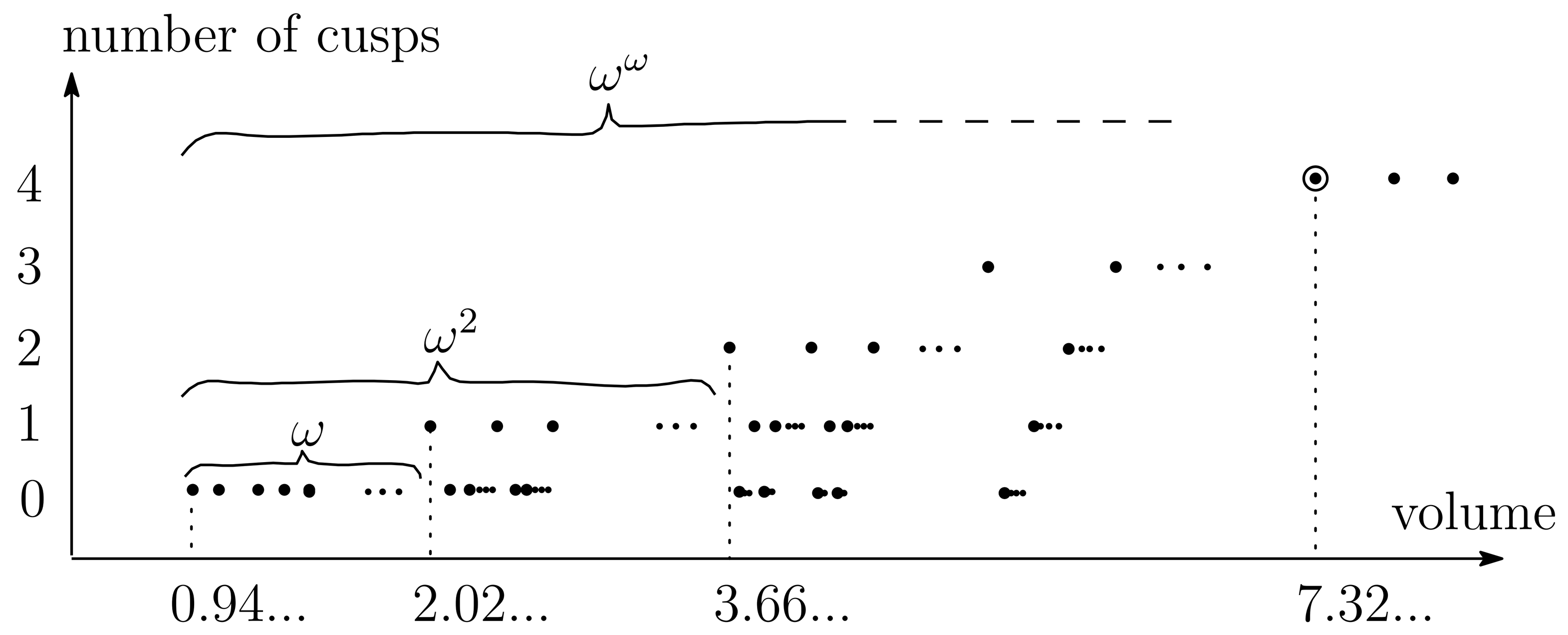

Figure 1: *The set of the volumes of orientable hyperbolic 3-manifolds*

This theorem gives rise to a problem: **What are the minimal volume orientable hyperbolic 3-manifolds with n cusps?** The answers are known in the cases where $0 \le n \le 2$ ([1], [3], [5]). Agol [1] conjectured which manifolds have the minimal volume in the cases where $n \ge 3$. We present the result that we determined it in the case where $n = 4$.

Main Theorem [7]
The minimal volume orientable hyperbolic 3-manifold with 4 cusps is homeomorphic to the 8^4_2 link complement. Its volume is $7.32... = 2V_8$, where V_8 is the volume of the ideal regular octahedron.

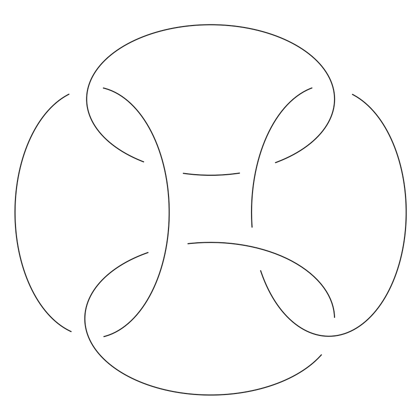

Figure 2: *The 8^4_2 link*

The 8^4_2 link complement is obtained from two ideal regular octahedra by gluing along the faces. Hence we need a lower bound on the volume of an orientable hyperbolic 3-manifold with 4 cusps. The proof relies on Agol's argument used to determine the minimal volume hyperbolic 3-manifolds with 2 cusps [1].

Let M be a finite volume hyperbolic 3-manifold, and let X be a (non-necessarily connected) essential surface in M. After we cut M along X, the relative JSJ decomposition can be performed by cutting along characteristic annuli. The obtained components are characteristic or hyperbolic.

- characteristic part — $T^2 \times I, S^1 \times D^2$ or I-bundle
- hyperbolic part — "guts"

The union of hyperbolic components is called the guts of $M - X$ and denoted by $\mathrm{Guts}(M - X)$. The guts admit another hyperbolic metric with totally geodesic boundary. Then we can obtain a lower bound of the volume of M.

Theorem (Agol-Storm-Thurston [2])
Let L be the guts of $M - X$. Then $\mathrm{vol}(M) \ge \mathrm{vol}(L) \ge \frac{V_8}{2}|\chi(\partial L)|$, where $\mathrm{vol}(L)$ is defined with respect to the hyperbolic metric of L with totally geodesic boundary. Moreover, L is obtained from ideal regular octahedra by gluing along the faces if the equality holds.

Therefore it is sufficient that we estimate the Euler characteristic of the boundary of guts. Let M be a finite volume hyperbolic 3-manifold with 4 cusps $T_0, \dots, T_3$.

Proposition
There exists an essential surface X in M such that $\mathrm{Guts}(M - X)$ have 4 (torus or annulus) cusps.

(sketch of proof.) We start from an essential surface X_0 which does not intersect $T_1, \dots, T_3$. Its existance is proved by character variety [4]. After performing annular compressions (i.e. surgery along an annulus between ∂M and X_0) for X_0 to $T_1, \dots, T_3$ as many times as possible, we obtain an essential surface X_1 such that $\mathrm{Guts}(M - X_1)$ intersect $T_1, \dots, T_3$.
If $\mathrm{Guts}(M - X_1)$ is not a component of $M - X_1$, there is more one cusp of $\mathrm{Guts}(M - X_1)$.
If $\mathrm{Guts}(M - X_1)$ is a component of $M - X_1$, we perform annular compression for the component of X_1 intersecting $\mathrm{Guts}(M - X_1)$ as many times as possible. Then we obtain an essential surface X such that $\mathrm{Guts}(M - X)$ intersect $T_0, \dots, T_3$. $\square$

We need to estimate the volume of a hyperbolic manifold L with totally geodesic boundary and 4 cusps.

Theorem
Let L be a hyperbolic manifold with totally geodesic boundary and 4 cusps. Then $\mathrm{vol}(L) \ge 2V_8$.

(sketch of proof.) If $\chi(\partial L) \le -4$, $\mathrm{vol}(L) \ge 2V_8$. Assume that $\chi(\partial L) = -2$.
Purely homological arguement shows that L has a non-separating essential surface Y_0. Beginning from this surface, we construct an essential surface Y in L such that $\chi(\partial \mathrm{Guts}(L - Y)) \le -4$. Since Agol-Storm-Thurston's inequality holds for L by doubling L, $\mathrm{vol}(L) \ge 2V_8$.

The geodesic boundary of L must intersect guts or I-bundle. $\chi(\partial \mathrm{Guts}(L - Y_0) \cap \partial L) = 0, -1$ or -2.
Suppose that $\chi(\partial \mathrm{Guts}(L - Y_0) \cap \partial L) = -2$. Then $\chi(\partial \mathrm{Guts}(L - Y_0)) \le -4$ since $\partial \mathrm{Guts}(L - Y_0)$ contains more than the part in ∂L.
Suppose that $\chi(\partial \mathrm{Guts}(L - Y_0) \cap \partial L) = 0$. Then we can modify the surface Y_0 and remove I-bundle on a half of ∂L. Therefore it is reduced to the case $\chi(\partial \mathrm{Guts}(L - Y_0) \cap \partial L) = -1$.
Suppose that $\chi(\partial \mathrm{Guts}(L - Y_0) \cap \partial L) = -1$ and $\chi(\partial \mathrm{Guts}(L - Y_0)) = -2$. Then we can modify the surface Y_0 and remove $Y_0 \cap \partial \mathrm{Guts}(L - Y_0)$. After performing this construction as many times as possible, we obtain an essestial surface Y such that $\chi(\partial \mathrm{Guts}(L - Y)) \le -4$. $\square$

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